

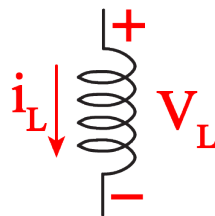
1 Introduction

We now know about resistor and capacitors. We know capacitance introduces “time” into our circuits. Why?

- The I-V relationship of capacitors ($i = Cdv/dt$) has the concept of time in it.
- Intuitively, capacitors store energy and any stored energy cannot be moved around infinitely fast. Doing so would require infinite power, since $power = d(Energy)/dt$.

2 Inductors

Inductors have a dual relationship in terms of voltage and current (I-V) as compared to capacitors (i.e., V being proportional to change in I as opposed to I being proportional to change in V). The schematic symbol of an inductor is drawn below:

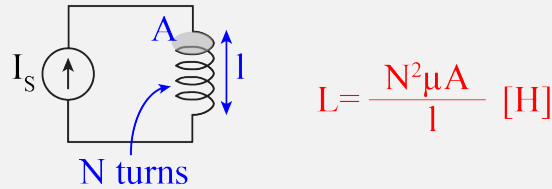


1. The unit of inductance is Henry (H).
2. $v_L(t) = L \frac{di_L(t)}{dt}$
3. $i_L(t)$ cannot change instantly (why?).
4. Inductors look like short circuits at DC (makes sense since they are wires!)
5. The energy stored in an inductor is: $U = \frac{1}{2}Li_L^2(t)$

While inductors are introduced in this course only as a circuit symbol and a mathematical construct, we are not spending time on the physics behind them, they have multiple applications in the real world and you will study them in future courses.

Inductors store energy by setting up a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an \vec{E} field, anytime we flow current down a conductor, this creates a magnetic field (\vec{B}). Likewise, the magnetic field can store energy. Their behavior can be described using Faraday's law of induction.

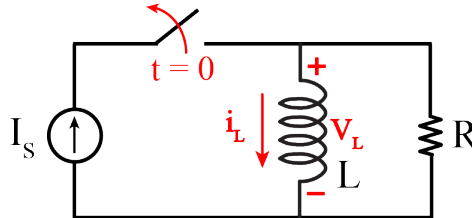
The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries if we are trying to create a useful inductance on purpose. A solenoid is a good example:



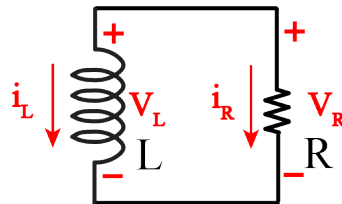
Note that the inductance (L) depends on geometry and a material property called permeability of the solenoid core material. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc.

3 RL Circuits

In the same way that one R and one C in a circuit can lead to a differential equation (DE) and a solution with an $e^{-t/RC}$ term in it, an RL circuit results in an DE and $e^{-Rt/L}$ terms. We will sketch an example below:



Note i_L for $t < 0$ is I_s because an inductor at DC looks like a short. For $t > 0$, the circuit looks like:



Let's solve this circuit for $i_L(t)$ for $t > 0$ by writing a KVL equation:

$$v_L = v_R = i_R R = -i_L R \Rightarrow v_L + i_L R = 0 \Rightarrow L \frac{di_L}{dt} + i_L R = 0 \Rightarrow \frac{di_L}{dt} + \frac{R}{L} i_L = 0 \quad (1)$$

This equation should be familiar to you! Correct! this is a 1st order DE ($\dot{x} + ax = 0$ with $a = R/L$). The solution is then:

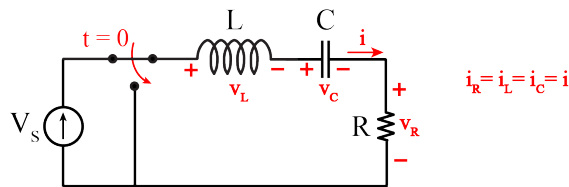
$$i_L(t) = i_L(0)e^{-\frac{R}{L}t} \quad (2)$$

And since $i_L(0) = I_S$, thus:

$$i_L(t) = I_S e^{-\frac{R}{L}t} \quad (3)$$

4 RLC Circuits

We could spend some time solving RL circuits same as RC circuits and they are useful (they come up a lot for motors and power generation machinery. Also, RL circuits are a good way to generate giant voltages to cause air breakdown and sparks (Look up how spark plugs work!). Here, however, we will focus on what happens when we put R,L, and C together! Consider the following circuit:



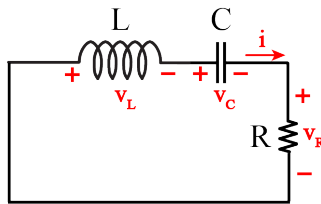
For $t < 0$, L looks like a short and C looks like an open so $i = 0$ and by KVL:

$$v_C(t < 0) = V_S$$

$$v_L(t < 0) = 0 \quad (\text{Short})$$

$$v_R(t < 0) = 0 \quad (\text{Since } i_R = 0)$$

Let's solve for $v_C(t)$ for $t > 0$ by writing the KVL and using I-V characteristics of R,L, and C:



$$v_L + v_C + v_R = 0$$

$$L \frac{di}{dt} + v_C + iR = 0$$

$$L \frac{d}{dt} \left(C \frac{dv_C}{dt} \right) + v_C + C \frac{dv_C}{dt} R = 0$$

$$LC \frac{d^2 v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = 0$$

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = 0 \quad (4)$$

Now we have a differential equation with a second derivative! As we saw previously, this is a 2nd order DE. Because R.H.S=0, it is called homogeneous. A general form is $\ddot{v} + a\dot{v} + bv = 0$. Here $a = R/L$ and $b = 1/LC$.

5 2nd Order D.E. Solution

Last lecture we taught you a method for solving 2nd order D.E.s using a very elegant technique that allows you to rewrite a 2nd order ODE as a system of coupled 1st order ODE's, which we can solve.

For now, we will just observe that the solution of $\ddot{v} + a\dot{v} + bv = 0$ can be written as:

$$v(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \quad (5)$$

where λ_1 and λ_2 are: $\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ with $\alpha = \frac{R}{2L}$ and $\omega_0 = \sqrt{\frac{1}{LC}}$. These two parameters depend only on circuit elements (RLC). K_1 and K_2 are constants.

In general, there are 3 possibilities for what type of number λ_1 and λ_2 can be.

1. If $\alpha > \omega_0$, then the $\alpha^2 - \omega_0^2$ term is positive and less than α , so λ_1 and λ_2 are both negative real numbers. So the solution is just the sum of two decaying exponentials, $v(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}$.
2. If $\alpha < \omega_0$, then the $\alpha^2 - \omega_0^2$ term is the square root of a negative number thus:

$$\begin{aligned} \lambda_1 &= -\alpha + j\omega_D \\ \lambda_2 &= -\alpha - j\omega_D \end{aligned}$$

where $\omega_D = \sqrt{\omega_0^2 - \alpha^2}$. What does this look like?:

$$\begin{aligned} v(t) &= K_1 e^{(-\alpha + j\omega_D)t} + K_2 e^{(-\alpha - j\omega_D)t} \\ &= K_1 e^{-\alpha t} e^{j\omega_D t} + K_2 e^{-\alpha t} e^{-j\omega_D t} \\ &= e^{-\alpha t} (K_1 e^{j\omega_D t} + K_2 e^{-j\omega_D t}) \end{aligned}$$

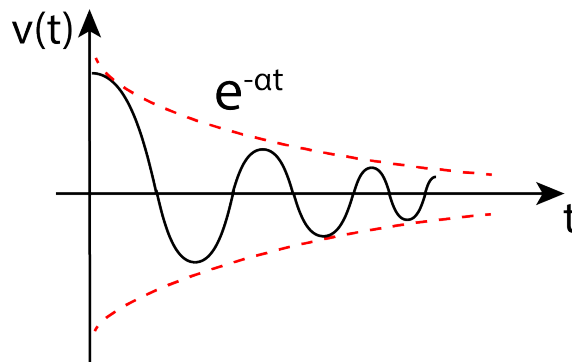
By Euler's formula:

$$\begin{aligned} e^{j\omega_D t} &= \cos(\omega_D t) + j\sin(\omega_D t) \\ e^{-j\omega_D t} &= \cos(\omega_D t) - j\sin(\omega_D t) \end{aligned}$$

Thus $v(t)$ can be reformulated as:

$$\begin{aligned} v(t) &= e^{-\alpha t} (K_1 \cos(\omega_D t) + jK_1 \sin(\omega_D t) + K_2 \cos(\omega_D t) - jK_2 \sin(\omega_D t)) \\ &= e^{-\alpha t} [(K_1 + K_2) \cos(\omega_D t) + j(K_1 - K_2) \sin(\omega_D t)] \end{aligned}$$

This looks like sinusoids with a frequency ω_D slowly decaying as they are multiplied by $e^{-\alpha t}$.



3. If $\alpha = \omega_0$, you get a special case. Then $\lambda_1 = \lambda_2 = \lambda$ and $v(t) = K_1 e^{\lambda t} + K_2 t e^{\lambda t}$.

In summary there are 3 cases:

1. **Over-damped ($\alpha > \omega_0$): Decays, no oscillation**
2. **Under-damped ($\alpha < \omega_0$): Decays with oscillation**
3. **Critically-damped ($\alpha = \omega_0$): Decays, no oscillation (this is the boundary)**

Note that if $R = 0$, i.e. there is no damping, then $\lambda = \pm j\omega_0$ and there is no decaying exponential. That is, the circuit only oscillates at ω_0 in this case and then it does so forever. In all other under-damped cases, the circuit oscillates at ω_D as it decays to 0.

Example: Consider the series RLC above with the following values. Find $v_C(t)$ and $i(t)$.

$$V_S = 16V, \quad R = 64\Omega, \quad L = 0.8H, \quad C = 2mF$$

Solution:

$$\alpha = \frac{R}{2L} = \frac{64}{2 \times 0.8} = 40 \quad [Nepers/s]$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.8 \times 2 \times 10^{-3}}} = 25 \quad [rad/s]$$

$\alpha > \omega_0$ and the circuit is overdamped.

$$\lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -40 + \sqrt{40^2 - 25^2} = -8.8 \quad [Np/s]$$

$$\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -40 - \sqrt{40^2 - 25^2} = -71.2 \quad [Np/s]$$

$$\Rightarrow v(t) = K_1 e^{-8.8t} + K_2 e^{-71.2t}$$

Now we only need to find K_1 and K_2 . For this, we will use the initial conditions. Since we know $v_C(t = 0) = V_S = 16V$, thus:

$$v_C(0) = K_1 + K_2 = 16$$

We need one more equation though; we know $i(t=0) = 0$ and also $i(t) = C \frac{dv_C}{dt}$, hence:

$$i(t) = 2 \times 10^{-3} \times (-8.8K_1 e^{-8.8t} + -71.2K_2 e^{-71.2t})$$

$$i(0) = 2 \times 10^{-3} \times (-8.8K_1 + -71.2K_2) = 0$$

Solving these two equations for the values of K s, we obtain: $K_1 = 18.2564$ and $K_2 = -2.25641$. Consequently, $v_C(t)$ and $i(t)$ can be expressed as:

$$v_C(t) = 18.2564e^{-8.8t} - 2.25641e^{-71.2t} \quad [V]$$

$$i(t) = 2 \times 10^{-3} \times (-8.8 \times 18.2564e^{-8.8t} + 71.2 \times 2.25641e^{-71.2t}) \quad [A]$$