This homework is due April 4, 2016, at Noon.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?

2. **Lecture Attendance**

   Did you attend live lecture this week? (the week you were working on this homework) What was your favorite part? Was anything unclear? Answer for each of the subparts below. If you only watched on YouTube, write that for partial credit.

   (a) Monday lecture
   (b) Wednesday lecture
   (c) Friday lecture

3. **Active Filter**

   Now consider the circuit shown below:

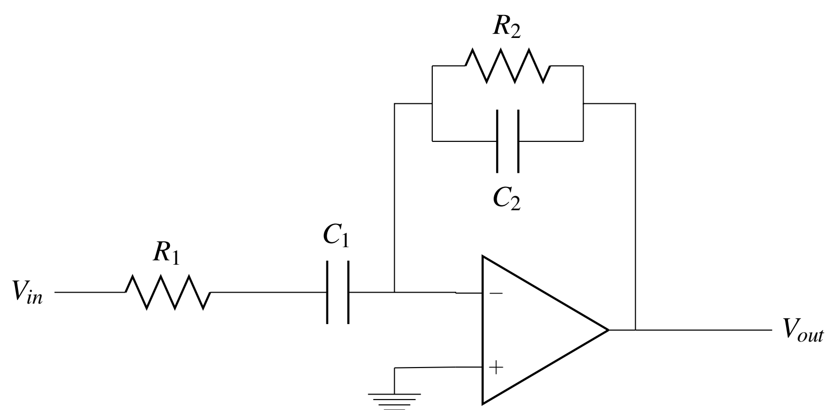

Figure 1: Active filter

   (a) Write down the transfer function of $\frac{V_{out}}{V_{in}}$
   (b) Assume $\frac{1}{R_2C_2} \gg \frac{1}{R_1C_1}$. Sketch the bode plot for the above circuit. What kind of a filter is it?
   (c) Say, we want a circuit with low and high pass frequencies being 1 kHz and 50 kHz and a gain of 100. Find a set of values for the resistance and capacitance to design the filter.

4. Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

$$\vec{x}(k+1) = A\vec{x}(k) + Bu(k)$$

where A is a 2×2 real matrix and B is a 2×1 real vector.

(a) Let $A = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$ with $a, c, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$. Find a B such that the system is controllable no matter what nonzero values a, c, d take on, and a B for which it is not controllable no matter what nonzero values are given for a, c, d. You can use the controllability rank test, but please explain your intuition as well.

(b) Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$. and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero a, d, f, g that make the system uncontrollable.

(c) We want to see if controllability is preserved under changes of coordinates. To begin with, let $\vec{z}(k) = V^{-1}\vec{x}(k)$, please write out the system equation with respect to $\vec{z}$.

(d) Show that $rank(MA) = rank(A)$ for any invertible matrix M. *(hint: recall that $rank(A) = k$ exactly when there (a) exists a set of k columns of A that are linearly independent; (b) All collection of $k+1$ columns of A are linearly dependent.).*

(e) Now show that controllability is preserved under change of coordinates.

5. Linearization

Suppose we would like to control a spacecraft near the surface of the moon. The spacecraft has two thrusters located at opposites sides of the spacecraft that are capable of generating vertical thrust (relative to the spacecraft). For simplicity, we will only discuss the problem in two dimensions.

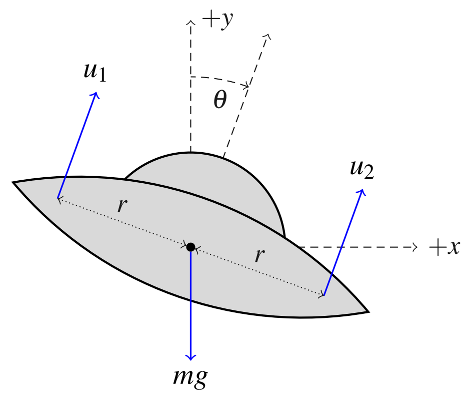

Figure 2: Spacecraft. The forces operating on the spacecraft are drawn in blue.

We use the following notation: the gravitation acceleration is g, the spacecraft has mass m, moment of inertia J, the distance between the thrusters and the center of mass is r, x is the position of the spacecraft along the x-axis, y is the position of the spacecraft along the y-axis, θ is the angle between the spacecraft and the y-axis, and u_1 and u_2 are the forces generated by the thrusters. The forces operating on the spacecraft can be seen in figure 2.

This problem uses some simple physics of the $F = ma$ variety. Please ask on Piazza if you don't know the simple physics involved, your fellow students will help you out. We'll start by modeling the behavior of the spacecraft with differential equations:

(a) Write a differential equation for Newton's second law in x-axis (relative to the moon's surface).

$$m\frac{d^2}{dt^2}x(t) = ?$$

(b) Write a differential equation for Newton's second law in the y-axis (relative to the moon).

$$m\frac{d^2}{dt^2}y(t) = ?$$

(c) Write the moment equation of the spacecraft around its center of mass.

$$J\frac{d^2}{dt^2}\theta(t) = ?$$

Next, we'll derive a state-space model for the system:

(d) Identify your state variables. How many do you have?

(e) Write the differential equations describing the system in standard form:

$$\frac{d}{dt}\vec{z}(t) = f(\vec{z}(t), \vec{u}(t)),$$

where $\vec{z}$ is the vector of state variables you have identified in the previous part.

The next step is to pick a point at which to linearize the system:

(f) Does the system have an operating point? That is, is there $\vec{z}$ such that $f(\vec{z}, \vec{0}) = \vec{0}$? What is the physical meaning of such a point?

(g) Choose some point $\vec{z_0}, \vec{u_0}$ where $\theta = 0$ such that the thursters counter the gravitational force exactly, and the spacecraft maintains the same position and rotation.

We now linearize the system and check if it is controllable.

(h) Linearize the system around the point $\vec{z_0}, \vec{u_0}$ by changing variables according to

$$\vec{\tilde{z}} = \vec{z} - \vec{z_0}$$
$$\vec{\tilde{u}} = \vec{u} - \vec{u_0},$$

and writing the linearized system in standard form

$$\frac{d}{dt}\vec{\tilde{z}}(t) = A\vec{\tilde{z}} + B\vec{\tilde{u}}$$

(i) Is the spacecraft controllable around $\vec{z_0}$ using only these two thrustors?
Assume that the test for continuous-time controllability is the same as in discrete-time.

(j) Suppose that due to a mechanical problem, the right thruster stopped responding and is stuck at the nominal thrust setting for that thrustor. Is the system still controllable in the neighborhood around z_0, u_0?

6. **Solutions of Differential Equations (optional, but in-scope for the exam)**

We have seen many real problems, such as first/second order circuits, can be described by systems of linear differential equations, i.e.,

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

where $A \in \mathbb{R}^2$, $\vec{x}(0) = \vec{x}_0$.

In this problem we will think about not only how to solve such systems of differential equations, but more importantly we will connect the properties of the solution to the eigenvalues of A. We will also study an illustration technique called "phase portraits."

(a) Assume A has distinct eigenvalues. Show that the general solution is $c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$, where λ_1, λ_2 are eigenvalues of A with corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. By setting the two constants c_1, c_2, we can match the initial condition.

(b) Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Solve $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ and plot the two component, $\vec{x}_1(t), \vec{x}_2(t)$, as a function of t (You can use $\vec{x}_0 = [0, -4]^T$ for this plot). What do you observe? What can you say about the evolution of the solution when we have $\lambda_1 < 0$ and $\lambda_2 > 0$?

(c) Now plot the solution on the 2D-$\{\vec{x}_1, \vec{x}_2\}$ plane. Start by looking at particular solutions when $c_2 = 0$. What does the solution look like on this plane when $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t}$? How about if we let $c_1 = 0$ and let c_2 be nonzero instead?

(d) For the two particular solutions you have drawn on the 2D-$\{\vec{x}_1, \vec{x}_2\}$ plane, how does each of them evolve when $t \to \infty$?

(e) Consider general solutions as the linear combination of the two particular solutions when we set $c_1 = 0$ or $c_2 = 0$, sketch their trajectory and their evolution. *(Hint: what is the limiting behavior of a general solution when $t \to \infty$ and when $t \to -\infty$).*

(f) Using the previous steps, solve the system of linear differential equations with $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, and plot each component of the solution, as well as the phase portrait.

(g) Can you design a matrix A, such that all solutions go inward to the origin $[0, 0]^T$ and all the eigenvalues of A are real.

(h) Now let us proceed to the case when the eigenvalues are a complex conjugate pair. Let the eigenvalues be $\lambda_\pm = \alpha \pm i\beta$ and the corresponding eigenvectors be $\vec{u} \pm i\vec{w}$. Show that the general solution has the form $e^{\alpha t}[c_1(\vec{u}\cos(\beta t) - \vec{w}\sin(\beta t)) + c_2(\vec{u}\sin(\beta t) + \vec{w}\cos(\beta t))]$.

(i) Use what you obtained so far, solve the system of linear differential equations with $A = \begin{bmatrix} 0.2 & 1 \\ -1 & 0.2 \end{bmatrix}$, and plot each component of the solution (You can use $\vec{x}_0 = [0, 1]^T$ for this plot), as well as the phase portrait. What do you observe?

(j) Once again, solve the system of linear differential equations with $A = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix}$, and plot each component of the solution, as well as the phase portrait. What do you observe?

7. **Boundary Value Problems (optional out-of-scope for the exam)**

Let's suppose that you are in a competition where you must launch a projectile at a target in the distance, and you want to hit the target at a specific time. You only have control over the initial vertical and horizontal velocities of the pojectile when you launch it. While it's in the air, its trajectory is governed by gravity. To make things easier for the competition, air resistance has been removed.

(a) The position of the projectile over time is determined by the acceleration $\vec{a}(t)$, where $\frac{d\vec{x}(t)}{dt} = \vec{v}(t)$ and $\frac{d\vec{v}(t)}{dt} = \vec{a}(t)$. The functions $\vec{x}(t)$ and $\vec{v}(t)$ are the position and velocity of the projectile at time t.

Let's restrict our problem to a 2D plane and define the origin $(0,0)$ as the starting position of the projectile. Your target is located at (j,h), and you want to hit the target T seconds after you launch the projectile. The acceleration due to gravity for the competition is set to Earth's gravity, $\vec{g} = [0, -9.8m/s^2]^T$. You are asked to hit a target located at $(100m, 10m)$ after $T = 5$ seconds from when you launch the projectile. Thus, our boundary conditions for our differential equation is

$$\begin{aligned} \vec{x}(0) &= [0,0]^T \\ \vec{x}(5) &= [100,10]^T \end{aligned} \tag{1}$$

Given that $\vec{a}(t) = [0, 9.8]^T$ for all t, solve the differential equation for $\vec{x}(t)$ that satisfies the boundary conditions and specify what initial velocity $\vec{v}(0)$ with which you will need to launch the projectile to hit the target. Verify your answer in the IPython notebook.

HINT: when acceleration is constant, position as a function of time is given by

$$\vec{x}(t) = \vec{x}(0) + \vec{v}(0)t + \frac{1}{2}\vec{a}t^2 \tag{2}$$

(b) Sometimes a boundary value problem cannot be solved analytically, and we therefore need to use a computer to help us find a solution. Let's first try solving the above problem in IPython to verify that our method closely approximates the analytic solution.

At each point in time, we require that $\frac{d^2\vec{x}(t)}{dt^2} = \vec{g}$, and we require that the boundary conditions in Equation 1 are satisfied. To solve for $\vec{x}(t)$ with a computer, we first pick a set of evenly spaced points in time $\{t_0, t_1, ..., t_n\}$ where $t_0 = 0$, $t_n = T$, and $t_i - t_{i-1} = h$ for all $1 \leq i \leq n$. We can now solve for $\vec{x}(t_i)$ for any i from 0 to n. This can be accomplished by creating a vector of unknowns:

$$\vec{x} = \begin{pmatrix} x_x(t_0) \\ x_y(t_0) \\ \vdots \\ x_x(t_i) \\ x_y(t_i) \\ \vdots \\ x_x(t_n) \\ x_y(t_n) \end{pmatrix} \tag{3}$$

where $x_x(t_i)$ is the x component of $\vec{x}(t_i)$, and $x_y(t_i)$ is the y component. This gives us $2(n+1)$ unknowns, and we will set up a system of linear equations to solve for them. The first four equations we create will be the boundary conditions: $x_x(t_0) = 0$, $x_y(t_0) = 0$, $x_x(t_n) = 100$, $x_y(t_n) = 10$.

For the remaining equations, we will use the fact that $\frac{d^2\vec{x}(t)}{dt^2} = \vec{a}$. The second derivative can be approximated as follows:

$$\frac{d^2\vec{x}(t)}{dt^2} \approx \frac{\vec{x}(t-h) - 2\vec{x}(t) + \vec{x}(t+h)}{h^2} \tag{4}$$

We can use this approximation to write the remaining $2(n-1)$ equations for the acceeleration at time t_1 through t_{n-1}.

We combine these equations in the form $A\vec{x} = \vec{b}$. Describe what A and $\vec{b}$ look like.

(c) Write the code to fill in the entries for A and $\vec{b}$ in the IPython notebook. Use the plot of the analytic solution and the approximated solution to verify that A and $\vec{b}$ are filled in correctly.

(d) To make things interesting in the contest, acceleration due to gravity now varies over time. You are told that the acceleration is given by

$$\vec{a}(t) = \begin{pmatrix} 0 \\ -9.8 + 2e^{t/3} + 8\sin(2t) \end{pmatrix} \quad (5)$$

You can find a plot of the second component over time in the IPython notebook. We can still solve for $\vec{x}(t)$ by using a system of linear equations. Describe what changes would need to be made to A and $\vec{b}$.

(e) Implement these changes in the IPython notebook to find the trajectory the projectile will follow to hit the target.

(f) Let's suppose that now acceleration due to gravity is now a function of the position of the particle instead of time. For example, acceleration is given by $\vec{a}(\vec{x}(t)) = -9.8 - \vec{x}_y(t)^2$. Can we still use a system of linear equations to solve for $\vec{x}(t)$ at our time points $t_0, t_1, \ldots, t_n$? Explain why or why not.

8. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student must submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

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