EE16B: DESIGNING INFORMATION DEVICES AND SYSTEMS II

LECTURE NOTES

Disturbances & Stability

Consider system:

$$\vec{x}(t+1) = a\vec{x}(t) + b\vec{u}(t) + \vec{\omega}(t)$$

where $\vec{\omega}(t)$ is a disturbance in the system and $\vec{x}(0) = \vec{x}_o$. To what extent does this disturbance cause trouble?

Open loop case:

$\vec{u}(t) = 0$, System is stable iff $|a| < 1$
Solution:

$$\vec{x}(t) = a^t \vec{x}_o + \sum_{\tau=0}^{t-1} a^{t-1-\tau} \vec{\omega}(\tau)$$

Without loss of generality, let $\vec{x}_o = 0$ and $i = t - 1 - \tau$

$$\vec{x}(t) = a^t \vec{x}_o + \sum_{i=0}^{t-1} a^i \vec{\omega}(t-1-i)$$

Assume $|\vec{\omega}(t)| \leq \epsilon$ (disturbance is bounded by some value ϵ):

$$|\vec{x}(t)| \leq \sum_{i=0}^{t-1} |a|^i \left|\vec{\omega}(t-1-i)\right| \leq \sum_{i=0}^{t-1} |a|^i \epsilon \leq \sum_{i=0}^{\infty} |a|^i \epsilon$$

Using the sum of a geometric series (if $a < 1$):

$$\sum_{i=0}^{\infty} |a|^i \epsilon = \frac{\epsilon}{1 - |a|}$$

What if $a > 1$? $\longrightarrow$ System is unstable.

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Does there exist a sequence $\vec{\omega}(t)$ (where $|\vec{\omega}(t)| \leq \epsilon$) such that $|\vec{x}(t)| \longrightarrow \infty$?
In general, $\vec{\omega}(t) = \frac{\epsilon a^t}{|a|}$ works.

In the vector case

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{\omega}(t)$$

A bounded distribution has $|\vec{\omega}(t)[j]| \leq \epsilon$ for all t, j.

If A is diagonalizable for all V such that $A = V\Lambda V^{-1}$ where Λ is a diagonal matrix and each eigenvalue λ in this matrix exists such that $|\lambda| < 1$:

Change coordinates to

$$\vec{z}(t) = V^{-1}\vec{x}(t)$$

So,

$$\vec{z}(t+1) = \Lambda\vec{z}(t) + V^{-1}\vec{\omega}(t)$$

The first term, $\Lambda\vec{z}(t)$, is bounded by the λ values in the diagonal matrix, and the term $V^{-1}\vec{\omega}(t)$ is bounded by ϵ and the maximum entries of V^{-1}.

$\longrightarrow$ *Bounded input implied bounded output if the matrix A is diagonalizable.*

What if the matrix A is not diagonalizable?

There exists a matrix Q of orthonormal vectors such that

$$A = Q \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & stuff & \\ & & \lambda_3 & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} Q^{-1}$$

This is known as *Schur Decomposition.*

If we change coordinates to $\vec{z}(t) = Q^{-1}\vec{x}(t)$,

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & stuff & \\ & & \lambda_3 & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ \\ \vdots \\ \\ \vec{z}_n(t) \end{bmatrix} + Q^{-1}\vec{\omega}$$

$$\vec{z}_n(t+1) = \lambda \vec{z}_n(t) + \vec{\omega}(t)$$

We know $|\vec{z}_n(t)| < C_n \epsilon$ where C_n is a constant obtained from the "stuff" part of the matrix.

$$\vec{z}_{n-1}(t+1) = \lambda_{n-1}\vec{z}_{n-1}(t) + \text{``}junk''\vec{z}_n(t) + \omega_{n-1}(t)$$

All three terms of the output are bounded, again showing that a bounded input produces a bounded output.