1. Controls

Consider the following system:

$$\frac{dx_1(t)}{dt} = -x_1(t)^2 + x_2(t)u(t)$$
$$\frac{dx_2(t)}{dt} = 2x_1(t) - 2x_2(t)u(t)$$

(a) Choose states and write a state space model for the system in the form $\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), u(t))$.

Answer:

Because the states should be related to their derivatives, we choose $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), u(t)) \\ f_2(\vec{x}(t), u(t)) \end{bmatrix} = \begin{bmatrix} -x_1(t)^2 + x_2(t)u(t) \\ 2x_1(t) - 2x_2(t)u(t) \end{bmatrix}$$

(b) Find the equilibrium $\vec{x}^*$ and input u^* when $x_2^* = 1$ and $u^* = 1$.

Answer:

Plugging in $x_2^* = 1$ and $u^* = 1$, we solve the system of equations for x_1^*.
Looking at the second equation, we get:

$$0 = 2x_1^* - 2 \implies x_1^* = 1$$

$$\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u^* = 1$$

(c) Linearize the system around the equilibrium state and input from the previous part. Your answer should be in the form $\frac{d\tilde{\vec{x}}(t)}{dt} = A\tilde{\vec{x}}(t) + B\tilde{u}(t)$.

Answer:

Recall that $\tilde{\vec{x}}(t) \triangleq \vec{x}(t) - \vec{x}^*$ and $\tilde{u}(t) \triangleq u(t) - u^*$.
From linearization, we get:

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \underbrace{[\nabla_{\vec{x}} f(\vec{x}^*(t), u^*(t))]}_{A}\tilde{\vec{x}}(t) + \underbrace{[\nabla_{\vec{u}} f(\vec{x}^*(t), u^*(t))]}_{B}\tilde{u}(t)$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1^* & u^* \\ 2 & -2u^* \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} x_2^* \\ -2x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(d) Is this system controllable? Is it stable?

Answer:

$$R_n = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 6 \end{bmatrix}$$

$$\det(A - \lambda I) = (-2-\lambda)^2 - 2 = \lambda^2 + 4\lambda + 2 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 8}}{2} = -2 \pm \sqrt{2}$$

The controllability matrix has rank 2, so this system is controllable. This system also only has negative eigenvalues, so it is stable.

(e) Convert this system into controller canonical form. Your answer should be in the form $\frac{d\vec{z}(t)}{dt} = \tilde{A}\vec{z}(t) + \tilde{B}\tilde{u}(t)$, where $\vec{z}(t) = T\vec{x}(t)$.

Answer:

$\tilde{A}$ should have the same characteristic polynomial as A.

$$\lambda^2 + 4\lambda + 2 = \lambda^2 - a_2\lambda - a_1$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(f) Find a state feedback controller $\tilde{K}$ to place both system eigenvalues at $\lambda = -1$, where $\tilde{u}(t) = -\tilde{K}\vec{z}(t)$.

Answer:

$$\tilde{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$\tilde{A} - \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 \\ -2-k_1 & -4-k_2 \end{bmatrix}$$

Comparing this characteristic polynomial with $(\lambda - (-1))^2$, we get

$$\lambda^2 - (-4-k_2)\lambda - (-2-k_1) = \lambda^2 + 2\lambda + 1$$

$$k_1 = -1, k_2 = -2$$

$$\tilde{K} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

(g) Convert this feedback controller back into the non-CCF domain, i.e., find K, such that $\tilde{u}(t) = -K\vec{x}(t)$.

(*Hint:* Remember that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.)

Answer:

$$\tilde{u}(t) = -\tilde{K}\vec{z}(t) = -\tilde{K}T\vec{\tilde{x}}(t)$$

$$R_n^{-1} = -\frac{1}{2}\begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

$$\vec{q}^T = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix}$$

$$T = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$K = \tilde{K}T = \begin{bmatrix} -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix}$$

(h) Is our system observable? Our observed output is $y(t) = -\tilde{x}_1(t)$.

Answer:

$$y(t) = -\tilde{x}_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}\vec{\tilde{x}}(t)$$

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

The observability matrix has rank 2, so this system is observable.

(i) Construct an observer system for this system.

Answer:

Remember that we use an estimate $\hat{x}(t)$ to approximate our state and then choose an L for this estimate to converge to our actual state.

$$\frac{d\vec{\hat{x}}(t)}{dt} = A\vec{\hat{x}}(t) + B\tilde{u}(t) + L(\hat{y}(t) - y(t))$$

$$\hat{y}(t) = C\vec{\hat{x}}(t)$$

(j) How does the error evolve over time?

Answer:

$$\vec{e}(t) \triangleq \vec{\hat{x}}(t) - \vec{\tilde{x}}(t)$$

$$\frac{d\vec{e}(t)}{dt} = (A + LC)\vec{e}(t)$$

(k) Pick L, such that our error signals converge to 0. Place both eigenvalues at $\lambda = -1$ again.

Answer:

$$
\begin{aligned}
A+LC &= \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} -l_1 & 0 \\ -l_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2-l_1 & 1 \\ 2-l_2 & -2 \end{bmatrix} \\
\det(A+LC-\lambda I) &= (-2-l_1-\lambda)(-2-\lambda)-(2-l_2) \\
&= \lambda^2 - \lambda(-2-2-l_1)+4+2l_1-2+l_2 \\
&= \lambda^2 + (4+l_1)\lambda + (2+2l_1+l_2) \\
&= \lambda^2 + 2\lambda + 1 \text{ (Characteristic polynomial with eigenvalues set to } \lambda=-1) \\
&\implies l_1 = -2, l_2 = 3 \\
L &= \begin{bmatrix} -2 \\ 3 \end{bmatrix}
\end{aligned}
$$

(l) Write out the entire closed-loop system with feedback control and observer.

Answer:

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \vec{\tilde{x}}(t) \\ \vec{e}(t) \end{bmatrix} &= \begin{bmatrix} A-BK & -BK \\ 0 & A+LC \end{bmatrix} \begin{bmatrix} \vec{\tilde{x}}(t) \\ \vec{e}(t) \end{bmatrix} \\
A-BK &= \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -2+1 & 1-\frac{1}{2} \\ 2-2 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix} \\
A+LC &= \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2+2 & 1 \\ 2-3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \\
\frac{d}{dt}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ e_1 \\ e_2 \end{bmatrix} &= \begin{bmatrix} -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ e_1 \\ e_2 \end{bmatrix}
\end{aligned}
$$

Contributors:

- Kevin Chen.