EE16B Designing Information Devices and Systems II **Spring 2016**

Lecture Notes: 38

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38.1 Polynomial Interpolation and DFT

Choose evaluation points $s_0, s_1 ... s_n$ *with* $|s_j| = 1$ *s.t* evaluating s_j^n won't blow up. Having a magnitude 1 will neither shrink nor blow up the polynomial.

Pick n^{th} root of unity. $s_h^n = 1$.

$$s_0 = e^{i\frac{2\pi}{n}0}$$
$$s_h = e^{i\frac{2\pi}{n}h}$$
$$s_{n-1} = e^{i\frac{2\pi}{n}(n-1)}$$

Evaluate polynomial at these points. Polynomials are linear combinations of vectors $t^0, t^1 ... t^{n-1}$ which we can treat as basis vectors $\vec{b}^0, \vec{b}^1 ... \vec{b}^{n-1}$.

$$\vec{b}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{n}} \\ e^{i\frac{2\pi}{n}2} \\ \vdots \\ 1 \\ e^{i\frac{2\pi}{n}(n-1)} \end{bmatrix}, \vec{b}_h = \begin{bmatrix} 1 \\ e^{i\frac{2\pi h}{n}} \\ e^{i\frac{2\pi h}{n}2} \\ \vdots \\ 1 \\ e^{i\frac{2\pi h}{n}(n-1)} \end{bmatrix}$$

Note that these vectors can be seen as the DFT basis.

If we know that $\vec{x}$ has the form

$$\vec{x} = \sum_{j=0}^{k-1} \alpha_j \vec{u}_j$$

Then, $\vec{x}$ can be recovered from any k samples of $\vec{x}$.

One problem with this form is that it will always be complex, but our signals are real. So we need a more general result.

Consider

$$x(t) = \alpha_0 t^a + \alpha_1 t^{a+1} + ... + \alpha_{k-1} t^{a+k-1}$$

Where a is known. We can show $x(t)$ is like a polynomial.

$$x(t) = t^a(\alpha_0 + \alpha_1 t^1 + ... + \alpha_k t^{k-1})$$

$$x(t) = t^a \tilde{x}(t) = t^a \sum_{j=0}^{k-1} \alpha_j t^j$$

Given k samples of $(t_i, y_i = x(t_i))$, can we reconstruct a point at $x(t)$? Moreover, given the sample of x can we get samples of $\tilde{x}$? If we know what $\tilde{x}$ is, we can reconstruct x. So what we want is $\tilde{y}_i = \tilde{x}(t_i)$.

$$y_i = x(t_i) = t_i^a \tilde{x}(t_i) = t_i^a \tilde{y}_i$$

$$\tilde{y}_i = \frac{y_i}{t_i^a}$$

$$\tilde{x}(t) = \sum_{i=0}^{k=1} \tilde{y}_i l_i(t)$$

This means

$$x(t) = t^a \tilde{x}(t) = \sum_{i=0}^{k-1} t^a \tilde{y}_i l_i(t) = \sum_{i=0}^{k-1} y_i \frac{t^a}{t_i^a} l_i(t)$$

Hence we can see that

If $\vec{x} = \sum_{j=0}^{k-1} \alpha_j \vec{u}_{j+a}$, then $\vec{x}$ can be recovered from any k samples.

In particular,

If $\vec{x} = \sum_{j=-m}^{m} \alpha_j \vec{u}_j$, them $\vec{x}$ can be recovered from $2m+1$ samples.