This homework is due Monday February 1, 2016, at Noon.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?

2. **Orthonormal Proofs**

   In this problem, we ask you to establish several important properties of orthonormal bases. This is designed to both sharpen your understanding of these properties as well as practice doing proofs/derivations.

   Let $U = \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{n-1} \end{bmatrix}$ be an n by n matrix, where its columns $\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}$ form an orthonormal basis. One example of this is the DFT basis.

   (a) Show that $U^{-1} = U^*$, where U^* is the conjugate transpose of U.

   (b) Show that U preserves inner products, i.e. if $\vec{v}, \vec{w}$ are vectors of length n, then

   $$\langle \vec{v}, \vec{w} \rangle = \langle U\vec{v}, U\vec{w} \rangle .$$

   Recall that the inner-product is defined to be $\langle \vec{v}, \vec{w} \rangle = \vec{v}^* \vec{w}$.

   Also remember that for any matrices A, B or appropriate size so that their multiplication makes sense, that $(AB)^* = B^* A^*$. (This latter fact can be seen by looking at the entry in the ith row and jth column of $(AB)^*$. This is complex conjugate of the entry in the jth row and ith column of AB. This entry is just $A_j \vec{b}_i$ where A_j is the jth row of A and $\vec{b}_i$ is the ith column of B. Notice that the complex conjugate of this is just $\vec{b}_i^* A_j^*$ since the complex conjugate of a product is just the product of complex conjugates (most easily seen in polar form) and the complex conjugate of a sum is the sum of complex conjugates. This is thus the ith row and jth column of the product $B^* A^*$.)

   This fact is called Parseval's relation when applied in signal processing contexts and it helps us see orthogonality as well as think about energy in different bases.

   (c) Show that $\vec{u}_0, \ldots, \vec{u}_{n-1}$ must be linearly independent.
   (Hint: Suppose $\vec{w} = \sum_{i=0}^{n-1} \alpha_i \vec{u}_i$, then first show that $\alpha_i = \langle u_i, w \rangle$. From here ask yourself whether a nonzero linear combination of the $\{\vec{u}_i\}$ could ever be identically zero.)

   This basic fact shows how orthogonality is a very nice special case of linear independence.

   (d) Let M be a matrix which can be diagonalized by U, i.e. $M = U \Lambda U^*$, where Λ is a diagonal matrix with the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ along the diagonal. Show that M^* has the same set of eigenvectors U, while the eigenvalues of M^* are $\lambda_0^*, \ldots, \lambda_{n-1}^*$. (Aside: think about in which other problem on this HW this fact might be useful.)

   (e) Let V be another n by n matrix, where the columns also form an orthonormal basis. Show that the columns of the product, UV, also form an orthonormal basis. (This will turn out to be very helpful when we are defining a two-dimensional DFT or thinking image processing generally.)

3. Circulant Properties

In this problem, we address several properties of circulant matrices.

We will begin by establishing an important property. Let S be a circulant matrix. From what we have learned in the DFT lectures, we know that S is diagonalizable with the DFT basis. Here we want to show that the converse also holds. That is, if U is the DFT basis and Λ is a diagonal matrix (a diagonal matrix must have zero entries everywhere except possibly along the diagonal), then $U\Lambda U^*$ must be a circulant matrix. The first few parts of this problem are about showing this.

(a) Show that $U\Lambda U^* = \sum_{i=0}^{n-1} \lambda_i \vec{u}_i \vec{u}_i^*$.

(b) Show that for each i, the matrix $\vec{u}_i \vec{u}_i^*$ is circulant. Here, you will want to use the fact that U is the DFT basis.

(c) Argue why any linear combination of circulant matrices must be circulant.

(d) Use what you have established so far to conclude that $U\Lambda U^*$ is circulant.

(e) Let A, B be circulant matrices with the same size. Show $AB = BA$. In other words, circulant matrices commute with each other. (This fact is extremely useful when designing algorithms working with circulant matrices as well as when reasoning about Linear-Time-Invariant transformations modeled by circulant matrices.)

(f) Let A, B be circulant matrices with the same size. Use diagonalization and the above to show AB is circulant. (This shows that the composition of Linear-Time-Invariant transformations is itself Linear and Time-Invariant. And more importantly, it shows how the use of the Frequency Domain lets us understand such transformations much more easily.)

4. Conjugate Symmetry

The DFT basis is naturally complex. However, many signals that we are interested in understanding are real-valued. It is natural to wonder if anything special happens to real-vectors viewed in the DFT basis.

Why are such questions natural? Because a complex number can be viewed as a pair of real numbers. So, when we take n real numbers and transform them into n complex numbers, it is natural to wonder where the extra n real numbers have come from? Here we will see how the DFT exhibits something that we can call "conjugate symmetry" and that really, there are only n real numbers that determine everything.

(a) Convert $\vec{h} = [1, 2, 1, 0]^T$ to the DFT basis by computing $U^*\vec{h}$, where U is the DFT basis with $n = 4$. Plot the components using both Cartesian and Polar coordinates. Do you see something?

(b) Do the same for $\vec{h} = [2, 1, -1]^T$ by computing $U^*\vec{h}$, where U is now the DFT basis with $n = 3$. Do you see something?

(c) Let $\vec{x}$ be a real vector of length n, and let $\vec{X} = U^*\vec{x}$ be $\vec{x}$ in the DFT basis. Show that the k-th component of $\vec{X}$ satisfies $X[k] = (X[n-k])^*$, for $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Check that this holds in parts (a) and (b).

(d) Show that $X[0]$ is real if $\vec{x}$ is real. Check that this holds in parts (a) and (b).

(e) If n is even, show that $X[\frac{n}{2}]$ is real if $\vec{x}$ is real. Check that this holds in part (a).

(f) Do you think that this property has anything to do with what you showed in the earlier problem about orthonormal bases and what happens to the eigenvalues of the conjugate transpose of a matrix that is diagonalizable in that basis?

5. **Echo-removal**

This problem requires a newer version of numpy. Before you start, please update your numpy package by running:

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conda update -y numpy
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You receive a signal and it has an undesirable echo in it. However, you don't know what signal was originally transmitted, just about the echo.

(a) We can model the echo as an LTI system with the following impulse response,

$$h[n] = \delta[n] + \alpha\delta[n - t_d]$$

Here t_d represents the echo delay and α represents the amplitude of the echo. Here $\vec{\delta}$ is the standard first basis vector — having a 1 in its first position and zero everywhere else. The time-shift by t_d should be interpreted modulo N, the total length of the signal.

With this model, find the eigenvalues of the circulant matrix that applies the echo defined by $\vec{h}$.

(b) How would you correct for the echo? Design a system response that will do this by looking at the system in the DFT basis. What eigenvalues do you want to have?

(c) The result we dervied above is simple, yet powerful! We will apply this to "de-echo" a signal in the IPython notebook. Comment on this process. What would you want to be able to do if you wanted to be able to remove echoes without having to wait so long?

6. **Denoising Signals using the DFT**

Professor Maharbiz is sad. He just managed to create a beautiful audio clip consisting of a couple pure tones with beats and he wants Professor Sahai to listen to it. He calls Professor Sahai on a noisy phone and plays the message through the phone. Professor Sahai then tells him that the audio is very noisy and that he is unable to truly appreciate the music. Unfortunately, Professor Maharbiz has no other means of letting Professor Sahai listen to the message. Luckily, they have you! You propose to implement a denoiser at Professor Sahai's end.

(a) In the IPython notebook, listen to the noisy message. Plot the time signal and comment on visible structure, if any.

(b) Take the DFT of the signal and plot the magnitude. In a few sentences, describe what the spikes you see in the spectrum are.

(c) There is a simple method to denoise this signal: Simply threshold in the DFT domain! Threshold the DFT spectrum by keeping the coefficients whose absolute values lie above a certain value. Then take the inverse DFT and listen to the audio. You will be given a range of possible values to test. Write the threshold value you think works best.

Yay, Professor Maharbiz is no longer sad!

7. **Your Own Problem**

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?