EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz **Discussion 3A**

Complex Numbers

A complex number z is an ordered pair (x, y), where x and y are real numbers, written as $z = x + iy$ such that $i = \sqrt{-1}$. The magnitude of a complex number $z = a + ib$ is denoted as $|z|$ and is given by,

$$|z| = \sqrt{x^2 + y^2}$$

The phase or argument of a complex number is denoted as θ and is defined to be,

$$\theta = \text{atan2}(x, y)$$

Here, $\text{atan2}(x, y)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point (x, y). A complex number can also be written in polar form as follows.

$$z = |z|e^{i\theta}$$

Euler's Identity is,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

With this definition, the polar representation of a complex number will make more sense. Note that,

$$|z|e^{i\theta} = |z|\cos(\theta) + i|z|\sin(\theta)$$

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case $|z|$ is the magnitude and $e^{i\theta}$ is the unit vector that defines the direction. The complex conjugate of a complex number z is another complex number z^* such that, if $z = x + iy$, $z^* = x - iy$.

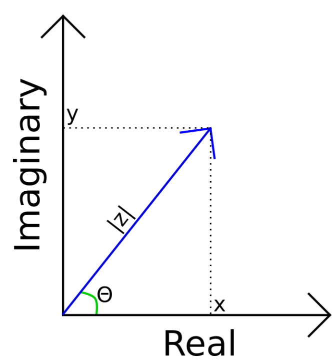


Figure 1: Complex number z represented as a vector in the complex plane.

Complex Number Properties

Rectangular vs polar forms: $z = x + iy = |z|e^{i\theta}$

where $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$, $\theta = \text{atan2}(x, y)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Complex conjugate: $z^* = x - iy = |z|e^{-i\theta}$.

$$(z+w)^* = z^* + w^*, (z-w)^* = z^* - w^*$$
$$(zw)^* = z^* w^*, (z/w)^* = z^*/w^*$$
$$z^* = z \Leftrightarrow z \text{ is real}$$
$$(z^n)^* = (z^*)^n$$

Complex Algebra

Let $z_1 = x_1 + iy_1 = |z_1|e^{i\theta_1}$, $z_2 = x_2 + iy_2 = |z_2|e^{i\theta_2}$.

(Note that we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$

Division: $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$

Power: $z_1^n = |z_1|^n e^{in\theta_1}$

$z_1^{1/2} = \pm|z_1|^{1/2}e^{i\theta_1/2}$

Useful Relations

$$-1 = i^2 = e^{i\pi} = e^{-i\pi}$$
$$i = e^{i\pi/2} = \sqrt{-1}$$
$$-i = -e^{i\pi/2} = e^{-i\pi/2}$$
$$\sqrt{i} = (e^{i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1+i)}{\sqrt{2}}$$
$$-\sqrt{i} = (e^{-i\pi/2})^{1/2} = \pm e^{i\pi/4} = \frac{\pm(1-i)}{\sqrt{2}}$$

Phasors

We consider sinusoidal voltages and currents of a specific form.

Voltage	$v(t) = V_0\cos(\omega t + \phi_v)$
Current	$i(t) = I_0\cos(\omega t + \phi_i)$

Where,

(a) V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

(b) ω is the frequency of oscillation.

(c) ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time.

We know from euler's identity that $e^{i\theta} = cos(\theta) + isin(\theta)$. Using this identity, we can obtain an expression for $cos(\theta)$ in terms of an exponential:

$$cos(\theta) = \mathfrak{Re}(e^{i\theta})$$

Extending this to our voltage signal from above:

$$v(t) = V_0\cos(\omega t + \phi_v) = V_0\mathfrak{Re}(e^{i\omega t + i\phi_v}) = V_0\mathfrak{Re}(e^{i\phi_v}e^{i\omega t})$$

Now, since we know the circuit will not change the the frequency of the signal, we can drop the $e^{i\omega t}$, as long as we remember that all signals will related to the voltage will be sinusoidal with angular frequency ω. The result is called the phasor form of this signal:

$$\boldsymbol{V} = V_0\mathfrak{Re}(e^{i\phi_v})$$

The phasor representation contains the magnitude and phase of the signal, but not the time-varying portion. Phasors let us handle sinusoidal signals much more easily, letting us use circuit analysis techniques that we already know to analyze AC circuits. Note, we can only use this if we know our signal is a sinusoid.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\boldsymbol{V}$ is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \mathfrak{Re}(\boldsymbol{V} e^{i\omega t})$$

The standard forms for voltage and current phasors are given below:

$$\begin{array}{r|l} \text{Voltage} & \boldsymbol{V} = V_0 e^{i\phi_v} \\ \text{Current} & \boldsymbol{I} = I_0 e^{i\phi_i} \end{array}$$

Phasor Relationship for Resistors

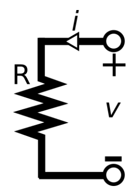


Figure 2: A simple resistor circuit

Consider a simple resistor circuit as in Figure 2, with current being,

$$i(t) = I_0 \cos(\omega t + \phi)$$

By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R \cos(\omega t + \phi) \end{aligned}$$

In phasor domain,

$$\boldsymbol{V} = R\boldsymbol{I}$$

Phasor Relationship for Capacitors

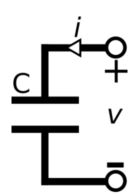


Figure 3: A simple capacitor circuit

Consider a capacitor circuit as in Figure 3, with voltage being,

$$v(t) = V_0 \cos(\omega t + \phi)$$

By the capacitor equation,

$$\begin{aligned}i(t) &= C\tfrac{\mathrm{d}v}{\mathrm{d}t}(t)\\ &= -CV_0\omega\sin(\omega t+\phi)\\ &= -CV_0\omega\left(-\cos\left(\omega t+\phi+\frac{\pi}{2}\right)\right)\\ &= CV_0\omega\cos\left(\omega t+\phi+\frac{\pi}{2}\right)\\ &= (\omega C)V_0\cos\left(\omega t+\phi+\frac{\pi}{2}\right)\end{aligned}$$

In phasor domain,

$$\boldsymbol{I} = \omega C e^{i\frac{\pi}{2}}\boldsymbol{V} = i\omega C\boldsymbol{V}$$

The impedence of a capacitor is an abstraction to model the capacitor as a resistor in the phasor domain. This is denoted $\boldsymbol{Z}_C$.

$$\boldsymbol{Z}_C = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{1}{i\omega C}$$

Questions

1. Complex Algebra

(a) Try to express the following values in polar forms: -1, i, $-i$, $\sqrt{i}$, and $\sqrt{-i}$.

(b) **Euler's identity.** Represent $\sin\theta$ and $\cos\theta$ using complex numbers.

(c) Show that $|z| = \sqrt{zz^*}$, where z^* is the complex conjugate of z.

Now let's tackle a numerical problem. Given two complex numbers, $V = 3 - i4$, $I = -(2+i3)$.

(d) Express V and I in polar form.

(e) Find VI, VI^*, V/I, and $\sqrt{I}$.

(f) What are the roots of $z^2 = 1$? What about $z^3 = 1$? How many roots does $z^n = 1$ have? What is the general form for the solutions of $z^n = 1$?

2. Proof of Induction

Given the voltage-current relationship of an inductor $V = L\frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

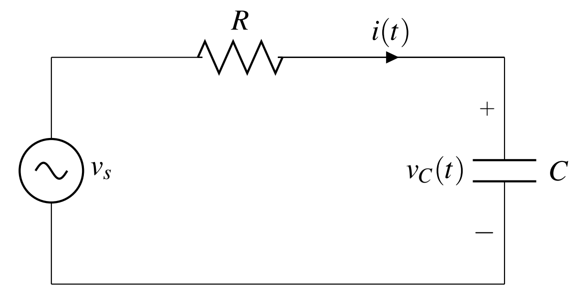
3. Phasor analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \mathfrak{Re}[Xe^{i\omega t}], \tag{1}$$

where X is a time-independent function called the phasor counterpart of $x(t)$. Thus, $x(t)$ is defined in the time domain, while its counterpart X is defined in the phasor domain.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 10^3$ rad/s, $R = \sqrt{3}\ k\Omega$, and $C = 1\ \mu F$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source v_s.

(a) **Step 1: Adopt cosine references**
All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert v_s into a cosine and write down its phasor representation V_s.

(b) **Step 2: Transform circuits to phasor domain**
The voltage source is represented by its phasor V_s. The current $i(t)$ is related to its phasor counterpart I by

$$i(t) = \mathfrak{Re}[Ie^{i\omega t}]. \tag{3}$$

What are the phasor representations of R and C?

(c) **Step 3: Cast KCL and/or KVL equations in phasor domain**
Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

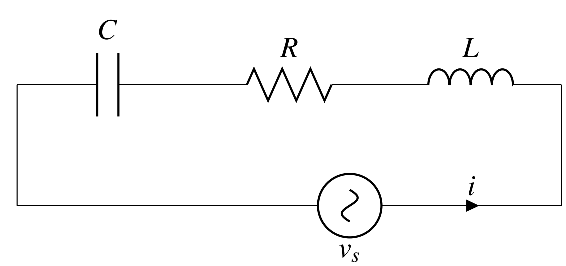
(d) **Step 4: Solve for unknown variables**
Solve the equation you derived in Step 3 for I and V_C. What is the polar form of I ($Ae^{i\theta}$, where A is a positive real number) and V_C?

(e) **Step 5: Transform solutions back to time domain**
To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?

4. RLC circuit in AC

We study a simple RLC circuit with an AC voltage source given by

$$v_s = B\cos(\omega t - \phi)$$



(a) Write out the phasor representation of v_s, R, C, L.

(b) Use Kirchhoff's laws to write down a loop equation relating the phasors in the previous part.

(c) Solve the equation in the previous step for the current I. What is the polar form of I?

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