EECS 16B Designing Information Devices and Systems II
Spring 2018 J. Roychowdhury and M. Maharbiz **Discussion 12B**

1 Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

The following are some of the possible properties that a system can have:

1.1 Linearity

A **linear system** has the properties below:

(a) **additivity**

$$x_1[n] + x_2[n] \longrightarrow \boxed{} \longrightarrow y_1[n] + y_2[n] \tag{1}$$

(b) **scaling**

$$\alpha x[n] \longrightarrow \boxed{} \longrightarrow \alpha y[n] \tag{2}$$

Here, α is some constant.

Together, these two properties are known as **superposition**:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

1.2 Time Invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x[n-n_0] \longrightarrow \boxed{} \longrightarrow y[n-n_0] \tag{3}$$

1.3 Causality

A **causal** system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not "look ahead."

1.4 Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $a[n]$ is bounded if there exists an A such that $|a[n]| \leq A < \infty \; \forall n$.

2 Linear Time-Invariant (LTI) Systems

A system is LTI if it is both linear and time-invariant. Let $h[n]$ be the **impulse response** of an LTI system. That is, $y[n] = h[n]$ if $x[n] = \delta[n]$, where $\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$ is the unit impulse.

An LTI system can be completely characterized by $h[n]$. The following properties hold:

- An LTI system is causal iff $h[n] = 0 \ \forall n < 0$.
- An LTI system is BIBO stable iff its impulse reponse is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

2.1 Convolution Sum

Consider the following LTI system with impulse reponse $h[n]$:

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

Notice that we can write $x[n]$ as a sum of impulses:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

In addition, we know that:

$$\delta[n] \longrightarrow \boxed{} \longrightarrow h[n]$$

By applying the LTI property of our system, we get that

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

The expression $\sum_{m=-\infty}^{\infty} x[m]h[n-m]$ is known as the **convolution sum** and can be written as $x[n] * h[n]$ or $(x*h)[n]$.

1. Time-Shift Systems

Imagine we have a system $S_{\rightarrow 2}$ that takes any length 5 input signal and shifts it by 2 steps. For example, $S_{\rightarrow 2}\left(\begin{bmatrix} 3 & 1 & 4 & 1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 & 3 & 1 & 4 \end{bmatrix}$.

(a) Is this system linear? That is, for any signals $\vec{x}$ and $\vec{y}$, does $S_{\rightarrow 2}$ fulfill properties (1) and (2)?

Answer: Yes.

(b) Is this system time-invariant? Does it fulfill property (3)?

Answer: Yes.

(c) What does $S_{\rightarrow 2}$ look like when written as a matrix?

Answer:

$$S_{\rightarrow 2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

2. Is it LTI?

Determine if the following systems are LTI:

(a) $y[t] = 2x[-2+3t] + 2x[2+3t]$

Answer:

Linear, not time-invariant.

Let $\hat{x}[t] = x[t-t_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[t]$ is equal to $2x[-2+3t-t_0] + 2x[2+3t-t_0]$.

However, we can see that $\hat{y}[t] \neq y[t-t_0] = 2x[-2+3(t-t_0)] + 2x[2+3(t-t_0)]$.

(b) $y[t] = 4^{x[t]}$

Answer:

Non-linear.

Let $\hat{x}[t] = 2x[t]$. Then $\hat{y}[t] = 16^{x[t]} \neq 2y[t]$.

(c) $y[t] - y[t-1] + y[t-2] = x[t] - x[t-1] - x[t-2]$

Answer:

LTI.

(d) $y[t] = x[t] + tx[t-1]$

Answer:

Not time-invariant.

(e) $y[t] = 2^t \cos\big(x[t]\big)$

Answer:

Not linear, not time-invariant.

3. Convoluted Convolution

Show that convolution is commutative. That is, show that $(x*h)[n] = (h*x)[n]$.

Answer:

$$
\begin{aligned}
(x*h)[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{k=-\infty}^{\infty} x[n-k]h[k] && \text{Let } k = n-m. \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= (h*x)[n]
\end{aligned}
$$

4. Mystery System

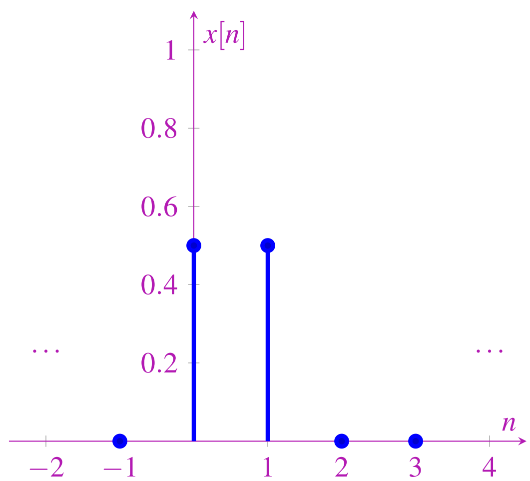
Consider an LTI system with the following impulse response:

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

(a) Create a sketch of this impulse reponse. Is this a finite or infinite impulse response system?

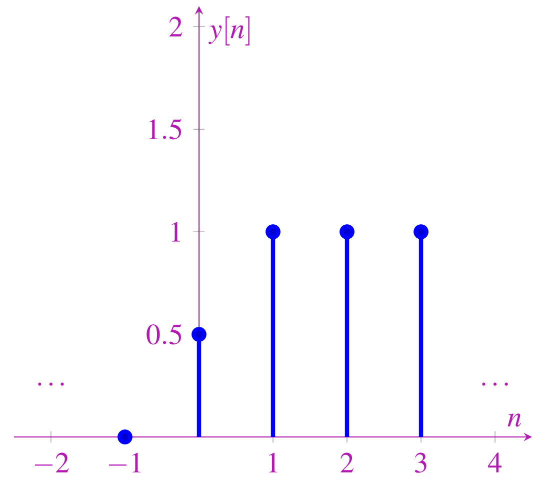
Answer:

This is an FIR system.



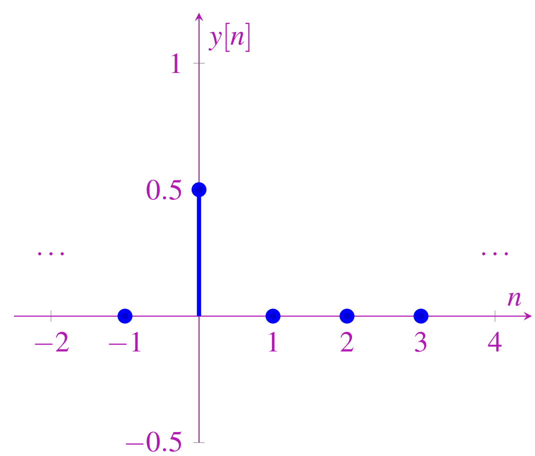
(b) What is the output of our system if the input is the unit step $U[n]$?

Answer:



(c) What is the output of our system if our input is $x[n] = (-1)^n U[n]$?

Answer:



(d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?

Answer:

The output of the system at each timestep n is the average of $x[n]$ and $x[n-1]$. This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter.

Contributors:

- Deborah Soung.
- Tianrui Guo.
- Brian Kilberg.