EECS 16B Designing Information Devices and Systems II
Fall 2017 **Note 3**

1 Second Order Differential Equations

Second order differential equations appear everywhere in the real world. In this note, we will walk through how to solve them in order to understand second order circuits.

1.1 Degree of a differential equation

Consider a differential equation of the form,

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t) + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}(t) + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}t}(t) + a_0 y(t) = 0$$

The degree of the above differential equation is n.

1.2 Theorem: Existence and Uniqueness of Solutions to Differential Equations

Given a n^{th} order differential equation and n initial conditions,

$$y(0) = d_0, \tfrac{\mathrm{d}y}{\mathrm{d}t}(t_0) = d_1, \cdots, \tfrac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}(t_0) = d_{n-1} \tag{1}$$

there exists a singe unique solution (say, f). We will not prove this in this class.

1.3 First order differential equations

Consider the following simple equation.

$$\tfrac{\mathrm{d}y}{\mathrm{d}t}(t) = y(t) \text{ with } y(0) = 1 \tag{2}$$

This is our starting point. The solution to (2) will set the first building block to solving second order circuits. *We are looking for the "eigenvector" of the differentiation operator that corresponds to an eigenvalue of* 1. Given the linear nature of the derivative operator, attempting to characterize the eigenvector is a big step towards understanding its nature.

Getting back to the problem at hand, we see that the function $f(t) = e^t$ satisfies the equation in (2) as well as the initial conditions. The Existence and Uniqueness of Solutions to Differential Equations Theorem

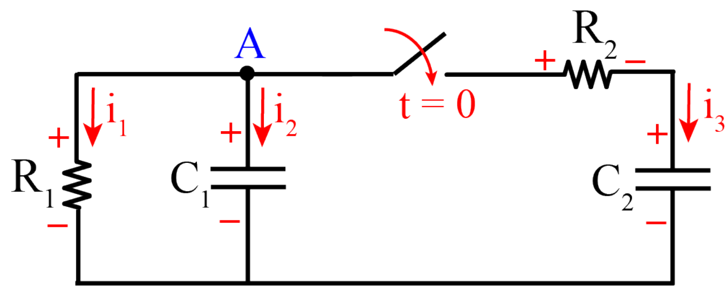

Figure 1: A second order circuit

states that there exists only one solution for a given initial condition, so we do not need to search for more. Furthermore, $f(t) = e^t$ is an eigenvector of the differentiation operator.

What about finding these "*eigenfunctions*" with some eigenvalue λ?

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = \lambda y(t) \text{ with } y(0) = c$$

The solution is,

$$y(t) = ce^{\lambda t}$$

With this, we have all the building blocks we need to solve much more complicated differential equations.

1.4 Example of a second order circuit

Consider a circuit like in Figure 1. Assume the switch is open up to time 0. Let the voltage of Capacitor 2 at time 0 be 0 and let the current at passing through Capacitor 2 at time 0 be 0. This tells us that,

$$V_{C_2}(0) = 0 \text{ and } \frac{\mathrm{d}V_{C_2}}{\mathrm{d}t}(0) = 0 \tag{3}$$

We want to figure out how the voltage across Capacitor 2 evolves after we close the switch at time 0.

To do so, we will apply KCL to Node A. We get that,

$$i_1 + i_2 + i_3 = 0 \tag{4}$$

This tells us that,

$$\frac{V_{R_1}}{R_1} + C_1 \frac{\mathrm{d}V_{C_1}}{\mathrm{d}t} + C_2 \frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} = 0 \tag{5}$$

We have used the capacitor equation. Note the negative signs. This is to take into account the manner in which we have drawn the voltage signs in the diagram. In this case, because of the zero, it does not matter but it is good practice to always keep in mind the direction of voltage drops according to the diagram.

Observe that,

$$V_{R_1} = V_{C_1} \tag{6}$$

This is because we are comparing the same two points across two different paths. Recall that the voltage difference between two points should always be the same regardless of the different paths between the two points. Using a similar argument,

$$V_{C_1} = V_{C_2} + i_3 R_2 \text{ where } i_3 = C_2 \frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} \tag{7}$$

Combining (5), (6) and (7), we get the following.

$$C_1 C_2 R_2 \frac{\mathrm{d}^2 V_{C_2}}{\mathrm{d}t^2} + \left(C_1 + C_2 + C_2 \frac{R_2}{R_1} \right) \frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} + \frac{V_{C_2}}{R_1} = 0 \tag{8}$$

Equations like (8) pop up all the time. We're going to now present a straight forward way of solving such differential equations using linear algebra!

1.5 Solving a general second order differential equation

Consider a general second order differential equation of the form,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t) + a_1 \frac{\mathrm{d}y}{\mathrm{d}t}(t) + a_0 y(t) = 0 \tag{9}$$

This can be written in matrix-vector form as,

$$\underbrace{\begin{bmatrix} \frac{\mathrm{d}y}{\mathrm{d}t}(t) \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t) \end{bmatrix}}_{\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(t) \\ \frac{\mathrm{d}y}{\mathrm{d}t}(t) \end{bmatrix}}_{\vec{x}} \tag{10}$$

Observe that $\vec{x}$ is a $2D$ vector consisting of $x(t)$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ as coordinates. (10) is written in short form as,

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} \tag{11}$$

We recast (9) to (11) in order to exploit the matrix structure of A and the linear properties of the differentiation operator $\frac{\mathrm{d}(\cdot)}{\mathrm{d}t}$. Particular, we are going to *diagonalize* A to exploit what we know about first order differential equations.

Let P be a matrix whose columns consist of eigenvectors of A. Let D be a diagonal matrix consisting of the eigenvalues (λ_1, λ_2) of A and let P^{-1} be the inverse of matrix P. We know that,

$$A = PDP^{-1}$$

Applying this to (11), we get,

$$P^{-1} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = DP^{-1}\vec{x} \tag{12}$$

Recall that multiplying vector $\vec{x}$ by P^{-1} finds the coordinates of $\vec{x}$ with respect to the columns of P. Define $\vec{z}$ as follows.

$$\vec{z} = P^{-1}\vec{x}$$

Then, we have that,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= A\vec{x}(t) \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= PDP^{-1}\vec{x}(t) \\
P^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= DP^{-1}\vec{x}(t) \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\left(P^{-1}\vec{x}\right)(t) &= DP^{-1}\vec{x}(t) \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\vec{z}\right)(t) &= D\vec{z}(t)
\end{aligned}$$

Note that we use the fact that A is independent of the time t and that P^{-1} and $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$ are linear functions on $\vec{x}(t)$ to conclude that,

$$P^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(P^{-1}\vec{x}\right)(t)$$

This transformation allows us to reach the following equation.

$$\frac{\mathrm{d}z_1}{\mathrm{d}t}(t) = \lambda_1 z_1(t) \text{ and } \frac{\mathrm{d}z_2}{\mathrm{d}t}(t) = \lambda_2 z_2(t)$$

Recall that λ_1 and λ_2 are the eigenvalues of A. *We have reduced a second order equation into two smaller first order equations that we know how to solve.* The closed form values of the eigenvalues are,

$$\lambda_1 = \frac{1}{2}\left(-a_1 - \sqrt{a_1^2 - 4a_0}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(-a_1 + \sqrt{a_1^2 - 4a_0}\right)$$

This tells us that,

$$z_1(t) = c_1 e^{\lambda_1 t}, z_2(t) = c_2 e^{\lambda_2 t} \tag{13}$$

After we solve of $\vec{z}$, we can get back $\vec{x}$ be observing that,

$$\vec{x} = P\vec{z}$$

Side note. We observe a lot of shared structure between λ_1 and λ_2. Particularly, if $a_1^2 - 4a_0 < 0$, we observe that the eigenvalues are complex and are in fact conjugates of each other.

$$\lambda_1 = \overline{\lambda_2}$$

1.6 A shortcut to solving second order differential equations

As you can imagine, these steps can get tedious sometimes. We present to you a shortcut to solving such equations that hinges on the fundamental theorem of differential equations to avoid using the matrices P and P^{-1}.

1. Cast a given differential equation into matrix form (11).
2. Find the eigenvalues of A. Let this be λ_1 and λ_2.
3. (a) If λ_1 and λ_2 are complex, they will be complex conjugates of each other. The solution will be,

   $$x(t) = c_1 e^{\sigma t}\cos(\omega t) + c_2 e^{\sigma t}\sin(\omega t)$$

   where,

   $$\lambda_1 = \sigma + j\omega, \lambda_2 = \sigma - j\omega$$

   (b) If λ_1 and λ_2 are real and distinct, the solution will be,

   $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

   This is similar to (13)
4. Use the initial conditions to solve for c_1 and c_2.

You might be wondering what happens when $\lambda = \lambda_1 = \lambda_2$. The solution is of the form,

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

The reason why is out of scope for this course as it requires the concept of Generalized Eigenvectors. You can verify that the above is indeed a solution using the fundamental theorem of solutions to differential equations.

1.7 Solving the second order circuit

Now that we have the theory, we can solve (8). We can recast this equation to,

$$\frac{\mathrm{d}^2 V_{C_2}}{\mathrm{d}t^2} + \left(\frac{C_1 + C_2 + C_2\frac{R_2}{R_1}}{C_1 C_2 R_2}\right)\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} + \frac{V_{C_2}}{C_1 C_2 R_1 R_2} = 0 \tag{14}$$

We get that,

$$a_1 = \left(\frac{C_1 + C_2 + C_2\frac{R_2}{R_1}}{C_1 C_2 R_2}\right) \text{ and } a_0 = \frac{1}{C_1 C_2 R_1 R_2}$$

Let us simplify things and plug in values. Let $C_1 = C_2 = 100\mu F, R_1 = R_2 = 1k\Omega$. Then,

$$a_1 = 30, a_0 = 100$$

This tells us that,

$$\lambda_1 \approx -26, \lambda_2 \approx -4$$

We know that,

$$V_{C_2}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

and,

$$\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t}(t) = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$$

We can now use the initial conditions in (3) to solve for c_1 and c_2.