EE16B Designing Information Devices and Systems II **Spring 2016**

Lecture Notes: 26

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26.1 Definition Overview

Controllability Given the ability to control the input, can we get the system to do what you want.

Observability Given the observation from the outputs can we figure out what is going on in the system

Stability When they system is left to itself, will the system blow up?

26.2 Observability

Previously we worked with the discrete time system

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) + \vec{w}(t)$$

Where $\vec{x}(t)$ is the current state, $\vec{u}(t)$ is the input and disturbance $\vec{w}(t)$.

We saw that

$$\vec{x}(t) = A^t\vec{x}_0 + \sum_{\tau}^{t-1} A^{t-1-\tau}B\vec{u}(\tau)$$

If we want to control $\vec{x}$ using $\vec{u}$, we are interested in finding a linear combination of the columns of $A^{t-1-\tau}B$ which we can choose by controlling $\vec{u}$. Hence we are interested in the $span(B, AB, A^2B, A^3B, ...)$.

If $span(B, AB, A^2B, A^3B, ...) = n - dim$ space, then all points are reachable through the choices of the control inputs.

If we want to check if each of $(B, AB, A^2B, A^3B, ...)$ are all $n - dim$ space, we can check if

$$rank(B, AB, A^2B, A^3B, ...) = n$$

The , notation here means concatenation of columns. It turns out we only need to check up to n

$$rank(B, AB, A^2B, A^3B, ..., A^{n-1}B) = n$$

If it does not reach rank n by then it is never going to reach a rank of n. We can think of it as, since there are at least n columns, if we add another column, it becomes linearly dependent.

We want to show that as soon as we observe a column that is not linearly independent then the rank cannot grow again.

Claim: If A^kB is a linear combination of $A^0B, A^1B, ...A^{k-1}B$, then $A^{k+1}B$ is also a linear combination of $A^0B, A^1B, ...A^{k-1}B$.

Proof:

$$\exists\ \alpha_\tau \quad s.t \quad A^kB = \sum_{\tau}^{k-1} \alpha_\tau\ A^\tau B$$

$$A^{k+1}B = A\sum_{\tau}^{k-1} \alpha_\tau A^\tau B$$

$$= \sum_{\tau}^{k-1} \alpha_\tau A^{\tau+1}B$$

$$= \alpha_{k-1}A^kB + \sum_{\tau}^{k-2} \alpha_\tau A^{\tau+1}B$$

$$= \alpha_{k-1}(\sum_{j}^{k-1} \alpha_j A^j B) + \sum_{\tau}^{k-2} \alpha_\tau A^{\tau+1}B$$

New Claim: If the columns of A^kB are linear combinations of $A^0B, A^1B, ...A^{k-1}B$ then $A^{k+1}B$ is also a linear combination of $A^0B, A^1B, ...A^{k-1}B$.

$$\exists\ U_\tau \quad s.t \quad A^kB = \sum_{\tau}^{k-1} A^\tau BU_\tau$$

Where U_τ is $p \times p$

Examples (Controllability):

(Example 1)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{u}(t)$$

This example is controllable. We can apply 2 dimensional control to control the 2 dimension states since B is fully rank.

(Example 2)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

We apply control to input 2. Can we move the state 1? We see that

$$rank(B, A^1B) = 2$$

Hence we can indeed control state 1 by controlling input 2. Therefore, this example is controllable.

(Example 3)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

We apply control to input 1. Can we move the state 2? We see that

$$rank(B, A^1 B) = 1$$

Hence we cannot change state 2 by controlling the input 1. We see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ hence the input can only control the state in its direction. Thus, this example is uncontrollable.

(Example 4)

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$rank(B, A^1 B) = 2$$

Same argument. This is controllable.

26.3 Observability

By observing the output can we understand the system.

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

$$\vec{y}(t) = C\vec{x}(t)$$

We can measure the system using some matrix C and observe the output $\vec{y}$. Only uncertainty we have is the uncertainty of $\vec{x}(0)$. Since $\vec{y}$ is an observation we are interested in the rows of C.

If $rank\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = rank(C, CA, ..., CA^{n-1}) = n$ then it is observable. Where the comma (,) represents the concatenation of rows.

26.4 Stability

We are often interested in figuring out if the system blows up. If the values blow up we say that the system is unstable. If the system states converges to 0 we say the system is stable. In discrete case if all the eigenvalues of A. $|\lambda| < 1$ the system is stable. If $|\lambda| > 1$ it is unstable and if $|\lambda| = 1$ it is marginally unstable.