EECS 16B Designing Information Devices and Systems II
Spring 2017 Murat Arcak and Michel Maharbiz **Homework 7**

## This homework is due March 15, 2017, at 17:00.

### 1. LLR circuit

Consider the circuit in Figure 1 driven by a voltage source with voltage $u(t)$. The output $y(t)$ is the current through the resistor and the state variables are the inductor currents as marked in the circuit diagram.

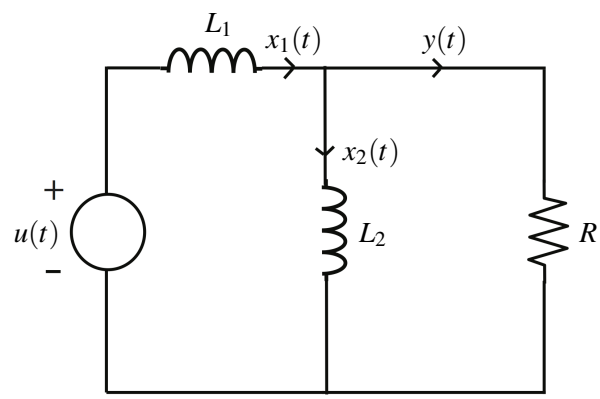



Figure 1: LLR circuit

(a) Write a state model for this circuit.

(b) Find all equilibrium points when $u(t) = 0$ for all $t$.

(c) Determine if the system is controllable.

(d) Determine if the system is observable.

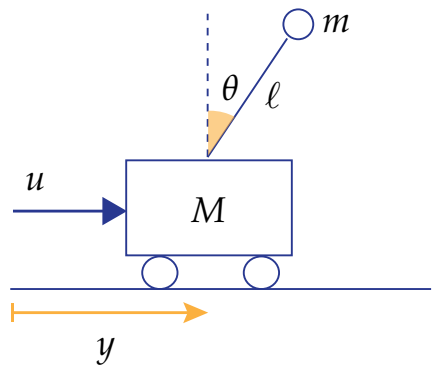
(e) If your answer to part (c) or (d) is no, explain the physical reason for lack or controllability or observability, whichever is applicable.

### 2. Inverted pendulum

Consider the inverted pendulum depicted below, whose equations of motion are

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right).$$




(a) Write the state model using the variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x_3(t) = \dot{y}(t)$. We do not include $y(t)$ as a state variable because we are interested in stabilizing the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.

(b) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and indicate the resulting $A$ and $B$ matrices.

(c) Show that the linearized model is controllable.

(d) Suppose $M = 1$, $m = 0.1$, $l = 1$, and $g = 10$, and design a state feedback controller,

$$u(t) = k_1\theta(t) + k_2\dot{\theta}(t) + k_3\dot{y}(t),$$

such that the eigenvalues of $A + BK$ (the "closed-loop eigenvalues") are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

(e) Suppose we set $k_2 = k_3 = 0$ and vary only $k_1$; that is, the controller uses only $\theta(t)$ for feedback. Does there exist a $k_1$ value such that all closed-loop eigenvalues have negative real parts?

**3. Continuous-time analog observer design: ship autopilots**

Modern ships use autopilots for steering. The main task of the autopilot is to maintain constant heading. A common system model used for ship steering controllers is the *Nomoto* first-order model. It is described using the following differential equation:

$$T\ddot{\psi} + \dot{\psi} = K\delta,$$

where $\psi$ is the ship heading, $\delta$ is the rudder angle, and $K$ and $T$ are constants that are empirically estimated during sea trials. The "dot" notation used here is the physics convention (Newton's notation) that is very convenient for problems where nothing more that a second derivative is needed.

The only sensor is a gyrocompass, which reports the ship's current heading $y(t) = \psi(t)$. We would also like to provide a good estimate of an additional important parameter, the rate of turn — the derivative of the ship's current heading.

The input of the ship model is the rudder angle $\delta$, and the output is the heading $\psi$, as measured by the gyrocompass.

*(Note for the curious: undoubtably, some of you are wondering why we don't just take the derivative of the measurement and be done with it. The reason is that although we are describing everything without any noise, in the real-world, all measurements are noisy. Taking the derivative of noise is a very bad idea because it is in the nature of noise to shake a lot and so the derivative gets swamped by the shaking of the noise.)*

In this problem you'll construct an analog continous-time observer, and then analyze its behaviour.

(a) Choose your state variables so that you have a two-dimensional state.

(b) Write down the system as a state-space model with a two-dimensional state.

(c) Is the system observable?

(d) Write down a model for the observer in matrix form using $\vec{\ell}$ to represent how you weigh the difference between the observed output $y(t)$ and the estimated output $\hat{y}(t)$ coming from within your observer.

(e) Find $\vec{\ell} = \begin{bmatrix} l_0 \\ l_1 \end{bmatrix}$ to place both the eigenvalues of the estimation error evolution at $-2$.

Now that we have designed the output-feedback and placed the eigenvalues of the estimation error. We'll design a circuit implementing the observer.

We will represent the state variables as voltages. Each input, output, and state variable will be implemented as a node in our circuit. The output of the original systems (the gyrocompass) would be an input of this system, and so would the rudder angle.

Recall that in EE16A and previously in EE16B, you have seen how to implement the following operations using simpler circuit elements (mainly resistors, capacitors and op-amps): differentiation, integration, scaling, addition and negation. This will be enough to implement the observer.

(f) Design a circuit whose output is the integral of its input with respect to time.

(g) Design a circuit whose output is a scaled version by a constant $a_0$ of its input.

(h) Design a circuit whose output is the negation of its input.

(i) Design a circuit whose output is the sum of its two inputs.

Now that we have the basic circuit elements. We'll implement the observer as a circuit.

(j) Use the circuits you designed above to construct the observer as a circuit driven by the output of the gyrocompass.

**4. Observability**

Consider the following continuous time system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

We want to construct an estimate $z$ of the system state $x$. To do so, we construct a pretend system with the same $[A, B, C, D]$ models, the same input and the output of the last system along with an $L$ system matrix. We do this to try and exploit the difference between the output of our pretend state and the actual output, with $L$ being the "knob" that we can control.

$$\dot{z}(t) = Az(t) + Bu(t) - L(Cz(t) - y(t)) \tag{2}$$

Define $e(t) = z(t) - x(t)$. This is the error term as a function of time.

(a) Using the two systems defined above, construct a system of the form,

$$\frac{\mathrm{d}e}{\mathrm{d}t}(t) = (A - LC)e(t) \tag{3}$$

(b) We want,

$$\lim_{t \to \infty} e(t) = 0$$

What does that imply about (3)?

(c) Does the initial value of the guess $z(0)$ matter in the long term?

**Contributors:**

- Murat Arcak.
- Baruch Sterin.
- Siddharth Iyer.