This homework is due April 18, 2016, at Noon.

1. **Homework process and study group**

   (a) Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.)

   (b) How long did you spend working on this homework? How did you approach it?

2. **Lecture Attendance**

   Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

   (a) Monday lecture

   (b) Wednesday lecture

   (c) Friday lecture

3. **Disturbance rejection** Consider the system

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{w}(t)$$

   where A is an $n \times n$ matrix and $\vec{w}(t)$ is a bounded disturbance that satisfies, for some $\varepsilon > 0$, $|\vec{w}(t)[i]| \leq \varepsilon$ for all $t \geq 0$ and $0 \leq i \leq n-1$. In the lectures, we learned that if the eigenvalues of A have magnitude less than 1, then $\vec{x}(t)$ is bounded for all t. In this problem, we will compute explicit bounds for $\vec{x}(t)$.

   (a) Let $A = \begin{pmatrix} 1 & \frac{3}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}$. Compute the diagonalization $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix with the eigenvalues of A along its diagonal in decreasing order, and the columns of V are eigenvectors of A. For convenience, choose eigenvectors such that the entries are integers with no common factor, and such that the bottom row of V is positive.

   (b) Consider the change of coordinates $\vec{z}(t) = V^{-1}\vec{x}(t)$. First, suppose that the initial conditions are $|\vec{z}(0)[i]| = 0$ for $i = 0,1$. Following the proof in lecture, compute a constant upper bound on $|\vec{z}(t)[i]|$ in terms of ε that holds for all $t \geq 0$, for $i = 0,1$ (*Hint*: Consider the largest term in V^{-1}.)

   (c) Now suppose instead that the the initial conditions satisfy $|\vec{z}(0)[i]| \leq \alpha$ for $i = 0,1$ and for some $\alpha > 0$. Update the previous part to compute an upper bound on $|\vec{z}(t)[i]|$ in terms of ε, α that holds for all $t \geq 0$, for $i = 0,1$.

   (d) Find an upper bound for $|\vec{x}(t)[i]|$ for $i = 0,1$, by considering the largest term in V and using the bound from the previous part.

   (e) Consider all 2×2 matrices with the same eigenvalues as A. Can the method described above give an upper bound on $|\vec{x}(t)|$ that holds for all such matrices? Why or why not?

(f) We now consider the case where A has repeated eigenvalues. Let $A = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{3}{2} & 2 \end{pmatrix}$. Compute Q, U such that $A = QUQ^{-1}$, where Q has orthonormal columns and U is upper triangular. Choose the columns of Q such that the entries in the left column and bottom row are positive. (*Hint:* First compute an eigenvector of A as the first column of Q, and choose a vector orthonormal to this eigenvector as the second column. Then solve for the non-diagonal entries of U.)

(g) Repeat parts (b) and (c) for this new choice of A.

(h) Repeat part (d) for this new choice of A.

(i) Repeat part (e) for this new choice of A.

4. Controllable Canonical Form and Eigenvalue Placement

Consider a linear discrete time system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $\vec{b} \in \mathbb{R}^n$).

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t)$$

If the system is *controllable*, then there exists a transformation $\vec{z} = T\vec{x}$ (where T is an invertible $n \times n$ matrix) such that in the transformed coordinates, the system is in *controllable canonical form*, which is given by

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

The characteristic polynomials of the matrices A and $\widetilde{A}$ are the same and given by

$$\det(\lambda I - A) = \det(\lambda I - \widetilde{A}) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_0 \tag{1}$$

(a) Show that $\widetilde{A} = TAT^{-1}$ and $\widetilde{\vec{b}} = T\vec{b}$.

(b) Show that A and $\widetilde{A}$ have the same eigenvalues (*Hint: let $\vec{v}$ be an eigenvector of A; use $T\vec{v}$ for $\widetilde{A}$*)

(c) Let the controllability matrices C and $\widetilde{C}$ be $C = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{n-1}\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} & \cdots & \widetilde{A}^{n-1}\widetilde{\vec{b}} \end{bmatrix}$, respectively. Show that $\widetilde{C} = TC$, which is equivalent to $T = \widetilde{C}C^{-1}$.

Now, consider the specific system

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t) = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \tag{2}$$

(d) Show that the system (2) is controllable.

Since the system is controllable, there exists a transformation $\vec{z} = T\vec{x}$ such that

$$\vec{z}(t+1) = \widetilde{A}\vec{z}(t) + \widetilde{\vec{b}}u(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3}$$

(e) Compute the matrix $\widetilde{A}$.

(f) Compute the controllability matrices $C = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix}$ and $\widetilde{C} = \begin{bmatrix} \widetilde{\vec{b}} & \widetilde{A}\widetilde{\vec{b}} \end{bmatrix}$.

(g) Compute the transformation matrix $T = \widetilde{C}C^{-1}$.

(h) Show that the system (2) is *unstable* in open-loop.

Now, we want to make the system *stable* by applying state feedback for the system in the canonical form (3). That is, let $u(t)$ be $u(t) = -\vec{k}^T\vec{z}(t) = \begin{bmatrix} -k_0 & -k_1 \end{bmatrix}\vec{z}(t)$. After applying state feedback, the systems (2) and (3) have the form

$$\vec{x}(t+1) = A_{cl}\vec{x}(t)$$
$$\vec{z}(t+1) = \widetilde{A}_{cl}\vec{z}(t)$$

(i) Compute A_{cl} and $\widetilde{A}_{cl}$ in terms of k_0 and k_1.

(j) Compute $\vec{k}$ so that $\widetilde{A}_{cl}$ has eigenvalues $\lambda = \pm\frac{1}{2}$ (*Hint: use Formula (1)*).

(k) Using the $\vec{k}$ you derived in the previous part, show that A_{cl} also has eigenvalues $\lambda = \pm\frac{1}{2}$ by explicit calculation.

5. Observers

So far we have used *state feedback* to construct the controls and hence the closed-loop system. State feedback uses the values of the state variables to set the values of the control inputs. What we have so far neglected to mention, is that the the state of the system is not always easily determined.

Directly measuring all the state variables is often impossible or very expensive. Instead, we have a more limited set of outputs that can tell us something about the state inputs. Fortunately, as we've seen previously, if the system is observable, it is possible to determine the state from the ongoing sequence of outputs over time.

In this problem you will learn how to construct a *state estimator* (also called an *observer*) that reconstructs the state of the system from the values of its outputs. You will also learn about the connection, or duality, between observability and controllability.

We start with an observable linear discrete-time system:

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t) \end{aligned} \tag{4}$$

The system has n state variables, n_i inputs, and n_o outputs. In other words $\vec{x}(t) \in \mathbb{R}^n$, $\vec{u}(t) \in \mathbb{R}^{n_i}$, and $\vec{y}(t) \in \mathbb{R}^{n_o}$.

We would like to build an observer for $\vec{x}(t)$. The observer is an additional system (that we make computationally) that monitors the inputs and outputs of the system and produces an estimate $\vec{\hat{x}}$ of $\vec{x}$, the state of the original system:

$$\vec{\hat{x}}(t+1) = A\vec{\hat{x}}(t) + B\vec{u}(t) - \underbrace{L\left(\vec{\hat{y}}(t) - \vec{y}(t)\right)}_{\text{output feedback}} \tag{5}$$

$$\vec{\hat{y}}(t) = C\vec{\hat{x}}(t)$$

The observer has an additional term, the *output feedback*. When the observer output diverges from the system output, the output feedback will nudge the observer so that it will eventually match the system output better. If L is chosen the right way, it will ensure that the estimate $\vec{\hat{x}}$ tracks the state $\vec{x}$.

(a) Let us define the estimation error at time t as $\vec{e}(t) := \vec{\hat{x}}(t) - \vec{x}(t)$. Show that the estimation error follows the recurrence

$$\vec{e}(t+1) = (A - LC)\vec{e}(t).$$

(b) Under what conditions is $\vec{e}(t)$ guaranteed to converge to zero? How fast does it happen? Recall results from lecture and discussion on similar systems. What property of systems does it remind you of?

(c) Under what condition is $\vec{e}(t)$ guaranteed to converge to zero in n steps? *HINT: Consider a nilpotent matrix. Recall that an $n \times n$ matrix M is nilpotent if $M^n = 0$, and that a matrix is nilpotent iff 0 is its only eigenvalue.*

(d) If the estimation error $\vec{e}$ goes to zero, what can you say about the estimate $\vec{\hat{x}}$?

We would like to choose the output feedback matrix L in such a way that the eigenvalues of $A - LC$ are placed where we want them to. This is very similar to choosing the state feedback, where we choose an F so that the eigenvalues of $A - BF$ are placed in the position we want them to be. The difficulty is that L appears to the right of C, while F appears to the left of B in the formula.

The way to resolve this difficulty is to introduce yet another system, the *dual* system:

$$\vec{z}(t+1) = A^T \vec{z}(t) + C^T \vec{v}(t) \tag{6}$$

Note that we don't really care how the state $\vec{z}$ of this system evolves, and how the inputs $\vec{v}$ of this system are set. We only care about how to place its eigenvalues, as you will see shortly.

(e) Show that the dual system is controllable, if and only if the original system is observable. *HINT: A matrix and its transpose have the same rank.*

We would like to close the loop of the dual system and place the eigenvalues of the closed loop dual system to appropriate values.

(f) Let us define $\vec{v}(t) = -L^T \vec{z}(t)$. What is the closed-loop matrix of the dual system? How does it relate to $\vec{e}(t)$?

(g) Assuming that we know how to find a state-feedback matrix that places the eigenvalues of the closed loop system at $\lambda_0, \ldots, \lambda_{n-1}$. Show how can the output-feedback L can be set such that the eigenvalues of $A - LC$ are $\lambda_0, \ldots, \lambda_{n-1}$. *HINT: recall that a matrix and its transpose have the same eigenvalues.*

You have seen how to use algorithms for computing state-feedback to compute the output-feedback matrix needed for building an observer to estimate the state of a dynamic system.

6. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

Contributors:

- Lynn Chua.
- Yen-Sheng Ho.
- Baruch Sterin.
- Ioannis Konstantakopoulos.