EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz **Discussion 9A**

1. **Solutions of Differential Equations**

Consider the following system of linear differential equations

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}(t)$$

and denote the system matrix by A.

We know from the homework that the solution can be obtained by finding parameters of general solutions $c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$, where λ_1 and λ_2 ($\vec{v}_1$ and $\vec{v}_2$) are eigenvalues (eigenvectors) of A. In this problem we will consider the case when A has complex eigenvalues. Also we will introduce a technique called phase portraits for illustrating solutions.

(a) Verify that $\lambda = i$ and $\vec{v} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} i \end{bmatrix}$ is an eigenvalue/eigenvector pair for A . What is the other one? to the system.

(b) What is the solution of the system with the initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? Plot this as a function of time.

(c) Now treat the solution above as being parameterized by t and sketch the solution on the 2D-$\{\vec{x}_1, \vec{x}_2\}$ plane using an arrow to show the direction of flow (This is called the "phase portrait" when done for lots of initial conditions at once.)). How about if we have different initial conditions? such as $\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, or $\vec{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$?

(d) Let us do something more interesting, let $A = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix}$. What are the eigenvalue/eigenvectors? What is the general solution?

(e) Again plot the solution on the 2D-$\{\vec{x}_1, \vec{x}_2\}$ plane, with $\vec{x}(0) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$, what do you observe? Relate your observation to the properties of the eigenvalue.

2. **Controllability test**

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u[t]$$

(a) Is this system controllable?

(b) Consider running the system from zero state $\vec{x}[0] = \vec{0}$, what is the set of state that the system can reach assuming we can choose arbitrary u?

(c) If we change coordinates for $\vec{x}$, should controllability be changed? Or is this a property that must hold in all choice of coordinates?

(d) Come up with examples that you think are qualitatively different for when things are not controllable and when things are controllable. First consider the case of A being a diagonal matrix.

3. Controllability test in 2D: alternative derivation

For simplicity, consider the control of some two dimensional linear discrete time system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t]$$

where A is a 2×2 real matrix and B is a 2×1 real vector.

(a) As the system evolves from $t = 0$ to $t = m \geq 2$, show that

$$\vec{x}[m] - A^m\vec{x}[0] = \sum_{i=0}^{m-1} A^{m-i-1}Bu[i]$$

(b) Let the initial condition $\vec{x}[0] = [0,0]^T$, observe that the final state $\vec{x}[m]$ is a linear combination of the columns of the matrix $\begin{bmatrix} A^{m-1}B, & A^{m-2}B, & \cdots, & AB, & B \end{bmatrix}$ with coefficient $u[0], u[1], \cdots, u[m-1]$.

(c) Assume that we can apply arbitrary control inputs $\begin{bmatrix} u[0] & u[1] & \cdots & u[m-2] & u[m-1] \end{bmatrix}$, what are the possible values of the final state $\vec{x}[m]$?

(d) Recall that the range of a matrix is the span of its column vectors. Now we are going to show that all possible values of the final state $\vec{x}[m]$ is just $range([B, AB])$. To begin with, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, verify that

$$A^2 - (a+d)A + (ad-bc)I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(e) Show that all higher powers A^j, $j \geq 2$ can be written in terms of I and A.

(f) Use your results from part (c) and (e) to show that all possible values of $\vec{x}[m]$ is $range([B, AB])$.

(g) Connect what you have obtained to the controllability of the system.

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