1 Complex Inner Product

For the complex vector space $\mathbb{C}^n$, we can no longer use our conventional real dot product as a valid inner product for $\mathbb{C}^n$. This is because the real dot product is no longer positive-definite for complex vectors.

For example, let $\vec{v} = \begin{bmatrix} j \\ j \end{bmatrix}$. Then, $\vec{v} \cdot \vec{v} = j^2 + j^2 = -2 < 0$.

Therefore, for two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, we define the complex inner product to be:

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i \overline{v}_i = \vec{u}^T \overline{\vec{v}} = \vec{v}^* \vec{u}$$

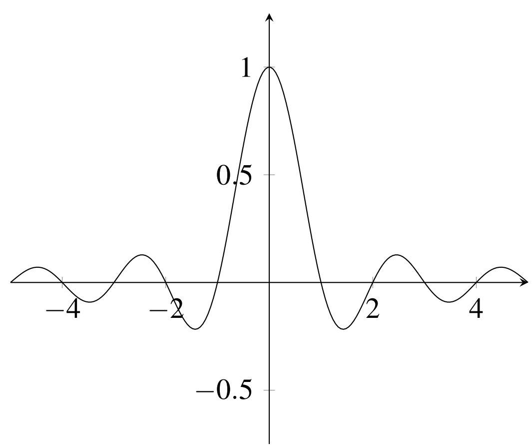
where $\overline{\cdot}$ denotes the complex conjugate and $\vec{\cdot}^*$ denotes the complex conjugate transpose. Recall that the complex conjugate of a complex number $z = a + jb = re^{j\theta}$ is equal to $\overline{z} = a - jb = re^{-j\theta}$. The conjugate transpose of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is $\vec{v}^* = \begin{bmatrix} \overline{v}_1 & \overline{v}_2 & \cdots & \overline{v}_n \end{bmatrix}$.

Note that this inner product is no longer symmetric but conjugate-symmetric, i.e., $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$.

2 Sinc Interpolation

For sinc interpolation of periodic functions, we use the following basis function:

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$



Note that $\text{sinc}(x)$ is a valid basis function because $\text{sinc}(0) = 1$ and $\text{sinc}(x) = 0$ for all integer values of x.

3 Convolution

Recall that we can represent any input sequence as a sum of impulses $\delta[t]$:

$$u[t] = \cdots + u[-1]\delta[t+1] + u[0]\delta[t] + u[1]\delta[t-1] + \cdots = \sum_{i=-\infty}^{\infty} u[i]\delta[t-i]$$

Since the system is linear time-invariant, the output is a sum of impulse responses $h[t]$:

$$y[t] = \sum_{i=-\infty}^{\infty} u[i]h[t-i]$$

If the system is causal, i.e. $h[t] = 0$ for all $t < 0$, then:

$$y[t] = \sum_{i=-\infty}^{t} u[i]h[t-i]$$

Furthermore, if we assume that $u[t] = 0$ for all $t < 0$, then:

$$y[t] = \sum_{i=0}^{t} u[i]h[t-i]$$

4 Causality

A system is causal if it does not respond to an input before the input is applied.

Assume that we have some input $\hat{u}[t]$ that produces the output $\hat{y}[t]$. We pick an arbitrary time τ and create a new input sequence $u[t]$, such that $u[t] = \hat{u}[t]$ for all $t < \tau$ but not necessarily for $t \geq \tau$.

We apply the new input $u[t]$ to the system and record its output $y[t]$. If $y[t] = \hat{y}[t]$ for all $t < \tau$, then the system is causal.

An LTI system is causal if and only if $h[t] = 0$ for all $t < 0$.

1. Linear State Space System

(a) Consider a general discrete-time system with input $\vec{u}[t]$, output $\vec{y}[t]$, and the following state space representation:

$$\begin{aligned} \vec{x}[t+1] &= A\vec{x}[t] + B\vec{u}[t] \\ \vec{y}[t] &= C\vec{x}[t] + D\vec{u}[t] \end{aligned}$$

Prove rigorously that such a system is linear time-invariant.

(b) Consider a general continuous-time system with input $\vec{u}(t)$, output $\vec{y}(t)$, and the following state space representation:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

Prove rigorously that such a system is linear time-invariant. Note that you only need to show that the first state space equation is linear time-invariant since the second state space equation is linear time-invariant as proven in part (a).

2. Linear State Space System with Initial Condition

Consider the following discrete-time system with $\vec{u}[t] = \vec{0}$ for $t < 0$:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

$$\vec{y}[t] = C\vec{x}[t]$$

(a) Find an expression for $\vec{y}[t]$ that only depends on the input at each time step $\vec{u}[t]$ and the intial condition $\vec{x}[0]$. (*Hint:* Find a general expression for $\vec{x}[t]$ and plug it into the $\vec{y}[t]$ equation.)

(b) Given the initial condition $\vec{x}[0] = \vec{0}$, is this system linear?

(c) If the initial condition $\vec{x}[0] = \vec{0}$, is this system time-invariant?

(d) Now the initial condition $\vec{x}[0] \neq \vec{0}$. Does this affect the linearity of this system?

(e) How can we reconcile the results from parts (c) and (d) with that of part (a) of the first problem?

(f) Re-derive part (a) but without assuming that $\vec{u}[t] = \vec{0}$ for $t < 0$.

3. Stability and Causality

(a) Recall that for a general LTI system, the output $y[t]$ is equal to $y[t] = \sum_{i=-\infty}^{\infty} u[i]h[t-i]$. Also recall that a system is BIBO stable if its output is bounded given some bounded input.
Derive a condition on $h[t]$, such that the LTI system is BIBO stable.

(b) Explain how the condition derived in part (a) relates to the eigenvalues of the state space representation $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$.

(c) Derive the causality condition for LTI systems from the general definition of causality.

Contributors:

- Titan Yuan.
- Kyle Tanghe.