EE16B - Spring 2017 - Discussion 11A

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Sampling theorem

Given a function f that has frequencies from $-\omega_{\text{max}}$ to ω_{max}, we can sample at points,

$$\{k \times \Delta\}_{k\in\mathbb{Z}} \text{ where } \Delta < \frac{\pi}{\omega_{max}}$$

and reconstruct the function f as,

$$f(x) = \sum_{k\in\mathbb{Z}} f(k\Delta)\text{sinc}\left(\frac{x - k\Delta}{\Delta}\right)$$

```
In [1]: %pylab inline
```

```
Populating the interactive namespace from numpy and matplotlib
```

Question 1

Consider $f(t)$ defined as,

$$f(t) = \cos(2\pi t)$$

where t is in seconds. - What is ω_{max} in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.) - What is the smallest sampling Δ that would not result in a perfect reconstruction? - If I sample every Δ_s seconds, what is the sampling frequency?

Solutions

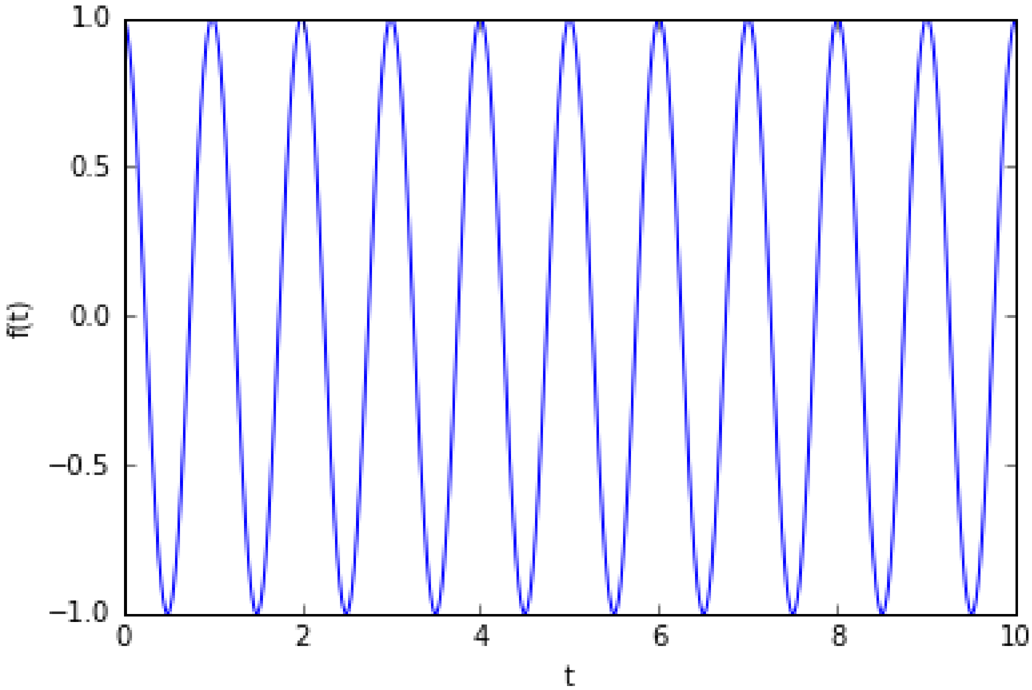
- $\omega_{\text{max}} = 2\pi$ in radians per second, which is 1 Hertz.
- $\Delta = \frac{1}{2}$. This is where $\Delta = \frac{\pi}{\omega_{\text{max}}}$.
- $\omega_s = \frac{2\pi}{\Delta s}$.

Now, we will see what happens as we vary the sample period Δ_s.

```
In [2]: def f(t):
            return np.cos(2 * np.pi * t)
```

```
In [3]: # Time axis
        t = np.array([k * 0.01 for k in range(0, int(10/0.01) + 1)])
        plot(t, f(t))
        xlabel('t')
        ylabel('f(t)')
```

Out[3]: <matplotlib.text.Text at 0x7fdf9bac5e80>



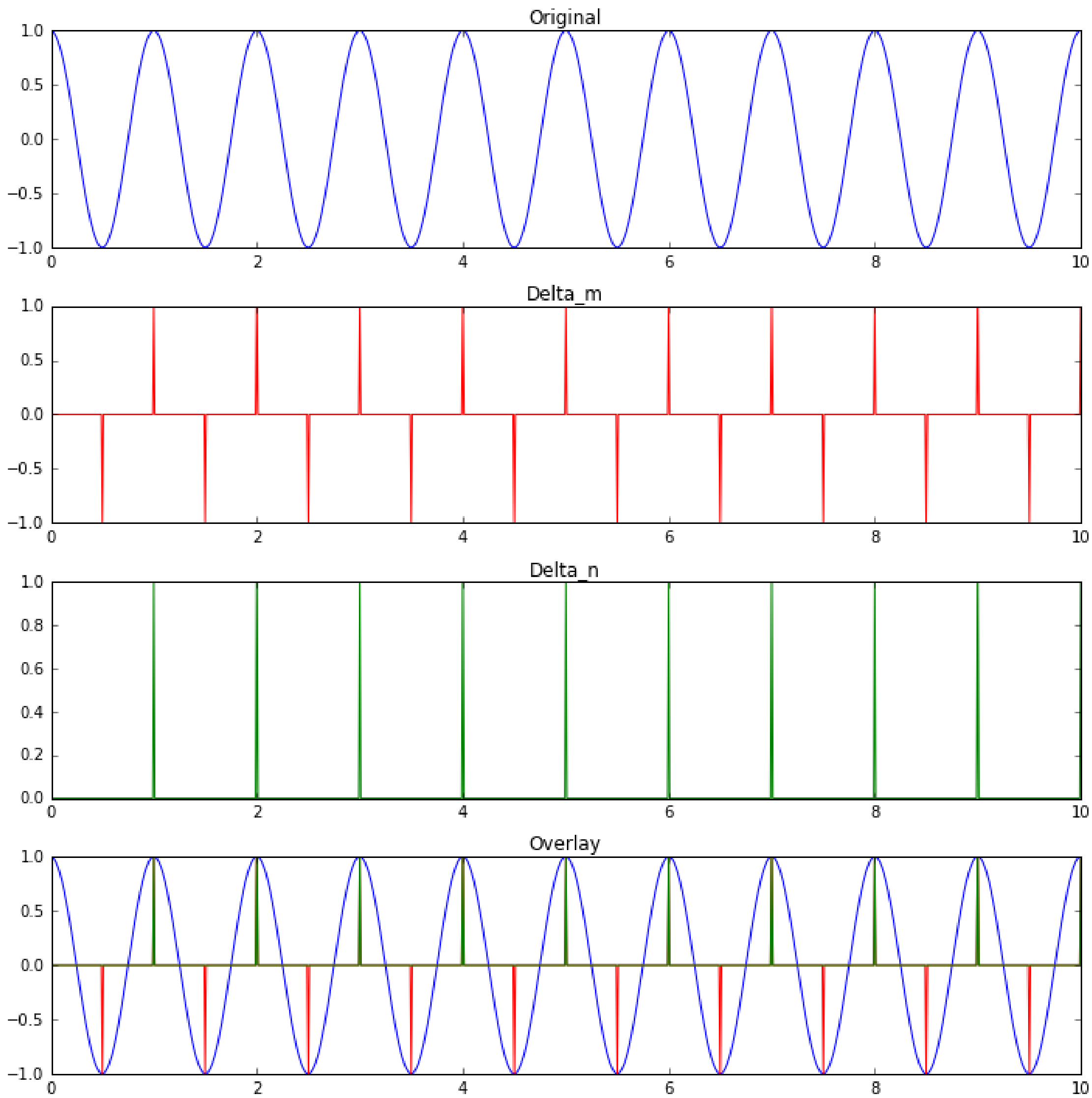
Question 2

We will sample f with $\Delta_m = \frac{1}{4}$ and $\Delta_n = 1$ and do a sinc interpolation on the resulting samples. Let the reconstructed functions be g_m and g_n.

In [5]:
```
Delta_m = 1/4
Delta_n = 1
gm = t * 0
gn = t * 0
for k in range(0, int(floor(10/Delta_m)) + 1):
    gm[int(k * Delta_m/t[1])] = f(k * Delta_m)
for k in range(0, int(floor(10/Delta_n)) + 1):
    gn[int(k * Delta_n/t[1])] = f(k * Delta_n)

fig, axes = plt.subplots(nrows=4, ncols=1)
fig.set_size_inches(10, 10)
plt.subplot(4, 1, 1)
plot(t, f(t))
title('Original')
plt.subplot(4, 1, 2)
plot(t, gm, color='red')
title('Delta_m')
```

```
plt.subplot(4, 1, 3)
plot(t, gn, color='green')
title('Delta_n')
plt.subplot(4, 1, 4)
plot(t, f(t))
plot(t, gm, color='red')
plot(t, gn, color='green')
title('Overlay')
fig.tight_layout()
```



- Have we staisfied the Nyquist limit in any case?
- What is the expected highest frequency of the sinc function,

$$\text{sinc}\left(\frac{t-\Delta_n k}{\Delta_n}\right)?$$

- Based on this answer, can you think of any periodic function that has a frequencies less than or equal to π that samples the same as g_n?

Solutions

- Δ_m satisfies Nyquist. Δ_n does not.
- The sinc functions used to reconstruct g_n is,

$$\left\{ \text{sinc}\left(\frac{t-k}{1} \right) \right\}_{k \in \mathbb{Z}}.$$

  These functions can represent a maximum frequency of π.
- Since the frequencies vary from 0 to π, the smallest period that can be represented is 2. That is to say, functions of period < 2 cannot be captured with the sinc function derived from Δ_n. Since the period must be greater than 2, no sine or cosin function can give the same samples as g_n. This means suggests looking into a fairly trivial kind of periodic function: a constant. In particular, the answer to this problem is the constant function that is 1 everywhere.

Question 3

Consider the function $f(x) = sin(0.2\pi x)$. - At what period T should we sample so that sinc interpolation recovers a function that is identically zero? - At what period T should we sample so that sinc interpolation recovers the function $g(x) = -\sin\left(\frac{\pi}{15}x\right)$?

Solutions

- T = 5
- T = 7.5