EE16B: DESIGNING INFORMATION DEVICES AND SYSTEMS II

LECTURE NOTES

Singular Value Decomposition (SVD)

Let A be an $m \times n$ matrix,

$$A = U[\Lambda|0]V^T$$

where V and U have orthonormal columns and Λ is a diagonal matrix.

Let,

$$S = AA^T$$
$$T = A^T A$$
$$S = U\Lambda_S U^T$$

$$(\Lambda_S \text{ contains eigenvalues of } S)$$

$$T = V\Lambda_T V^T$$

$$(\Lambda_T \text{ contains eigenvalues of } T)$$

$$\Lambda_T = \begin{bmatrix} \lambda_{T,i} & & 0 \\ & \ddots & 0 \\ 0 & 0 & \ddots \end{bmatrix} (n \times n)$$

We've proved that symmetric matrices have orthogonal eigenspaces.

On the next homework, you will prove that symmetric matrices have a full complement of eigenvectors.

You now know enough to prove that if there exists a real symmetric matrix, all of its eigenvalues must be real.

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Start with T. Consider

$$\vec{\tilde{u}}_i = \frac{A\vec{u}_i}{\sqrt{\lambda_{T,i}}} \text{ if } \lambda_{T,i} > 0$$

We can show that $\sqrt{\lambda_{T,i}}$ must be real, but how do we prove they are non-negative?

Consider

$$\vec{v}_i^T T \vec{v}_i = \lambda_{T,i}$$

$$\vec{v}_i^T T \vec{v}_i = (\vec{v}_i^T A^T)(A\vec{v}_i) = ||A\vec{v}_i||^2 = \lambda_{T,i}$$

All λ_T's must be real and non-negative, since the norm is always greater than or equal to zero.

Assert

$$\widetilde{U} \left[\begin{array}{cc|c} \sqrt{\lambda_{T,i}} & & 0 \\ & \ddots & 0 \end{array}\right] V^T = A$$

$$A\vec{v}_i = \widetilde{U}[\Lambda|0] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{\text{th}} \text{ position}$$

$$\widetilde{U} \begin{bmatrix} 0 \\ 0 \\ \sqrt{\lambda_{T,i}} \\ 0 \\ 0 \end{bmatrix} = A\vec{v}_i = \vec{\tilde{u}}_i \cdot \sqrt{\lambda_{T,i}}$$

Check orthonormality of $[\vec{\tilde{v}}_i]$: (using the substitutions $T = A^T A$ and $T = V\Lambda_T V^T$)

$$\vec{\tilde{u}}_i^T \vec{\tilde{u}}_j = \frac{\vec{v}_i^T A^T}{\sqrt{\lambda_{T,i}}} \cdot \frac{A\vec{v}_j}{\sqrt{\lambda_{T,j}}} = \frac{\vec{v}_i^T A^T A \vec{v}_j}{\sqrt{\lambda_{T,i}\lambda_{T,j}}} = \frac{\vec{v}_i^T T \vec{v}_j}{\sqrt{\lambda_{T,i}\lambda_{T,j}}} = \frac{\vec{v}_i^T V\Lambda_T V^T \vec{v}_j}{\sqrt{\lambda_{T,i} \cdot \lambda_{T,j}}} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$S = AA^T = \widetilde{U}[\Lambda|0]V^T V[\frac{\Lambda}{0}]\widetilde{U}^T = \widetilde{U}\Lambda^2\tilde{U}^T$$

$\rightarrow AA^T$ is diagonalized by $\vec{\tilde{u}}$, so $\vec{\tilde{u}}$ must be U.