EECS 16B Spring 2016 — Designing Information Devices and Systems II — Anant Sahai and Michel Maharbiz — **Discussion 6B**

1. Complex Algebra

We provide a review of complex algebra to prepare you for the introduction of the phasor domain technique.

(a) A complex number can be written in the rectangular form: $z = x + iy$, where x and y are the real and imaginary parts of z, respectively. How to express the following values in polar forms, $|z|e^{i\theta}$, where $|z$ and θ are the magnitude and phase, respectively: -1, i, $-i$, $\sqrt{i}$, and $\sqrt{-i}$.

(b) **Euler's identity.** How to represent $\sin\theta$ and $\cos\theta$ using complex numbers?

(c) Show that $|z| = \sqrt{zz^*}$, where z^* is the complex conjugate of z.

Now let's tackle a numerical problem. Given two complex numbers, $V = 3 - i4$, $I = -(2 + i3)$.

(d) Express V and I in polar form.

(e) Find VI, VI^*, V/I, and $\sqrt{I}$.

We have provided you with a table of useful properties of complex numbers.

Complex Number Representation

Rectangular vs polar forms: $z = x + iy = |z|e^{i\theta}$ (1)

where $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$ (2)

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \quad (3)$$

Complex conjugate: $z^* = x - iy = |z|e^{-i\theta}$. (4)

$$(z + w)^* = z^* + w^*, (z - w)^* = z^* - w^* \quad (5)$$

$$(zw)^* = z^* w^*, (z/w)^* = z^*/w^* \quad (6)$$

$$z^* = z \Leftrightarrow z \text{ is real} \quad (7)$$

$$(z^n)^* = (z^*)^n \quad (8)$$

Complex Algebra

Let $z_1 = x_1 + iy_1 = |z_1|e^{i\theta_1}$, $z_2 = x_2 + iy_2 = |z_2|e^{i\theta_2}$.

(Note that we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ (9)

Multiplication: $z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$ (10)

Division: $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}e^{i(\theta_1 - \theta_2)}$ (11)

Power: $z_1^n = |z_1|^n e^{in\theta_1}$ (12)

$$z_1^{1/2} = \pm|z_1|^{1/2} e^{i\theta_1/2} \quad (13)$$

Useful Relations

$$-1 = i^2 = e^{i\pi} = e^{-i\pi} \quad (14)$$

$$i = e^{i\pi/2} = \sqrt{-1} \quad (15)$$

$$-i = -e^{i\pi/2} = e^{-i\pi/2} \quad (16)$$

$$\sqrt{i} = (e^{i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1+i)}{\sqrt{2}} \quad (17)$$

$$-\sqrt{i} = (e^{-i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1-i)}{\sqrt{2}} \quad (18)$$

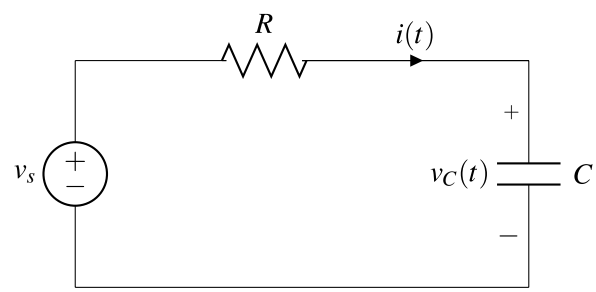
2. Phasor analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \mathfrak{Re}[Xe^{i\omega t}], \quad (19)$$

where X is a time-independent function called the phasor counterpart of $x(t)$. Thus, $x(t)$ is defined in the time domain, while its counterpart X is defined in the phasor domain.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s = 12\sin(\omega t - \frac{\pi}{4}), \tag{20}$$

with $\omega = 10^3$ rad/s, $R = \sqrt{3}\ k\Omega$, and $C = 1\ \mu F$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source v_s.

(a) **Step 1: Adopt cosine references**
All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert v_s into a cosine and write down its phasor representation V_s.

(b) **Step 2: Transform circuits to phasor domain**
The voltage source is represented by its phasor V_s. The current $i(t)$ is related to its phasor counterpart I by

$$i(t) = \mathfrak{Re}[Ie^{i\omega t}]. \tag{21}$$

What are the phasor representations of R and C?

(c) **Step 3: Cast KCL and/or KVL equations in phasor domain**
Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

(d) **Step 4: Solve for unknown variables**
Solve the equation you derived in Step 3 for I. What is the polar form of I ($Ae^{i\theta}$, where A is a positive real number)?

(e) **Step 5: Transform solutions back to time domain**
To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$?

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