1. DFT of pure sinusoids

We can think of a real-world signal that is a function of time $x(t)$. By recording its values at regular intervals, we can represent it as a vector of discrete samples $\vec{x}$, of length n.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[n-1] \end{bmatrix} \tag{1}$$

Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[n-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, that is

$$\vec{X} = U^{-1}\vec{x} = U^{*}\vec{x} \tag{2}$$

where U is a matrix of the DFT basis vectors ($\omega = e^{i\frac{2\pi}{n}}$).

$$U = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \tag{3}$$

Alternatively, we have that $\vec{x} = U\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{4}$$

In other words, $\vec{x}$ is a linear combination of the complex exponentials $\vec{u}_i$ with coefficients $X[i]$.

(a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

Compute $\vec{X}$ for this signal.

(b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

Repeat what you did above. What is $\vec{X}$ for this signal.

(c) Let's do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi k}{N}t\right)$, compute $\vec{X}$ of its vector form in discrete time, $\vec{x}$, of length $n = N$:

$$\vec{x} = \begin{bmatrix}\cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(n-1)\right)\end{bmatrix}^T.$$

2. The DFT basis and LTI systems

Suppose $\vec{x}$ is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $\vec{h}$. The output $\vec{y}$ is given by $C_{\vec{h}}\vec{x}$ where $C_{\vec{h}}$ is a circulant matrix with the first column given by $\vec{h}$. Suppose the DFT coefficients of $\vec{x}$ are given by

$$\vec{X} = \begin{bmatrix}X[0] & X[1] & \dots & X[n-1]\end{bmatrix}^T.$$

(a) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = 3$)

$$\vec{h} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$$

(b) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = 3$)

$$\vec{h} = \begin{bmatrix}\cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right)\end{bmatrix}^T$$

(c) Compute the DFT representation of the impulse response $\vec{h}$ as well as the eigenvalues of the circulant matrix that defines the system with that impulse response. ($n = N$)

$$\vec{h} = \begin{bmatrix}\cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(n-1)\right)\end{bmatrix}^T$$

3. Convolution and Duality

There is a term "convolution" that is often used in signal-processing. This is a very close sibling (how are they different?) to the term "correlation" that you have already learned in 16A. We call $C_{\vec{x}}\vec{y}$ the **circular convolution** of the signals $\vec{x}$ and $\vec{y}$, and this is sometimes denoted by $\vec{x} \circledast \vec{y}$.

(a) Let $\vec{z} = C_{\vec{x}}\vec{y}$. Let $\vec{X}, \vec{Y}, \vec{Z}$ be the DFT-basis representations of the signals $\vec{x}, \vec{y}, \vec{z}$. Use the properties of the DFT basis to relate $\vec{Z}$ to $\vec{X}$ and $\vec{Y}$.

(b) Now, let's do something seemingly strange. Let's try the same thing in the frequency-domain. Let $\vec{W} = C_{\vec{X}}\vec{Y}$. What is the relationship between $\vec{w}$ and $\vec{x}$ and $\vec{y}$?