1 Observability

Up to now, we have assumed that it is possible to know the state $\vec{x}[t]$ at any point in time, e.g. to implement full state feedback $\vec{u}[t] = -K\vec{x}[t]$. However, this assumption is not always physically feasible. For example, a car on a one-dimensional track has position $p(t)$ and velocity $v(t)$, but we have a driver who can only tell the velocity by glancing at his speedometer.

More generally, we formalize a system's output $\vec{y}[t]$ as a function of the current state $\vec{x}[t]$. For a system with state dynamics $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$, we represent the output as $\vec{y}[t]$ that is dependent on $\vec{x}[t]$.

$$\vec{y}[t] = C\vec{x}[t]$$

Clearly, if $\vec{y}[t] \in \mathbb{R}^n$ and $C = I$, we can know the state perfectly. This goal—equivalent to being able to determine $\vec{x}[0]$ from a finite series of measurements $\vec{y}[0], \vec{y}[1], \ldots, \vec{y}[k]$—is called *observability*. By the equation for $\vec{x}[t+1]$ and induction on t, we can then determine $\vec{x}[t]$ for all $t > 0$. We will search for necessary and sufficient conditions for a system to be observable as well as realize an upper bound for k.

1.1 Observability Matrix

To find $\vec{x}[0]$, let $\vec{u}[t] = 0$. At time $t = 0$, we are able to see the output $\vec{y}[0] = C\vec{x}[0]$. If we let the system evolve until time $t = k$, we observe the following values.

$$\begin{aligned}
\vec{y}[0] &= C\vec{x}[0] \\
\vec{y}[1] = C\vec{x}[1] &= CA\vec{x}[0] \\
\vec{y}[2] &= CA^2\vec{x}[0] \\
&\vdots \\
\vec{y}[k] &= CA^k\vec{x}[0]
\end{aligned}$$

Rewriting these equations, we get the following relation

$$\begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \\ \vec{y}[2] \\ \vec{y}[3] \\ \vdots \\ \vec{y}[k] \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^k \end{bmatrix} \vec{x}[0]$$

If we want to be able to uniquely determine the original state $\vec{x}[0]$, we want this matrix to be full rank. By the Cayley-Hamilton theorem, the rank of this matrix will not increase past $k = n-1$, where n is the dimension of our state space. Thus, we define the observability matrix $\mathscr{O}$ to be the following matrix.

$$\mathscr{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If the observability matrix $\mathscr{O}$ matrix is rank n, then the system is *observable*, and we can uniquely determine the initial state from a series of measured outputs. Notice the similarity to the controllability matrix. In fact, we call observability the mathematical dual of controllability.

2 Observers

We construct an additional system, called an *observer*, to estimate the state. We start with an initial guess for $\vec{x}[0]$, which we will call $\vec{\hat{x}}[0]$, and we will update this system as follows.

$$\vec{\hat{x}}[t+1] = A\vec{\hat{x}}[t] + B\vec{u}[t] + \underbrace{L\left(\vec{\hat{y}}[t] - \vec{y}[t]\right)}_{\text{output feedback}}$$

$$\vec{\hat{y}}[t] = C\vec{\hat{x}}[t] \qquad\qquad \vec{y}[t] = C\vec{x}[t]$$

The additional output feedback term tracks the difference between the estimated and real outputs. If the observer outputs deviate from the real outputs, our system nudges the estimated outputs closer to the real outputs. If L is chosen the right way, the estimate quickly converges to the actual state.

We define the error $\vec{e}[t] = \vec{\hat{x}}[t] - \vec{x}[t]$ to be the difference between the estimated state and the actual state.

$$\begin{aligned}
\vec{e}[t+1] &= \vec{\hat{x}}[t+1] - \vec{x}[t+1] \\
&= A\vec{\hat{x}}[t] + B\vec{u}[t] + L\left(\vec{\hat{y}}[t] - \vec{y}[t]\right) - (A\vec{x}[t] + B\vec{u}[t]) \\
&= A\left(\vec{\hat{x}}[t] - \vec{x}[t]\right) + L\left(\vec{\hat{y}}[t] - \vec{y}[t]\right) \\
&= A\left(\vec{\hat{x}}[t] - \vec{x}[t]\right) + L\left(C\vec{\hat{x}}[t] - C\vec{x}[t]\right) \\
&= (A + LC)\left(\vec{\hat{x}}[t] - \vec{x}[t]\right) \\
&= (A + LC)\vec{e}[t]
\end{aligned}$$

Note that we can now find values for the L matrix, such that the eigenvalues of $A + LC$ have a magnitude less than 1, in which case as $t \to \infty$, $\vec{e}[t] \to 0$, which means that that eventually, $\vec{\hat{x}}[t] = \vec{x}[t]$, telling us that our estimates for the states are in fact correct.

Note that this equation is similar to state feedback, but the order of LC vs. BK is different. To handle this difficulty, we introduce the **dual system**. It is a formal system—we do not care about its behavior—we just use it to find L.

$$\vec{z}[t+1] = A^T\vec{z}[t] + C^T\vec{v}[t]$$

Note that its controllability matrix is $\mathscr{C}_{\text{dual}} = \begin{bmatrix} C^T & A^T C^T & \cdots & \left(A^T\right)^{n-1} C^T \end{bmatrix}$, which is the transpose of the observability matrix of the original system $\mathscr{O}_{\text{orig}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$. This means that the dual system is controllable if the original system is observable because $\text{rank}\left(\mathscr{C}_{\text{dual}}\right) = \text{rank}\left(\mathscr{C}_{\text{dual}}^T\right) = \text{rank}\left(\mathscr{O}_{\text{orig}}\right)$.

By setting $\vec{v}[t] = L^T \vec{z}[t]$, we can put this system into feedback and set the eigenvalues of $A^T + C^T L^T$ as we please. Then, we can transpose the result and get $A + LC$ with the the same eigenvalues.

Finally, we can now write down the closed loop system with feedback and observer. Because of the added observer, the input is $\vec{u}[t] = -K\vec{\hat{x}}[t] = -K\left(\vec{x}[t] + \vec{e}[t]\right)$:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] = A\vec{x}[t] - BK\left(\vec{x}[t] + \vec{e}[t]\right) = (A - BK)\vec{x}[t] - BK\vec{e}[t]$$

Resulting in the block defined system

$$\begin{bmatrix} \vec{x}[t+1] \\ \vec{e}[t+1] \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} \vec{x}[t] \\ \vec{e}[t] \end{bmatrix}.$$

In continuous time, the math is the same, except for (1) we replace $\vec{x}[t+1]$ with $\frac{d}{dt}\vec{x}[t]$, and (2) we place the eigenvalues, such that their real part is less than 0.

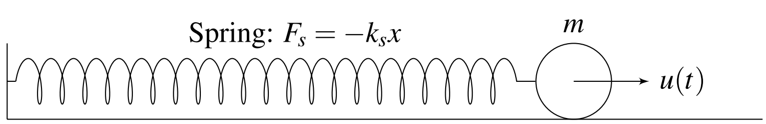
1. Springtime

We would like to model a self-propelled mass connected by a string. It is subject to the following forces:

- The spring applies a force $F_s(t) = -k_s x(t)$ (at $x = 0$, it does not apply any force).
- The mass is propelled by a force $u(t)$, the latest in perpertuum mobile technology. This is the only input of the system.

The only sensor the system has is a speedometer. That is, it can measure the speed of the system accurately, but it cannot directly measure its position.

Starting at an arbitrary position and speed, we would like to apply the correct inputs required to position the mass at $x = 0$.



(a) Model the system as a linear continuous-time state-space model:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Bu(t)$$

$$y(t) = C \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Write out A, B, and C.

(b) Given $m = 1\,\text{kg}$ and $k_s = 2$, plug these values into the system.

(c) Is the system observable?

(d) Write down the equations of the observer system.

(e) Write down the equations governing the estimation error:

$$\vec{e}(t) = \begin{bmatrix} e_0 \\ e_1 \end{bmatrix}$$

(f) Compute the output feedback matrix L that places both the eigenvalues of the system governing the estimation error at $\lambda = -2$.

(g) Write out the dual system of the original system.

(h) Is the dual system controllable?

(i) Write out the observer closed-loop system using L.

(j) Is the original system stable?

(k) Is the original system controllable?

(l) Derive a state feedback matrix K that places both the eigenvalues of the closed-loop system at $\lambda = -1$.

(m) Write down the equations for the closed-loop system, including the feedback and observer.

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