EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz **Discussion 1B**

1. **Cosine Transformation** Assume that we are dealing with signals of length n.

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[n-1] \end{bmatrix} = \sum_{p=0}^{n-1} X[p]\vec{u}_p = \begin{cases} \sum_{p=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} X[p]\vec{u}_p & \text{if } n \text{ odd} \\ \sum_{p=-\frac{n}{2}+1}^{\frac{n}{2}} X[p]\vec{u}_p & \text{if } n \text{ even} \end{cases} \tag{1}$$

In the lectures, we have seen that we can represent signals with sums of complex exponentials in the DFT basis. One non-intuitive aspect is that, even when the signal is real, the basis is complex. In this problem, we will explore a different representation in which real signals are written as linear combinations of periodic real signals.

Specifically, we will show how to derive

$$x[t] = \alpha_0 + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \alpha_m \cos\left(\frac{2\pi m}{n} t + \phi_m\right). \tag{2}$$

First, we need to understand cosines with phases.

(a) For a real signal $\vec{x}$, the DFT coefficients are conjugate symmetric, i.e. $X[m] = X[-m]^*$ (you will show this in the homework). Therefore, suppose $X[m] = re^{i\theta}$, what is $X[-m]$?

(b) Now assume that, for a real signal $\vec{x}$, its DFT coefficients are $X[m] = 0$ for $m \neq \pm 5$. Show that we can represent the t-th component of $\vec{x}$ by $x[t] = \alpha \cos(\frac{2\pi}{n} 5t + \phi)$. Find α and ϕ. Your answer should be in terms of $|X[5]|$, and $\angle X[5]$.

(c) Therefore, let $\vec{x}$ be an arbitrary signal of length n, where n is odd. Write it as a sum of cosines where the cosine scaling and phase are written in terms of the DFT coefficients, $X[-\frac{n-1}{2}], \ldots, X[\frac{n-1}{2}]$. You can use $\angle z$ and $|z|$ to refer to the angle and magnitude of a complex number, respectively (i.e. $z = |z|e^{i\angle z}$).

(d) How about when n is even?

2. **Phase response** Let $\vec{x}$ be a real signal of length n (assume n odd till the end). In the previous part, we showed that we could write

$$x[t] = \alpha_0 + \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \alpha_p \cos\left(\frac{2\pi p}{n} t + \phi_p\right).$$

where there are $1 + \frac{n-1}{2}$ different α_p parameters and $\frac{n-1}{2}$ parameters ϕ_p.

Let C be a circulant matrix with eignenvalues $\lambda_0, \ldots, \lambda_{n-1}$. In lecture, we have seen that the DFT basis diagonalizes C (which correspond to LTI systems). However, the basis is complex and the eigenvalues are usually complex. However, if we push a real signal through C, we will get a real signal back. Where do all the imaginary parts go, then?

Let $\vec{y}$ be the output of C with the input $\vec{x}$. The reason why the DFT basis is so useful is that, since C is diagonalized by the basis, we have

$$y[t] = \frac{1}{\sqrt{n}} \sum_{p=0}^{n-1} \lambda_p X[p] e^{i\frac{2\pi p}{n}t} \tag{3}$$

(a) Use the fact that the complex exponentials are eigenvectors of C to write out what the output of C is when given the input $\vec{x}$, for the specific case of $x[t] = \cos(\frac{2\pi p}{n}t + \theta)$.

(b) Using the fact that the eigenvalues λ_p for a real circulant matrix C exhibit conjugate symmetry (from the Homework), what cosine does this output correspond to?

(c) What does the system C do to cosines in terms of the effect on their magnitude, frequency, and phase?

(d) Write $y[t]$ entirely as a sum of cosines.

(e) What changes for n even? (Think about what $\lambda_{\frac{n}{2}}$ must be like for a real C matrix.)

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