Lecture Notes: 32

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32.1 CCF Example

This lecture we walk through an example of CCF.

We convert our system into Controllable Canonical Form since in this form we can control the system.

For an example, we start with the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

This matrix has the eigenvalues of

$$\lambda_1 = 16, \lambda_2 = -1.1, \lambda_3 = 0$$

This is a matrix that is not invertible but it has 3 unique eigenvalues. We use a control direction

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We can control the system with the vector $\vec{b}$ if it hits all the eigenvectors of the control states. We also know that the system is not controllable when the control direction does not hit all of the eigenvectors of A.

How do we check if $\vec{b}$ is hitting all the eigenvectors? We can take $\vec{b}$ to the eigenspace. We take the inverse of the eigenspace and dot it with the vector $\vec{b}$.

$$V^{-1}\vec{b} = \begin{bmatrix} -1.7 \\ -0.75 \\ 0 \end{bmatrix}$$

We see that this control matrix $\vec{b}$ will never be able to control the last state. We can double check by

$$det([\vec{b}, A\vec{b}, A^2\vec{b}]) = 0$$

If our matrix A has all distinct eigenvalues then as long as $\vec{b}$ can touch all the eigenvectors of A, the system is controllable.

We try a different controller $\vec{b}$

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

This control turns out to be controllable.

With the $A, \vec{b}$ that we found controllable, we want to convert the system into controllable canonical form , CCF. We need to find the values a_i in the CCF.

To do this we want to see how $A^n\vec{b} = A^3\vec{b}$ would look like in the G basis where G is

$$G = [\vec{b}, A\vec{b}, A^2\vec{b}]$$

Where we know G is full rank

$$P = G^{-1}A^3\vec{b}$$
$$P = \begin{bmatrix} 0 \\ -18 \\ -15 \end{bmatrix}$$

This corresponds to a polynomial

$$p = 0 - 18\lambda - 15\lambda^2 + \lambda^3$$

Looking at the root

$$roots = [-1.12, 0.0, 16.12]$$

These roots corresponds to the coefficients in the CCF. Note that when A has a nullspace greater than 1, the system is no longer controllable.

To see what A looks like in the basis of G we apply the following

$$\tilde{A} = G^{-1}AG$$
$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 18 \\ 0 & 1 & 15 \end{bmatrix}$$
$$\tilde{A}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 18 & 15 \end{bmatrix}$$

Now we can

$$\tilde{G} = [\vec{b}, \tilde{A}^T\vec{b}, (\tilde{A}^T)^2\vec{b}]$$
$$T = \tilde{G}^{-1}\tilde{A}^T\tilde{G}$$

We see that

$$T = \tilde{A}$$

The new basis that will take us from the beginning to where we want is given by

$$T_{basis} = G\tilde{G}^{-1}$$

Then we have

$$A_{control} = TAT^{-1}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 18 & 15 \end{bmatrix}$$

Now we can place the eigenvalues where we want. We choose the eigenvalues

$$\lambda_1 = 0.5, \lambda_2 = 0.25, \lambda_3 = 0.125$$

and look for the new desired charasteristic polynomial

$$p_{desired} = (\lambda - \frac{1}{2})(\lambda - \frac{1}{4})(\lambda - \frac{1}{8})$$

We see that the corresponding polynomial coefficients are

$$-0.016, 0.219, -0.875, 1.000$$

We set the control vector to move from original to the desired is

$$\vec{u}_{control} = [0.0156, -17 - 0.2186, -15 + 0.875]$$

$$= [0.0156, -18.22, -14.13$$

We then solve for A_{cl}

$$A_{cl} = A_{control} + \vec{b}_{control}\vec{u}_{control}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.0156 & -0.218 & 0.875 \end{bmatrix}$$

Solving for the eigenvalues we get

$$\lambda_1 = 0.125, \lambda_2 = 0.25, \lambda_3 = 0.5$$

$$\vec{u}_{original} = \vec{u}_{control}T$$

$$= [-2.77, -3.53, -4.29]$$

Now we check the eigenvalues of the original control

$$A + \vec{b}\vec{u}_{original}$$

$$\lambda_1 = 0.125, \lambda_2 = 0.25, \lambda_3 = 0.5$$