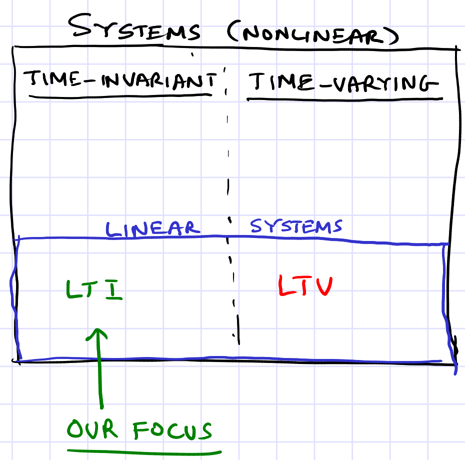
LINEAR TIME-INVARIANT (LTI) SYSTEMS

1. INTRODUCTION
2. RECAP OF LINEARITY
3. TIME INVARIANCE
4. EXAMPLES
5. LTI SYSTEMS: IMPULSE RESPONSE CHARACTERIZATION (CAUSALITY) — DISCRETE
6. ILLUSTRATION OF DISCRETE CONVOLUTION
7. VERY SIMILAR RESULT FOR C.T.

---

1. INTRODUCTION



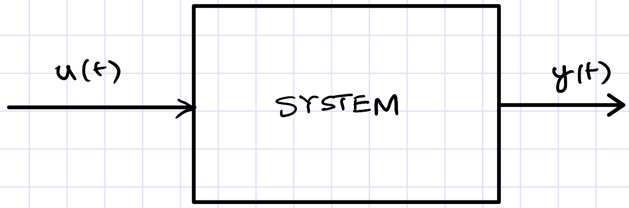
— WHY FOCUS ON LTI SYSTEMS?

- — EASIEST TO ANALYSE AND UNDERSTAND → POWERFUL, ELEGANT INSIGHTS.
  - eg. impulse responses
- — VERY USEFUL APPROXIMATION OF REAL WORLD SYSTEMS
- — WITHOUT KNOWING LTI WELL, CAN'T PROGRESS TO LTV/NONLINEAR

— EXAMPLES OF LTI SYSTEMS

- — PRETTY MUCH EVERY SYSTEM WE HAVE DONE IN THIS CLASS!
  - — RC, RLC CIRCUITS
  - — (LINEARIZED) PENDULUM → NONLINEAR PENDULUM WAS TIME-INVARIANT, THOUGH NOT LINEAR
  - — CO-OPERATIVE CAR CONTROL

2. RECAP OF LINEARITY



"LEADS TO"

$u(t) \mapsto y(t)$

INPUT — OUTPUT

→ LINEAR IFF ("IF AND ONLY IF"):

1. SCALING: IF $u(t) \xrightarrow{\text{"LEADS TO"}} y(t)$, then $(\alpha u(t)) \mapsto (\alpha y(t))$ $\forall \alpha, \forall u(t)$

AND

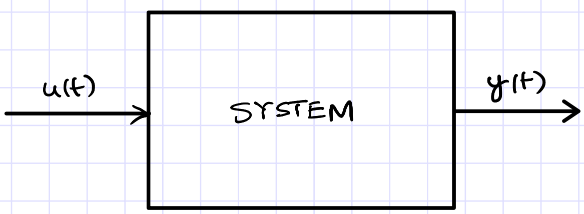
2. SUPERPOSITION: IF $u_1(t) \mapsto y_1(t)$, $u_2(t) \mapsto y_2(t)$, then

$$(u_1(t) + u_2(t)) \mapsto (y_1(t) + y_2(t)), \quad \forall u_1(t), u_2(t)$$

→ EXAMPLE: deferred to a little later.

---

3. TIME INVARIANCE



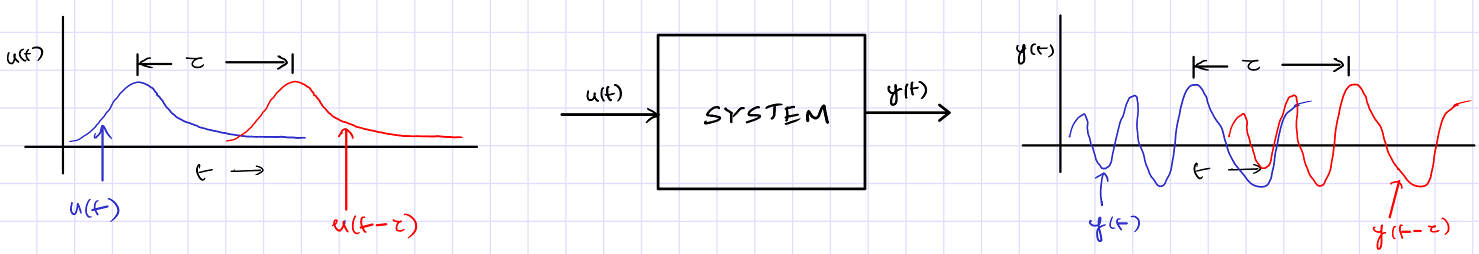
$u(t) \mapsto y(t)$

→ IN WORDS: SHIFTING THE INPUT (IN TIME) SHIFTS THE OUTPUT (BY THE SAME AMOUNT)

→ IN EQUATIONS: if $u(t) \mapsto y(t)$, then $u(t-\tau) \mapsto y(t-\tau)$ $\quad \forall \tau \in \mathbb{R}, \forall u(t)$

(τ IS A CONSTANT, DOES NOT DEPEND ON t)

→ IN PICTURES



4. EXAMPLES:

→ LINEAR BUT NOT TIME INVARIANT:

→ $y(t) = 2\sin(2\pi t)\, u(t)$

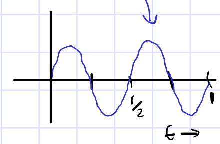
→ LINEAR? CHECK SCALING & SUPERPOSITION: YES

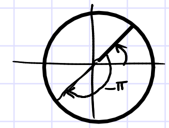
→ TI: TRY $u(t) = \cos(2\pi t)$

[$\sin(A+B) = \sin A \cos B + \cos A \sin B$]

[$\Rightarrow \sin(2A) = 2 \sin A \cos(A)$]

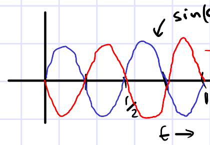
→ $y(t) = 2\sin(2\pi t)\cos(2\pi t) = \sin(4\pi t)$





→ Shift $u(t)$ by $\tau = 0.5$: $u(t-\tau) = \cos(2\pi(t-0.5)) = \cos(2\pi t - \pi) = -\cos(2\pi t)$

→ $y_{new}(t) = 2\sin(2\pi t)\, u(t-\tau) = -2\sin(2\pi t)\cos(2\pi t) = -\sin(4\pi t)$



→ if TI, then $y_{new}(t)$ should be $= y(t-0.5) = \sin(4\pi(t-0.5))$

$= \sin(4\pi t - 2\pi)$

$= \sin(4\pi t)$

→ BUT IT IS NOT!

→ HENCE NOT TI.

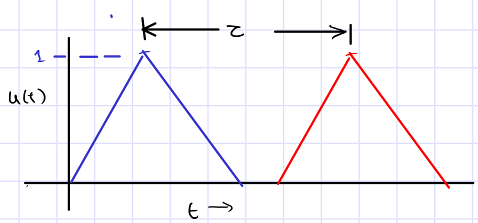
→ EXAMPLE: NOT LINEAR, BUT TIME-INVARIANT

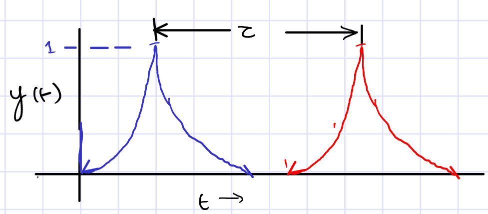
→ $y(t) = u^2(t)$

→ LINEAR? SCALING: try $u(t) \equiv 1 \Rightarrow y(t) \equiv 1$; $\alpha = 2 \Rightarrow y_{new}(t) = 4 \neq 2y(t)$! NOT LINEAR

→ TI? $u_{new}(t) \triangleq u(t-\tau)$

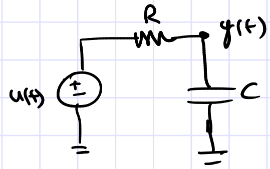
$$\left.\begin{array}{l} \rightarrow y_{new}(t) = u^2_{new}(t) = u^2(t-\tau) \\ \rightarrow y(t-\tau) \equiv u^2(t-\tau) \end{array}\right] \rightarrow \text{THE SAME} \Rightarrow \text{TI.}$$





→ LINEAR AND TIME INVARIANT (LTI)

→ EXAMPLE:



$$C\,\dot{y}(t) = \frac{u(t)-y(t)}{R}$$

→ LINEAR?

1. SCALING: if $u_{new}(t) = \alpha\, u(t)$ and $y_{new}(t) = \alpha\, y(t)$, is the eqn. satisfied?

$$\rightarrow\ ??\ C\,\dot{y}_{new}(t) = \frac{u_{new}(t) - y_{new}(t)}{R}\ ??$$

$$\equiv\ C\,\cancel{\alpha}\,\dot{y}(t) = \frac{\alpha\, u(t) - \alpha\, y(t)}{R} = \frac{\cancel{\alpha}\,(u(t)-y(t))}{R}$$

$$\equiv\ C\,\dot{y}(t) = \frac{u(t)-y(t)}{R}$$ ← which is true, from defn. of $y(t)$

2. SUPERPOSITION: ALSO YES (LEFT AS AN EXERCISE)

→ TI?

→ Take $u_{new}(t) = u(t-\tau)$

→ Does $y_{new}(t) \triangleq y(t-\tau)$ satisfy the system equation?

↳ IN WORDS: $y_{new}(t)$ is just $y(s)$ evaluated at $s = t-\tau$

or "$y(t)$ evaluated at $t = t+\tau$"

$$\rightarrow\ \frac{d}{dt} y_{new}(t) = \frac{d}{ds} y(s)\Big|_{s=t-\tau}$$

$$\rightarrow\ C\,\frac{d}{ds} y(s) = \frac{u(s)-y(s)}{R}$$

$$\rightarrow\ C\,\frac{d}{ds} y(s)\Big|_{s=t-\tau} = \left(\frac{u(s)-y(s)}{R}\right)\Big|_{s=t-\tau}$$

$$\rightarrow\ C\,\underbrace{\frac{d}{ds} y(s)\Big|_{s=t-\tau}}_{\frac{dy_{new}(t)}{dt}} = \frac{u(t-\tau)-y(t-\tau)}{R} = \frac{u_{new}(t)-y_{new}(t)}{R}$$

$$\rightarrow\ C\,\frac{dy_{new}(t)}{dt} = \frac{u_{new}(t)-y_{new}(t)}{R}$$ ← SHIFTED WAVEFORMS $u_{new}(t)$ and $y_{new}(t)$ SATISFY THE SYSTEM EQUATION ⇒ IT IS TI

— ASIDE: A USEFUL WAY TO INTERPRET (DIFFERENTIAL) EQUATIONS

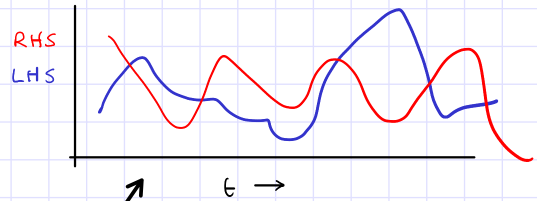
→ SUPPOSE YOU HAVE AN EQUATION LIKE:

$$\longrightarrow \underbrace{C\frac{dy(t)}{dt}}_{\text{LHS}} = \underbrace{\frac{u(t)-y(t)}{R}}_{\text{RHS}}, \quad \text{with some given input } u(t)$$

→ PICK SOME $y(t)$

→ PLOT THE LHS

→ PLOT THE RHS



→ DO THEY MATCH?

→ YES: $y(t)$ is a solution

→ NO: $y(t)$ is NOT a solution (TRY AGAIN)

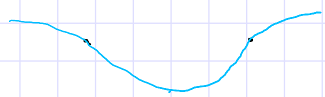
→ REVISIT TIME INVARIANCE PROOF:

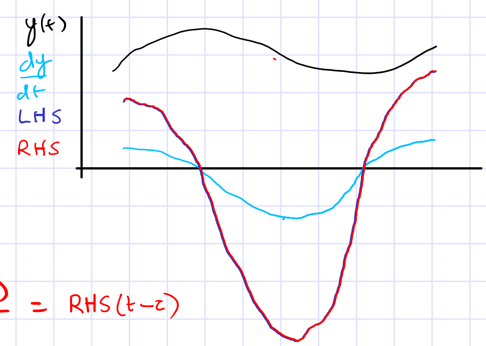
$$\underbrace{C\frac{dy(t)}{dt}}_{\text{LHS}} = \underbrace{\frac{u(t)-y(t)}{R}}_{\text{RHS}}$$

1. START WITH A SOLUTION: $u(t), y(t)$

2. $\text{RHS} = \dfrac{u(t)-y(t)}{R}$

   $\text{LHS} = C\dfrac{dy(t)}{dt}$





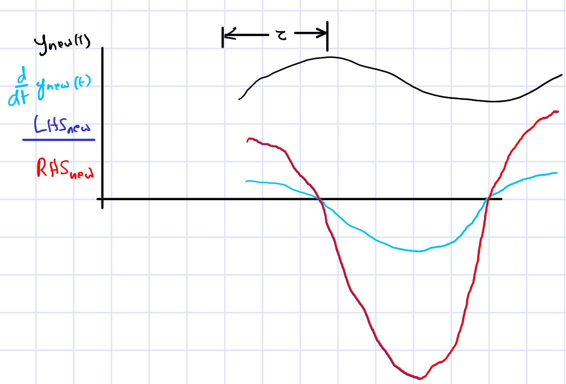
3. DEFINE $u_{new}(t) = u(t-\tau)$

   $y_{new}(t) = y(t-\tau)$

$$\text{RHS}_{new}(t) \equiv \frac{u(t-\tau)-y(t-\tau)}{R} = \text{RHS}(t-\tau)$$

$$\text{LHS}_{new}(t) = C\frac{d}{dt}y(t-\tau)$$

↳ DERIVATIVE OF THE WAVEFORM $y(t-\tau)$



→ STANDARD LINEAR(IZED) STATE-SPACE EQUATION IS ACTUALLY LTI

→ $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, $\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$ (C.T.)

→ $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$, $\vec{y}[t] = C\vec{x}[t] + Du[t]$ (D.T.)

→ EXACTLY THE SAME REASONING AS FOR THE RC CKT EXAMPLE SHOWS IT IS LTI

→ LEFT AS EXERCISES

→ POINT TO PONDER ON YOUR OWN: HOW DOES THE INITIAL CONDITION FIGURE IN LINEARITY AND TIME INVARIANCE?

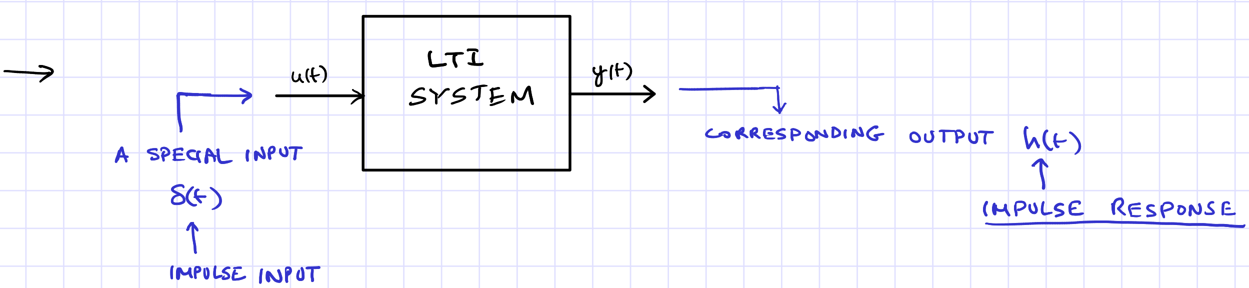
→ HINT: WORK IT OUT FIRST FOR THE RC CKT EXAMPLE.

→ (THERE MAY BE A HW ON THIS)

---

5. IMPULSE RESPONSES OF LTI SYSTEMS

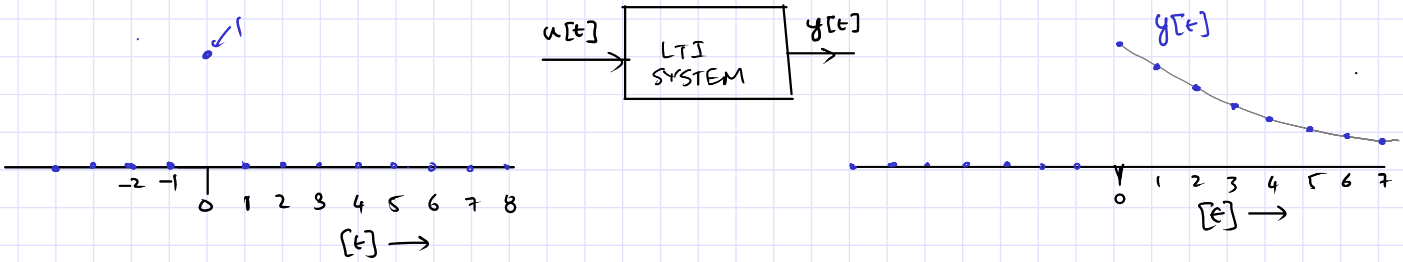
→

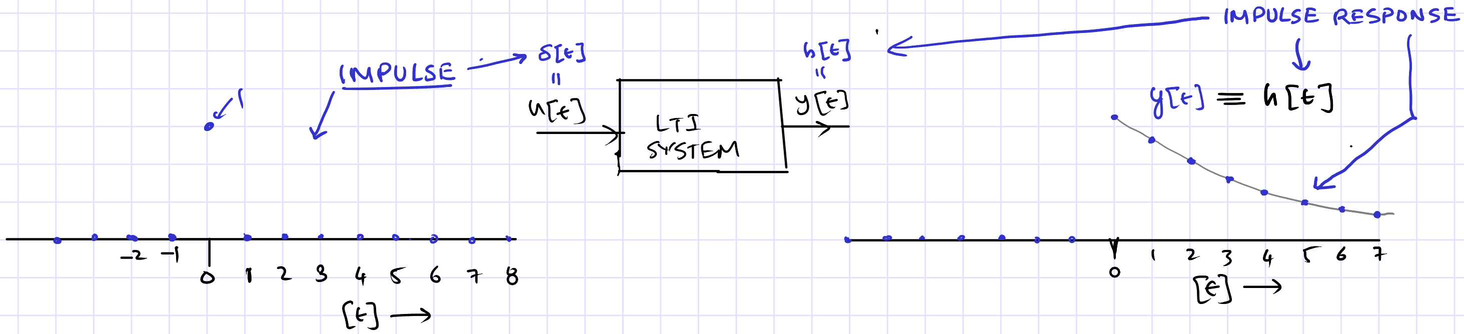


→ AMAZING FACT: IF YOU KNOW AN LTI SYSTEM'S IMPULSE RESPONSE, YOU CAN CALCULATE ITS RESPONSE TO ANY INPUT

→ WHAT IS THE IMPULSE RESPONSE?

(NEXT PAGE)





$$\rightarrow \delta[t] = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

: DISCRETE-TIME IMPULSE (OR DELTA FUNCTION)

CAUSALITY: ANOTHER IMPORTANT SYSTEM PROPERTY

→ **IN WORDS**: THE SYSTEM'S RESPONSE CAN ONLY COME **AFTER** AN INPUT HAS BEEN APPLIED

→ **IN EQNS**:

1) TAKE ANY INPUT $\hat{u}[t]$, and corresponding OUTPUT $\hat{y}[t]$ (i.e., $\hat{u}[t] \mapsto \hat{y}[t]$)

2) TAKE ANY NUMBER τ

3) DEVISE A NEW INPUT $u[t]$ THAT MATCHES $\hat{u}[t]$ UP TO $t = \tau$, BUT IS DIFFERENT FOR $t \geq \tau$.

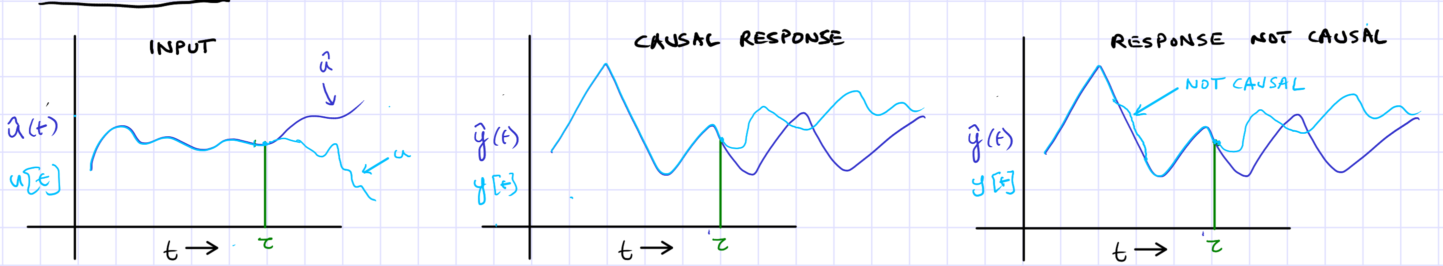
   → $u[t] = \hat{u}[t]$ for $t < \tau$, DIFFERENT THEREAFTER.

4) APPLY $u[t]$ to the system to get $y[t]$: $u[t] \mapsto y[t]$

5) CHECK: IS $y[t] = \hat{y}[t]$ for $t < \tau$?

   → IF YES — FOR **ALL CHOICES** OF $\hat{u}[t]$, τ AND $u[t]$ — THEN THE SYSTEM IS CAUSAL

→ **IN PICTURES**

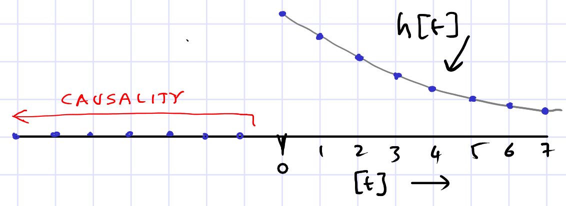


→ CHECKING A SYSTEM FOR CAUSALITY CAN BE TEDIOUS

→ BUT FOR LTI SYSTEMS, IT IS VERY EASY

→ JUST FIND THE IMPULSE RESPONSE $h[t]$.

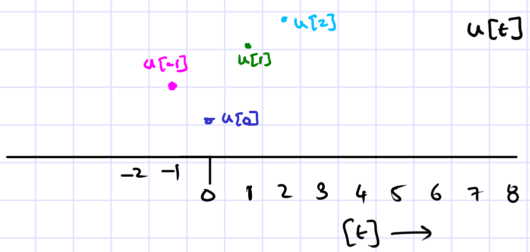
→ $h[t] = 0 \quad \forall t < 0 \iff$ CAUSAL



→ PROOF ? LEFT AS EXERCISE

CLAIM: IF YOU KNOW $h[t]$ FOR AN LTI SYSTEM, YOU CAN CALCULATE $y[t]$ FOR ANY GIVEN INPUT $u[t]$.

→ PROOF: GIVEN ANY $u[t]$, WE CAN WRITE IT AS A SHIFTED AND SCALED SUM OF IMPULSE FUNCTIONS



$$u[t] = \underbrace{u[0]}_{\text{JUST A NUMBER}}\ \underbrace{\delta[t]}_{\text{IMPULSE}} + \underbrace{u[1]}_{\text{JUST A NUMBER}}\ \underbrace{\delta[t-1]}_{\text{SHIFTED IMPULSE (BY 1)}} + u[2]\ \delta[t-2] + u[-1]\ \delta[t+1] + \cdots$$

$$= \sum_{i=-\infty}^{\infty} \underbrace{u[i]}_{\text{JUST NUMBERS (SCALING)}}\ \underbrace{\delta[t-i]}_{\text{SHIFTED IMPULSE FUNCTIONS}}$$

→ RESPONSE OF SYSTEM TO $\delta[t-i]$?

→ BY TIME INVARIANCE: IT IS $h[t-i]$, i.e., $\delta[t-i] \mapsto h[t-i]$

→ RESPONSE TO $u[i]\,\delta[t-i]$?

→ BY SCALING: $u[i]\,\delta[t-i] \mapsto u[i]\,h[t-i]$

→ RESPONSE TO $\sum_{i=-\infty}^{\infty} u[i]\,h[t-i] = u[t]$?

→ BY SUPERPOSITION: $\sum_{i=-\infty}^{\infty} u[i]\,h[t-i] \longmapsto \sum_{i=-\infty}^{\infty} u[i]\,h[t-i]$

→ OR: IF $u[t] \mapsto y[t]$, then $y[t] = \sum_{i=-\infty}^{\infty} u[i]\,h[t-i]$

— IF THE SYSTEM IS CAUSAL, THEN: $h[t] = 0$ if $t < 0$

$$\Longrightarrow \quad y[t] = \sum_{i=-\infty}^{t} u[i]\,h[t-i] \quad = \quad \sum_{j=0}^{\infty} u[t-j]\,h[j]$$

↳ THIS IS A DISCRETE-TIME CONVOLUTION

→ NOTATION: $y[t] = u[t] \circledast h[t]$

— FURTHER: IF $u[t] = 0 \;\; \forall\, t < 0$, then

$$y[t] = \sum_{i=0}^{t} u[i]\,h[t-i] = \sum_{j=0}^{t} u[t-j]\,h[j]$$

→ EXAMPLE: $x[t+1] = a\,x[t] + b\,u[t]$ (with zero initial condition, i.e., $x[0]=0$)

$y[t] = c\,x[t] + d\,u[t]$

→ impulse input: $u[t] = \begin{cases} 1, & t=0 \\ 0, & \text{otherwise} \end{cases}$

$y[0] = d\,u[0] = d$

$x[1] = a\,x[0] + b\,u[0] = b$;	$y[1] = c\,x[1] + d\,u[1] = cb$
$x[2] = a\,x[1] + b\,u[1] = ab$;	$y[2] = cab$
$x[3] = a\,x[2] + b\,u[2] = a^2 b$;	$y[3] = ca^2b$
$\vdots$	$\vdots$
$x[t] = a^{t-1} b$	$y[t] = c\,a^{t-1} b$

→ THUS $h[t] = \begin{cases} d, & \text{if } t=0 \\ c\,a^{t-1} b, & \text{if } t>0 \end{cases}$

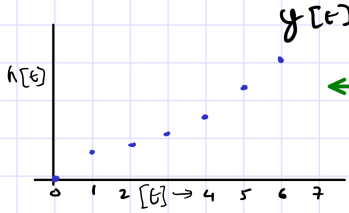
→ COMPOUND INTEREST EXAMPLE (FROM LONG AGO): $S[t+1] = S[t](1 + r/12) + u[t]$

$y[t] = S[t]$

→ $a = (1 + r/12)$, $b = 1$, $c = 1$, $d = 0$

→ $h[t] = (1 + r/12)^{t-1}$, $t > 0$

$= 0$ otherwise.

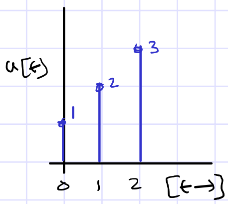


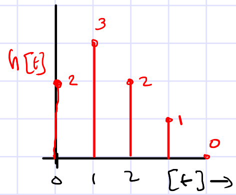
← UNSTABLE (BUT WE DON'T MIND IN THIS CASE)

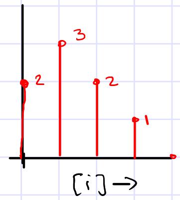
→ ANOTHER QUESTION TO PONDER: CAN WE CHARACTERIZE BIBO STABILITY/INSTABILITY IN TERMS OF $h[t]$ ALONE (WE JUST KNOW $h[t]$ – nothing else).

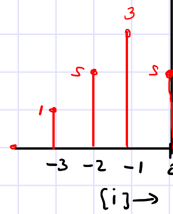
6. GRAPHICAL ILLUSTRATION OF CONVOLUTION

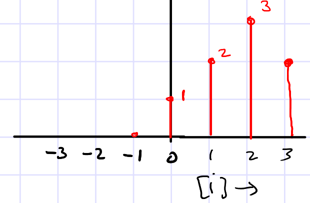
$\rightarrow y[t] = u[t] \circledast h[t] = \sum_{i=0}^{t} u[i]\,h[t-i]$ (assuming causality + $u[t<0]=0$)



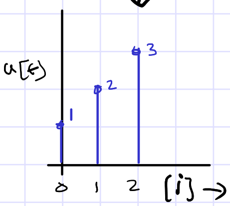




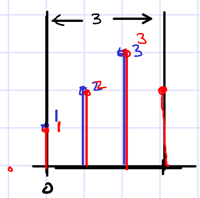




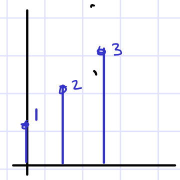
$\rightarrow y[3] = \sum_{i=0}^{3} u[i]\,h[3-i]$

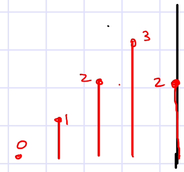


MULTIPY and ADD

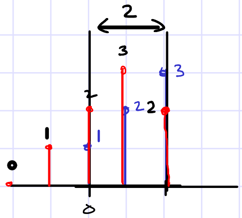


$= 1 + 4 + 9 + 0 = 14$



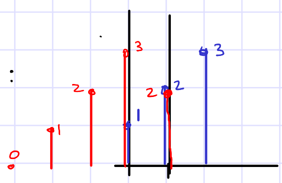


$\rightarrow y[2]:$



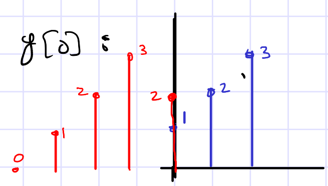
$\rightarrow 2 + 6 + 6 = 14$

$\rightarrow y[1]:$



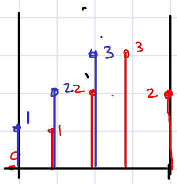
$\rightarrow 4 + 3 = 7$

$\rightarrow y[0]:$



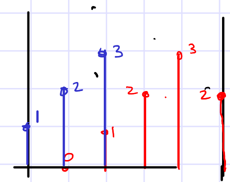
$\rightarrow 2$

$\rightarrow y[4]:$

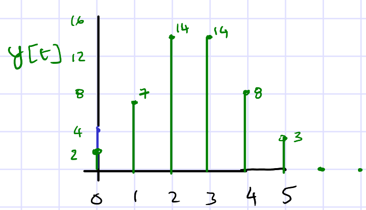


$= 2 + 6 = 8$

$\rightarrow y[5]$



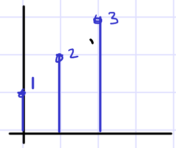
$= 3$



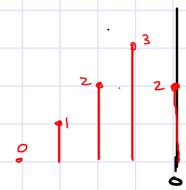
$\rightarrow y[6]$ and above : 0

→ SUMMARY OF GRAPHICAL CONVOLUTION

1. KEEP A COPY OF $u[t]$ HANDY :

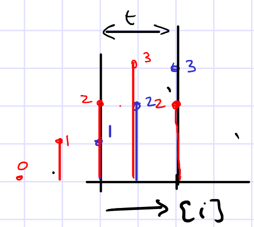


2. MIRROR $h[t]$ AROUND $t=0$ AND KEEP HANDY ($h[-i]$):



3. TO GET $y[t]$:

3a: SHIFT $h[-i]$ TO THE RIGHT BY t and PLACE OVER $u[t]$.

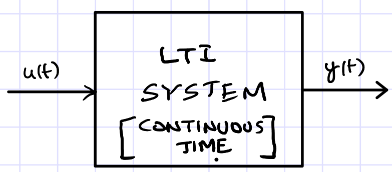


3b: MULTIPLY AND ADD UP TO GET $y[t]$.

4. REPEAT FOR EVERY t.

---

7. IMPULSE RESPONSE AND CONVOLUTION FOR C.T. SYSTEMS



→ Apply Dirac. δ "function" $\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise} \end{cases}$, satisfying $\int_{-\epsilon}^{+\epsilon} \delta(\tau)\,d\tau = 1$, any $\epsilon > 0$

→ Record output $h(t)$: this is the C.T. impulse response.

→ Then, given any $u(t)$, with $u(t) \mapsto y(t)$, $y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau = u(t) \circledast h(t)$

C.T. convolution.

→ Proof: Analogous to D.T. case, using properties of Dirac δ.

→ Causality : $h(\tau) = 0$ for $\tau < 0$ $\Longrightarrow$ $y(t) = \int_{-\infty}^{t} u(\tau)\,h(t-\tau)\,d\tau$