

1 RC Circuit Theory

The RC circuit is a fundamental component of any real world circuit. Many electronic systems' specifications, like clock speed and bandwidth, are direct results of RC circuits. We will use differential equation methods to find the time domain behavior of RC systems. We first set up our problem by defining two functions of time: $I_C(t)$ is the current into the capacitor at time t , and $V_C(t)$ is the voltage across the capacitor at time t .

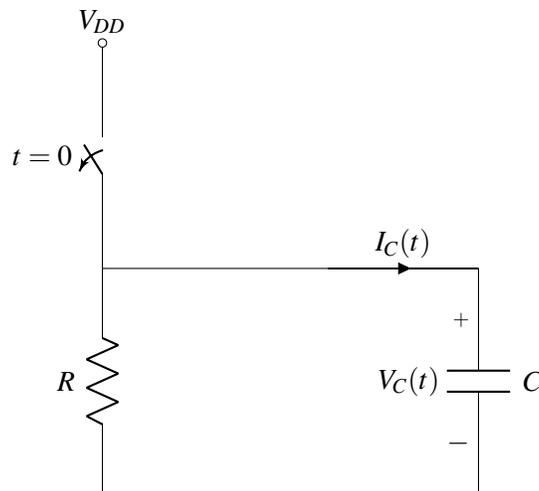


Figure 1: RC Circuit

Let's consider the RC circuit above in Figure 1. Assume that the capacitor is fully charged to V_{DD} , and at time $t = 0$, the switch is opened. Now, we essentially have a pull-down network bringing the capacitor voltage to 0. After $t = 0$, the voltage in the cap is still V_{DD} , but current starts to flow out of the capacitor through the resistor. As the current flows out, the charge stored in the capacitor decreases. This causes the voltage across the capacitor to decrease. How can we describe this behavior mathematically?

Real life is continuous, so we need to use differential equations. Let's start with the relationship between the charge and the voltage in a capacitor, which is shown below.

$$Q(t) = CV_C(t)$$

We know that there is a current flowing within the capacitor because it is connected to a resistor. Since current is simply the change of the charge over time, we can relate $Q(t)$ to $I_C(t)$.

$$I_C(t) = \frac{dQ(t)}{dt}$$

Taking the derivative of the both sides, we get

$$\frac{dQ(t)}{dt} = C \frac{dV_C(t)}{dt} = I_C(t)$$

This is the differential equation that relates the current and voltage of a capacitor.

We know that

$$V_C(t) = -I_C(t)R$$

Therefore, we substitute it in for $I_C(t)$.

$$C \frac{dV_C(t)}{dt} = -\frac{V_C(t)}{R}$$

We end up with $-\frac{V_C(t)}{R}$ because the current through the resistor is flowing against the direction we defined for $I_C(t)$.

Rearranging the term, we get

$$\begin{aligned} \frac{dV_C(t)}{dt} &= -\frac{1}{RC}V_C(t) \\ \frac{d}{dt}V_C(t) &= -\frac{1}{RC}V_C(t) \end{aligned}$$

The differentiation operator is a linear operator, so we can view this equation as a linear system $\mathbf{A}V_C(t) = \lambda V_C(t)$, where λ is the eigenvalue of the system and \mathbf{A} is the differentiation operator. Essentially, we need to find a function $V_C(t)$ that, when operated on by \mathbf{A} , results in that same function multiplied by a scaling factor. In linear systems, this function is called the eigenfunction of an operator. The eigenfunction of $\frac{d}{dt}$ is $Ke^{\lambda t}$. This means that $V_C(t)$ is of the form $Ke^{\lambda t}$.

When we operate on it with $\frac{d}{dt}$, we get

$$\frac{d}{dt}Ke^{\lambda t} = K\lambda e^{\lambda t} = \lambda Ke^{\lambda t}$$

and λ is the corresponding eigenvalue.

In our RC system, the constant K is defined by the initial conditions and $\lambda = -\frac{1}{RC}$. Plugging our eigenvalue into our eigenfunction gives us

$$V_C(t) = Ke^{-\frac{t}{RC}}$$

To solve for K , we need to take into account the initial conditions of our problem. At $t = 0$, $V_C(t) = V_{DD}$, so

$$V_C(0) = Ke^{-\frac{0}{RC}} = V_{DD}$$

$$K = V_{DD}$$

Finally, we have

$$V_C(t) = V_{DD}e^{-\frac{t}{RC}}$$

Now, we can evaluate how long it takes to discharge half the voltage.

$$t_{\text{half life}} = \ln(2)RC \approx 0.693RC$$

The equation is derived by setting $V_C(t) = \frac{1}{2}V_{DD}$ and solving for t . We see that bigger the values of R and C , the longer it takes for the voltage to drop. RC is also called the time constant τ . It's useful to have a general idea of how many τ it takes for a capacitor to reach its final steady state value. After one τ , the capacitor voltage is within 36.8% of its final steady state value. After 5τ , it is within 1% of its final steady state value.

1. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor V_C over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t , $V(t)$ is the voltage across the circuit at time t , and $V_C(t)$ is the voltage across the capacitor at time t .

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.

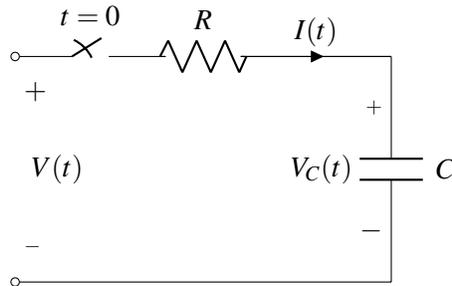


Figure 2: Example Circuit

- First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.
- Using Kirchhoff's law, write an equation that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.
- So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we found in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.
- Let's suppose that for $t < 0$, the capacitor is charged to a voltage V_{DD} and that $V(t) = 0 \forall t \geq 0$, i.e. shorted to ground. Assuming that we close the switch at $t = 0$, use the fact that $V_C(0) = V_{DD}$ to solve this differential equation for $V_C(t)$.
- Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Assuming that the switch is closed at $t = 0$, use your differential equation to solve for $V_C(t)$.
- Now that you know how the voltage across a capacitor acts over time in an RC circuit, how does the charge in the capacitor act over time? Write your answer as a function of $Q(t)$, and remember that $V_C = \frac{Q}{C}$.

2. RC Circuit of Inverter Input

Let's now consider a slightly more complicated RC circuit.

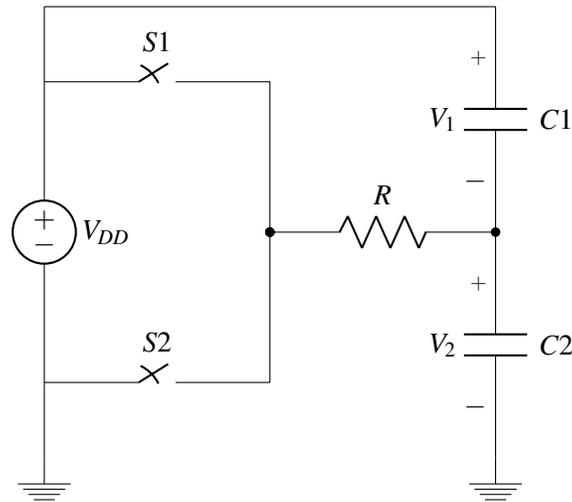


Figure 3: Inverter Input

In this problem, we will explore what happens when we change the voltage in between the capacitors.

- Suppose that $S1$ has been closed and $S2$ has been open long enough that the voltage across $C1$ and $C2$ have settled to constant values. Suppose at $t = 0$, we open $S1$ and close $S2$. State the initial conditions of the differential equation (i.e. $V_1(0)$ and $V_2(0)$), and express the voltages $V_1(t)$ and $V_2(t)$ in both capacitors as a function of time using the equations you derived from the previous problem.
- Suppose that $S2$ has been closed and $S1$ has been open long enough that the voltage across $C1$ and $C2$ have settled. Suppose at $t = 0$, we open $S2$ and close $S1$. State the initial conditions of the differential equation (i.e. $V_1(0)$ and $V_2(0)$), and express the voltage in both capacitors as a function of time using the equations you derived from the previous problem.

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