This homework is due February 2, 2017, at 17:00.

1. **Fundamental Theorem of Solutions to Differential Equations**

   In this question, you shall discover the power of the fundamental theorem of solutions to differential equations. For convenience, we shall restate the theorem here.

   Theorem. Consider a differential equation of the form,

   $$\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t)+\alpha_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t)+\cdots+\alpha_1\frac{\mathrm{d}y}{\mathrm{d}t}(t)+\alpha_0 y(t)=0$$

   Given n initial conditions of the form,

   $$y(t_0)=a_0,\frac{\mathrm{d}y}{\mathrm{d}t}(t_0)=a_1,\cdots,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t_0)=a_{n-1},$$

   there exists a singe unique solution (say, f).

   (a) Consider the following 2 functions.

   $$\phi_1(x)=e^x,\phi_2(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

   Prove that $\phi_1(x)=\phi_2(x)$ by showing that both functions satisfy the following differential equation:

   $$\frac{\mathrm{d}f}{\mathrm{d}x}(x)=f(x)\text{ with }f(0)=1$$

   Side note: Assume $0^0=1$.

   (b) Consider the following 2 functions.

   $$\phi_1(x)=\cos(x),\phi_2(x)=\cos(-x)$$

   Prove that $\phi_1(x)=\phi_2(x)$ by showing that both functions satisfy the following differential equation:

   $$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)=-f(x)\text{ with }f(0)=1,\frac{\mathrm{d}f}{\mathrm{d}x}(0)=0$$

2. **Second Order Differential Equation**

   Find the solution to:

   $$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)=4f(x)\text{ such that}f(0)=1,\frac{\mathrm{d}f}{\mathrm{d}x}(0)=0$$

(a) What is the A matrix?

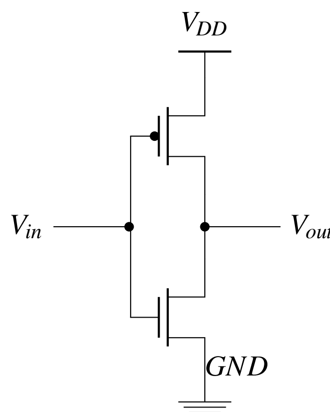
(b) What are the eigenvalues of A?

(c) Use the eigenvalues and the initial conditions to construct a solution to the differential equation.

3. From Transistors to Inverter

The following circuit is an inverter built with one pMOS and one nMOS transistor. We assume V_{out} is connected to the input of another identical inverter (not shown). We also assume there is a capacitance connected between V_{out} and GND. In real transistors, this capacitance arises from a combination of capacitances contributed by the transistors of both the inverter in question and any inverters connected to the output. For the moment, we'll just call this capacitance C_L (for load). The resistances associated with the pMOS and nMOS are R_{onP} and R_{onN} , respectively. Let the threshold voltage of the pMOS be V_{tp}, while that for the nMOS be V_{tn}. We will now model the action of the inverter as an RC circuit with two switches controlled by V_{in} as what we did in lectures.

(a) For starters, what are the on-off conditions of the two switches? Please draw the RC circuit modeling this inverter.



(b) Assume $V_{in} = V_{DD}$ for $t < 0$, and $V_{in} = 0$ for $t \geq 0$. In other words, we are assuming the input to our inverter can switch states infinitely fast (this is not true in real life, but gives us a good lower bound on how fast an inverter can switch). How much energy does it take to fully charge C_L?

(c) Given the same condition as in (b), write down the differential equation that describes $V_{out}(t)$ for $t \geq 0$.

(d) What is the solution to this differential equation? Plot $V_{out}(t)$ for $t > 0$.

(e) The term *propagation delay* is used to describe the amount of time it takes between when the input reaches $\frac{V_{DD}}{2}$ and when the output reaches $\frac{V_{DD}}{2}$. Calculate the propagation delay for our inverter above (keep in mind that the input to our inverter changes instantly). Is propagation delay a function of V_{DD}?

(f) Now consider a serial chain of inverters, each driving the one before it. If we assume that $|V_{tn}| = |V_{tp}| = \frac{V_{DD}}{2}$ what is the propagation delay for one of these inverters, given (d) and (e)? (If you like, ignore the first inverter and assume it is driven by an input as in (a)). Here we let $R_{onP} = R_{onN} = R$.

(g) Now let's consider the following scenario: there are N inverters on the chip in your cell phone. It takes **E** Joules of energy to charge all of the inverters at once (from zero to V_{DD}). What is the value of C_L?

(h) Here we interpret a voltage V as logic "1" when $V > \frac{V_{DD}}{2}$, and logic "0" when $V < \frac{V_{DD}}{2}$. Let's assume the maximum frequency, f, at which an inverter can switch back and forth between logic "0" and logic "1" at the output is the inverse of the propagation delay (i.e. we can only switch as fast as one propagation delay). Find an expression that links f, C_L, and R.

4. Two Capacitors

Consider the circuit below, assume that when $t \leq 0$, both capacitors are fully charged ($V_1(t=0) = V_s$ and $V_2(t=0) = V_s$). At $t = 0$, the switch closes. Assume $V_s = 10$ V, $R_1 = R_2 = 100\Omega$, and $C_1 = C_2 = 100\mu$F.

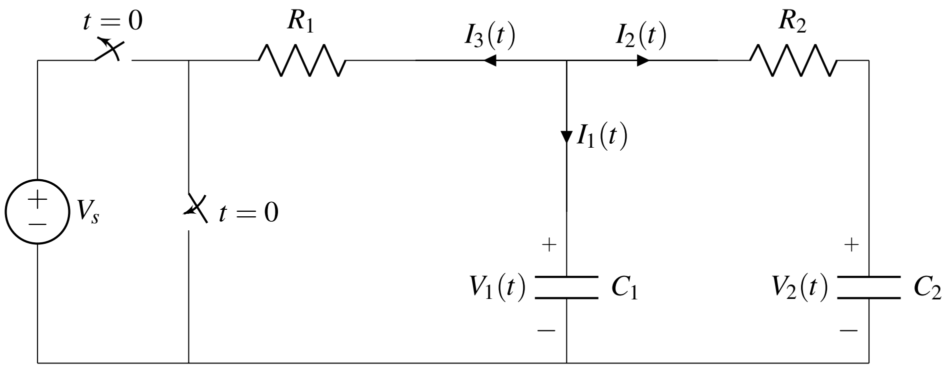


Figure 1: Two Capacitor Circuit with Voltage Source

(a) First, use Kirchoff's Laws and the capacitor equation ($I = \frac{dV}{dt}C$) to find the second order differential equation for this system in terms of $V_2(t)$

(b) Now cast this second order differential equation into the following form:

$$\frac{d\vec{v}}{dt} = A\vec{v}$$

where

$$\vec{v} = \begin{bmatrix} V_2(t) \\ \frac{dV_2(t)}{dt} \end{bmatrix}$$

(c) Find the eigenvalues of A. Are they real or complex?

(d) Using the initial conditions, what is the solution to the differential equation?

(e) Sketch the voltage vs time plots of $V_1(t)$ and $V_2(t)$.

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