EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz

Homework 10

This homework is due April 11, 2016, at Noon.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?

2. **Lecture Attendance**

   Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

   (a) Monday lecture

   (b) Wednesday lecture

   (c) Friday lecture

3. **Redo problem 3 on the midterm**

   (a)

   (b)

   (c)

   (d)

   (e)

   (f)

   (g)

4. **Redo problem 4 on the midterm**

   (a)

   (b)

   (c)

5. **Redo problem 5 on the midterm**

   (a)

   (b)

6. **Redo problem 6 on the midterm**

   (a)

(b)

(c)

7. Redo problem 7 on the midterm

(a)

(b)

8. Redo problem 8 on the midterm

(a)

(b)

(c)

(d)

(e)

(f)

9. Redo problem 9 on the midterm

(a)

(b)

10. Redo problem 10 on the midterm

11. Redo problem 11 on the midterm

12. Midterm comments

Please give your feedback and comments on the midterm. Did you find it fair? What could we have done to have made this exam better?

13. SVD for minimum energy control

Consider the open-loop discrete-time linear system with state $\vec{x}(t)$ and scalar control $u(t)$ defined by the recursive equation:

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t).$$

We want to drive the system from some initial state $\vec{x}_0$ to $\vec{x}_f$. We know that if A, B are controllable and the dimension is n, then clearly we can get to the desired $\vec{x}_f$ in n steps. However, suppose that we only need to get there by $m > n$ steps. We now have a lot of flexibility. How can we choose a best control to apply?

One choise is to ask for a control that gets us to the desired $\vec{x}_f$ using minimal energy. i.e., having minimal $\sum_{t=0}^{m-1} |u(t)|^2$.

This often can make sense when the underlying units of the $u(t)$ is an applied voltage or some other voltage-like quantity because then the energy consumed is proportional to the square of $u(t)$.

(a) Consider the system evolution equations from $t = 1$ to $t = m$, obtain an expression for the final state $\vec{x}(m)$ as a function of the initial state $\vec{x}_0$ and control inputs.

(b) Write out the above equation in a giant matrix form, with $\vec{u} = [u(0), u(1), \cdots, u(m-1)]^T$.

(c) Now you have obtained a linear equation in the form $\vec{y} = G\vec{u}$, where $\vec{y}$ and G were found in the previous parts. Recall that in HW4 problem 3, you have shown that the solution obtained by means of the psuedo-inverse (computed using the SVD) has a nice minimality property. Use this to derive the minimum energy control inputs $\vec{u}$.

(d) How could you extend the above to the case when $u(t)$ is a vector $\vec{u}(t)$?

(e) Let's revisit Beiber's segway from Homework 3 from EE16A. In that problem, we had a 4-dimensional state vector

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{1}$$

We modeled the motion of the segway as a cart-pole system, which was linearized to be a linear system. We are given the following information:

$$A = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix} \tag{2}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix} \tag{3}$$

$$\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.103227 \\ 0.9120005 \\ -24 \end{bmatrix} \tag{4}$$

Previously, we wanted to find the controls needed to reach the target state $\vec{x}_f = \vec{0}$ in exactly 4 time steps. Because we chose 4 steps, the problem had exactly one solution. However, if we now increase the number of steps to $m > 4$, then there are infinitely many solutions to this problem. Let's use what we learned from the previous parts of this question to find the minimum energy solution $\sum_{t=0}^{m-1} ||u(t)||^2$ when we set $m = 30$. Fill in the missing code in the IPython notebook to compute the minimum energy solution to this problem.

14. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}(t+1) = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$.

(a) Is this system controllable?

(b) Is this discrete-time linear system stable on its own?

(c) Suppose we use state feedback of the form $u(t) = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}(t)$

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) Now suppose we've got a seemingly different system described by the controlled scalar difference equation $z(t+1) = z(t) + 2z(t-1) + u(t)$. (Think back to the Fibonacci number problem you saw in 16A.) Write down the above system's representation in the following matrix form:

$$\vec{z}(t+1) = A_1 \vec{z}(t) + B_1 u(t). \tag{5}$$

Please specify what the vector $\vec{z}(t)$ consists of as well as the matrix A_1 and the vector B_1.

(e) We will now show how the initial matrix representation for $\vec{x}(t)$ can be converted to the canonical form for $\vec{z}(t)$ using a change of basis. Suppose we do a transform the coordinates of the state $\vec{x}(t)$ to $\vec{z}(t) = P\vec{x}(t)$. Write down the state-transition matrices of $\vec{z}(t)$ in terms of the state transition matrices of $\vec{x}(t)$, i.e., express A_1 and B_1 in terms of A, B, and P.

(f) For $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, confirm that the state space representation of $\vec{z}(t)$ is the same as part (d). Design a feedback $\begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Confirm that $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 & \bar{f}_2 \end{bmatrix} P$.

(g) Here, we gave you the P matrix. How would you have come up with the P matrix on your own? (Hint: start with the second column of P and ask where it might have come from. Then, is there a relationship between the coefficients of the difference equation in part (d) to the polynomial whose roots you need to find in part (b)?)

By following this technique, any controllable discrete-time system can be converted to the "controllable canonical form" shown in part (d) by finding the right change of coordinates.

(h) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and B from (a) for the system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + w(t)$$

into the IPython notebook and use the random input $w(t)$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}(t)$ evolves over time for the given A. What do you see happening to the norm of the state?

(i) Add the feedback computed in part (c) to the system in the notebook and explain how the norm of the state changes.

(j) Now we evaluate a system described by the following scalar system $z(t+1) = az(t) + u(t) + w(t)$ in the iPython notebook. Consider two values of a, one case with $a > 1$ and one with $a < 1$, to observe the difference in the evolution of $|z(t)|$ for the same error as part (g). Describe the differences between the two.

(k) Suppose that the disturbance is actually coming from observation noise. We assume $y(t) = x(t) + w(t)$ where $w(t)$ is some random noise. Add a state feedback $u(t) = ky(t)$ to the system so that the resulting closed loop system is described by $z(t+1) = (a+k)z(t) + kw(t)$. Say we know $a = -3$. For what values of k's will the result be bounded. Confirm with the norm of the closed loop system.

(l) Is it advisable to have $a+k$ close to zero if we want to minimize the magnitudes of the state $x(t)$? How does the effect of the noise in the observation influence this? Assign values of k close to $-a$ to see the effect in the IPython notebook.

15. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

Contributors:

- Ioannis Konstantakopoulos.
- Stephen Bailey.
- Varun Mishra.