EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz **Homework 12**

This homework is due April 25, 2016, at Noon.

1. Homework process and study group

(a) Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.)

(b) How long did you spend working on this homework? How did you approach it?

2. Lecture Attendance

Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

(a) Monday lecture

(b) Wednesday lecture

(c) Friday lecture

3. Tracking a Desired Trajectory in Continuous

Many problems in 16AB so far have treated closed-loop control as being about holding a system steady at some desired operating operating point, which was designated as zero in a linear model. This control used the actual current state (and in principle can use an estimate of the state from an observer) to apply a control signal designed to bring the state to zero. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how can we get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be buffeted by small disturbances throughout.

The key conceptual idea is to realize that we can change coordinates in a time-varying way so that "zero" is the desired "open-loop" trajectory.

Consider a linear continuous-time system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$).

$$\dot{\vec{x}} = A\vec{x}(t) + Bu(t)$$

where $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times 1}$.

Here we use the physics-style Newton convention of denoting time derivatives by placing dots above variables.

(a) What is the condition for the system to be controllable?

Now, consider the system

$$\dot{\vec{x}} = A\vec{x}(t) + Bu(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{1}$$

(b) Is the given system controllable?

(c) Now, consider that we supposed that we started at $\vec{x}(0) = \vec{0}$ and had a nominal control signal $u_n(t)$ that would make the system follow the desired trajectory $\vec{x}_n(t)$ that satisfies (1) together with $u_n(t)$.

Change variables to $\vec{x}(t) = \vec{x}_n(t) + \vec{v}(t)$ and $u(t) = u_n(t) + u_v(t)$ and write out what (1) implies for the evolution of the trajectory deviation $\vec{v}(t)$ as a function of the control deviation $u_v(t)$.

(d) Now, add a bounded disturbance term $\vec{w}(t)$ to the original state evolution in (1) and see if you can absorb that entirely within an evolution equation for $\vec{v}(t)$ based on $u_v(t)$. Write out the resulting equation for the dynamics as:

$$\dot{\vec{v}} = A_v\vec{v} + B_v u_v + \vec{w} \tag{2}$$

What are A_v and B_v ?

(e) Based on what you have found above, how will the system behave over time? If there is some small distubance, will we end up following the intended trajectory $\vec{x}_n(t)$ closely if we just apply the control $u_n(t)$?

Now, we want to apply state feedback control to the system to get it to more or less follow the desired trajectory.

(f) Just looking at the $\vec{v}(t), u_v(t)$ system, how would you apply state-feedback to choose $u_v(t)$ as a function of $\vec{v}(t)$ that would place both the eigenvalues of the closed-loop $\vec{v}(t)$ system at -10.

(g) Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x}_n(t)$, the nominal controls $u_n(t)$, and the actual measurement of the state $\vec{x}(t)$, come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.

4. Continuous-Time Analog Observer Design: Ship Autopilots

Modern ships use autopilots for steering. The main task of the autopilot is to maintain constant heading. A common system model used for ship steering controllers is the *Nomoto* first-order model. It is described using the following differential equation:

$$T\ddot{\psi} + \dot{\psi} = K\delta,$$

where ψ is the ship heading, δ is the rudder angle, and K and T are constants that are empirically estimated during sea trials. The "dot" notation used here is the physics convention (Newton's notation) that is very convenient for problems where nothing more that a second derivative is needed.

The only sensor is a gyrocompass, which reports the ship's current heading $y(t) = \psi(t)$. We would also like to provide a good estimate of an additional important parameter, the rate of turn — the derivative of the ship's current heading.

The input of the ship model is the rudder angle δ, and the output is the heading ψ, as measured by the gyrocompass.

(Note for the curious: undoubtably, some of you are wondering why we don't just take the derivative of the measurement and be done with it. The reason is that although we are describing everything without any noise, in the real-world, all measurements are noisy. Taking the derivative of noise is a very bad idea because it is in the nature of noise to shake a lot and so the derivative gets swamped by the shaking of the noise.)

In this problem you'll construct an analog continous-time observer, and then analyze its behaviour.

(a) Choose your state variables so that you have a two-dimensional state.

(b) Write down the system as a state-space model with a two-dimensional state.

(c) Is the system observable?

(d) Write down a model for the observer in matrix form using $\vec{\ell}$ to represent how you weigh the difference between the observed output $y(t)$ and the estimated output $\hat{y}(t)$ coming from within your observer.

(e) Find $\vec{\ell} = \begin{bmatrix} l_0 \\ l_1 \end{bmatrix}$ to place both the eigenvalues of the estimation error evolution at -2.

Now that we have designed the output-feedback and placed the eigenvalues of the estimation error. We'll design a circuit implementing the observer.

We will represent the state variables as voltages. Each input, output, and state variable will be implemented as a node in our circuit. The output of the original systems (the gyrocompass) would be an input of this system, and so would the rudder angle.

Recall that in EE16A and previously in EE16B, you have seen how to implement the following operations using simpler circuit elements (mainly resistors, capacitors and op-amps): differentiation, integration, scaling, addition and negation. This will be enough to implement the observer.

(f) Design a circuit whose output is the integral of its input with respect to time.

(g) Design a circuit whose output is a scaled version by a constant a_0 of its input.

(h) Design a circuit whose output is the negation of its input.

(i) Design a circuit whose output is the sum of its two inputs.

Now that we have the basic circuit elements. We'll implement the observer as a circuit.

(j) Use the circuits you designed above to construct the observer as a circuit driven by the output of the gyrocompass.

5. Lagrange interpolation by polynomials

Given n distinct points and the corresponding evaluations/sampling of a function $f(x)$, $(x_i, f(x_i))$ for $0 \leq i \leq n-1$, the Lagrange interpolating polynomial is the polynomial of the least degree which passes through all the given points.

Given n distinct points and the corresponding evaluations, $(x_i, f(x_i))$ for $0 \leq i \leq n-1$, the Lagrange polynomial is

$$P_n(x) = \Sigma_{i=0}^{i=n-1} f(x_i) L_i(x),$$

where

$$L_i(x) = \Pi_{j=0;j\neq i}^{j=n-1} \frac{(x-x_j)}{(x_i-x_j)} = \frac{(x-x_0)}{(x_i-x_0)} \cdots \frac{(x-x_{i-1})}{(x_i-x_{i-1})} \frac{(x-x_{i+1})}{(x_i-x_{i+1})} \cdots \frac{(x-x_{n-1})}{(x_i-x_{n-1})}$$

.

Here is an example: for two data points, $(x_0, f(x_0))$ = (0, 4), $(x_1, f(x_1))$ = (-1, -3), we have

$$L_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-(-1)}{0-(-1)} = x+1$$

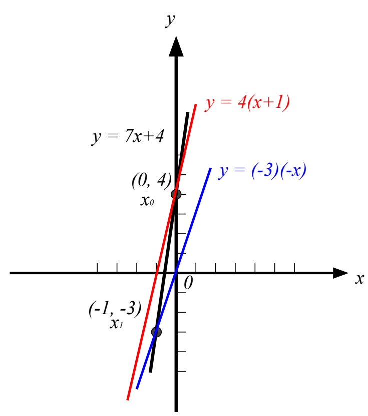
and

$$L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-(0)}{(-1)-(0)} = -x$$

. Then

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) = 4(x+1) + (-3)(-x) = 7x+4$$

We can sketch those equations on the 2D plane as follows:



(a) Given three data points, (2, 3), (0, -1) and (-1, -6), find a polynomial $f(x) = ax^2 + bx + c$ fitting the three points. Do this by solving a system of linear equations for the unknowns a, b, c. Is this polynomial unique?

(b) Like the monomial basis $\{1, x, x^2, x^3, \ldots\}$, the set $\{L_i(x)\}$ is a new basis for the subspace of degree n or lower polynomials. $P_n(x)$ is the sum of the scaled basis polynomials. Find the $L_i(x)$ corresponding to the three sample points in (a). Show your steps.

(c) Find the Lagrange polynomial $P_n(x)$ for the three points in (a). Compare the result to the answer in (a). Are they different from each other? Why or why not?

(d) Sketch $P_n(x)$ and each $f(x_i)L_i(x)$ on the 2D plane.

(e) Show that the Lagrange interpolating polynomial must pass through all given points. In other words, show that $P_n(x_i) = f(x_i)$ for all x_i. Do this in general, not just for the example above.

6. The vector space of polynomials

A polynomial of degree at most n on a single variable can be written as

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots p_nx^n$$

where we assume that the coefficients $p_0, p_1, \ldots, p_n$ are real. Let P_n be the vector space of all polynomials of degree at most n.

(a) Consider the representation of $p \in P_n$ as the vector of its coefficients in $\mathbb{R}^{n+1}$.

$$\vec{p} = \begin{bmatrix} p_0 & p_1 & \dots & p_n \end{bmatrix}^T$$

Show that the set $\mathscr{B}_n = \{1, x, x^2, \dots, x^n\}$ forms a basis of P_n, by showing the following.

- Every element of P_n can be expressed as a linear combination of elements in $\mathscr{B}_n$.
- No element in $\mathscr{B}_n$ can be expressed as a linear combination of the other elements of $\mathscr{B}_n$.
(*Hint*: Use the aspect of the fundamental theorem of algebra which says that a nonzero polynomial of degree n has at most n roots, and use a proof by contradiction.)

(b) Suppose that the coefficients $p_0, \dots, p_n$ of p are unknown. To determine the coefficients, we evaluate p on $n+1$ points, $x_0, \dots, x_n$. Suppose that $p(x_i) = y_i$ for $0 \le i \le n$. Find a matrix V in terms of the x_i, such that

$$V \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

(c) For the case where $n = 2$, compute the determinant of V and show that it is equal to

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i).$$

Conclude that if $x_0, \dots, x_n$ are distinct, then we can uniquely recover the coefficients $p_0, \dots, p_n$ of p. This holds for $n > 2$ in general, but consider only the case where $n = 2$ for now.

(d) (optional) Argue using Lagrange interpolation that indeed such matrices V above must always be invertible if the x_i are distinct.

(e) We can define an inner product on P_n by setting

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) \, dx.$$

Show that this satisfies the following properties of a real inner product. (We would have to put in a complex conjugate on p if we wanted a complex inner product.)

- $\langle p, p \rangle \ge 0$, with equality if and only if $p = 0$.
- For all $a \in \mathbb{R}$, $\langle ap, q \rangle = a \langle p, q \rangle$.
- $\langle p, q \rangle = \langle q, p \rangle$.

(f) Now that we have an inner product on P_n, we can consider orthonormality. If $\mathscr{B} = \{b_0, b_1, \dots, b_n\}$ is a basis for P_n, we say that it is an *orthonormal* basis if

- $\langle b_i, b_j \rangle = 0$ if $i \ne j$.
- $\langle b_i, b_i \rangle = 1$.

We can also compute projections. For any $p, u \in P_n, u \ne 0$, the projection of p onto u is

$$\text{proj}_u p = \frac{\langle p, u \rangle}{\langle u, u \rangle} u.$$

Consider the case where $n = 2$. From part (a), we have the basis $\{1, x, x^2\}$ for P_2. Convert this into an orthonormal basis using the Gram-Schmidt process.

(g) (optional) An alternative inner-product could be placed upon real polynomials if we simply represented them by a sequence of their evaluations at $0, 1, \ldots, n$ and adopted the standard Euclidean inner product on sequences of real numbers. Can you give an example of an orthonormal basis with this alternative inner product?

7. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

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