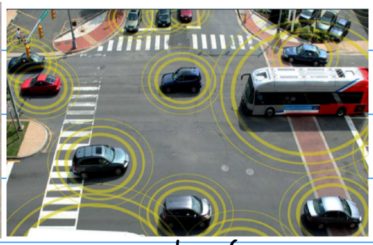
WEEK 4 LEC 1: STATE SPACE REPRESENTATIONS

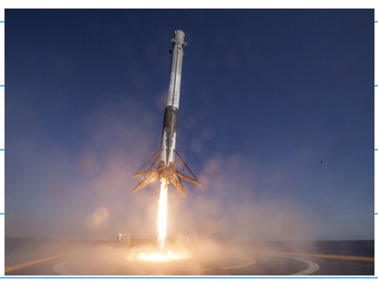
- So far: circuits
- Now: SYSTEMS – a broader concept.

UAVs

Autonomous vehicles



SPACEX

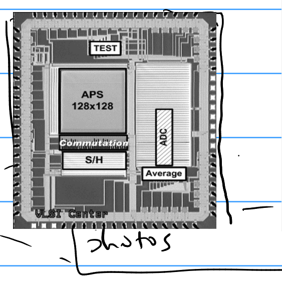


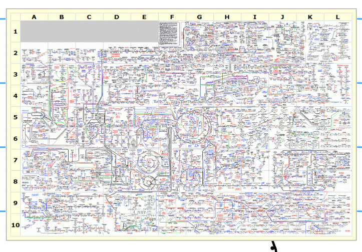
Electrical/mechanical/aero/chemical/optical

"intelligence"/ control/ computing/ communication → sensors/ actuators/ environment

motor

→

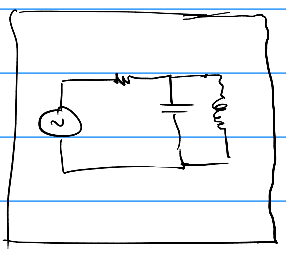




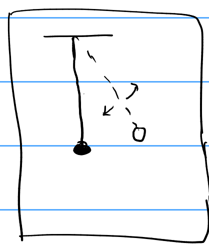
photos

- Electrical (+ some micromechanical + electrochemical (batteries))
- huge sizes (billions of transistors)

→ small, single domain (easier to analyse and understand at first)



little circuits



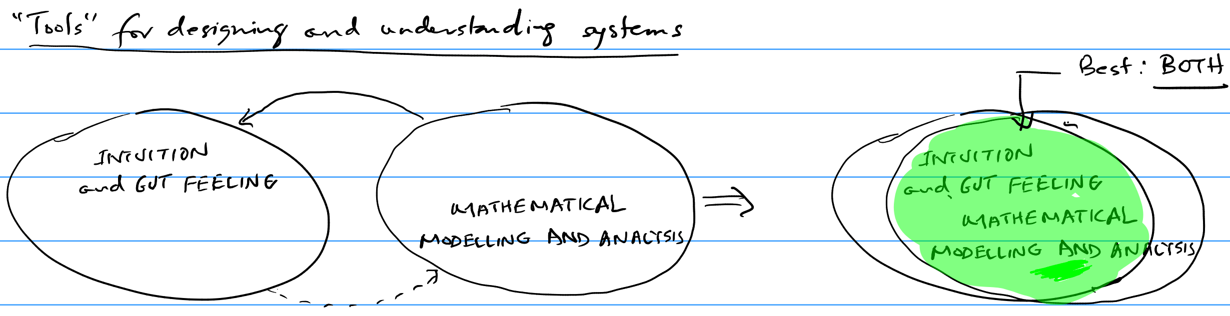
mechanical

$4H + O_2 \rightleftarrows H_2O$

chemical

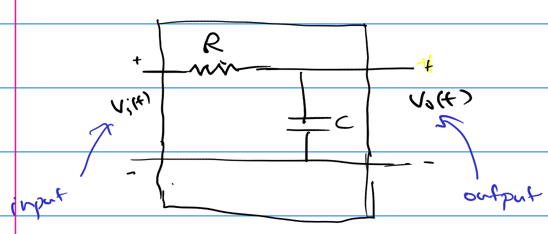
CAN DO INTERESTING/COMPLICATED THINGS

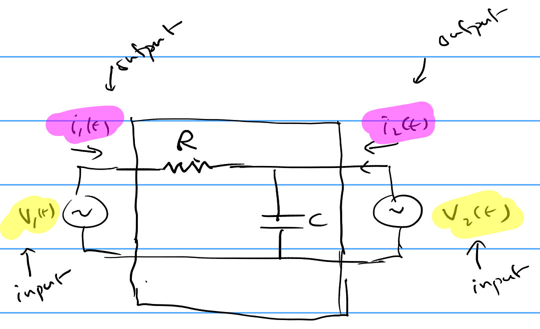
→ "Tools" for designing and understanding systems



NOT INDEPENDENT OF EACH OTHER!

- INPUTS AND OUTPUTS



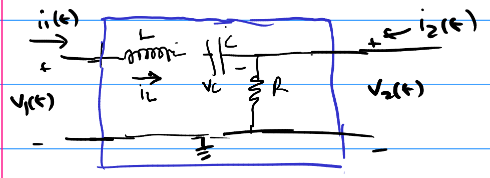


$\vec{u}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$; $\vec{y}(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$

vector of inputs — vector of outputs

$\boxed{\text{INPUTS AND OUTPUTS CAN BE ORGANIZED AS VECTORS}}$

- INTERNAL UNKNOWNS or STATE



$\vec{u}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$; $\vec{y}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}$

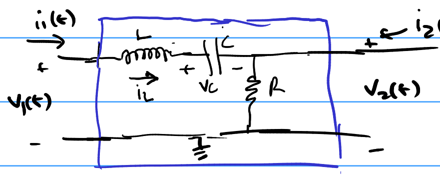
capacitor: $C\frac{dv_C(t)}{dt} - i_L(t) = 0$

inductor + KVL: $L\frac{di_L(t)}{dt} + v_C(t) + R(i_L(t) + i_2(t)) - v_1(t) = 0$

- write the internal unknowns also as a vector:

$\vec{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ ← "state"

- WRITING THE SYSTEM'S EQUATIONS USING $\vec{x}, \vec{u}$ and $\vec{y}$



$$\begin{bmatrix} C & \\ & L \end{bmatrix} \underbrace{\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\frac{d\vec{x}}{dt}} + \begin{bmatrix} & -1 \\ 1 & R \end{bmatrix} \underbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \begin{bmatrix} & \\ -1 & R \end{bmatrix} \underbrace{\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}}_{\vec{u}(t)} = \vec{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} C & \\ & L \end{bmatrix}^{-1} \left(\begin{bmatrix} & +1 \\ -1 & -R \end{bmatrix} \overbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}^{\vec{x}(t)} + \begin{bmatrix} & \\ 1 & -R \end{bmatrix} \overbrace{\begin{bmatrix} v_1(t) \\ i_L(t) \end{bmatrix}}^{\vec{u}(t)} \right)$$

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 1/C & \\ & 1/L \end{bmatrix} \left(\begin{bmatrix} & 1 \\ -1 & -R \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} & \\ 1 & -R \end{bmatrix} \begin{bmatrix} v_1(t) \\ \end{bmatrix} \right)$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} & \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}}$$

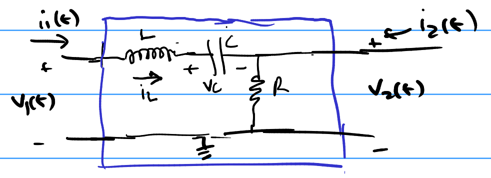
STATE SPACE FORMULATION

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$$

$$\vec{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}; \quad \vec{u}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\vec{f}(\vec{x}(t), \vec{u}(t)) \equiv \begin{bmatrix} & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} & \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

– what about the outputs $\vec{y}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}$?



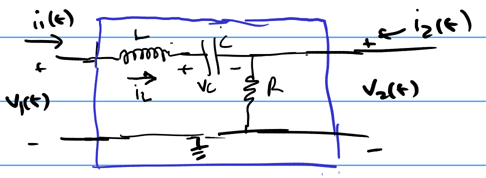
$$i_1(t) = i_L(t); \quad v_2(t) = \frac{i_L(t) + i_2(t)}{R}$$

$$\boxed{\vec{y}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/R \\ 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}^{\vec{x}} + \begin{bmatrix} 0 & 1/R \\ 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}}^{\vec{u}}}$$

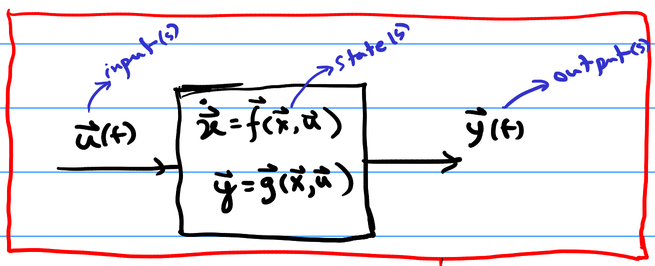
$$\boxed{\vec{y}(t) = \vec{g}(\vec{x}(t), \vec{u}(t))}$$

→ Recap: what have we just done?

– We have shown that this circuit



can be expressed in system form as:



→ Also: an initial condition (or an equivalent condition)

– why is this important or useful?

- – not just this ckt, but ANY ckt, however complicated, can be written in the SAME form
  - – $\vec{u}, \vec{x}, \vec{y}, \vec{f}(\cdot), \vec{g}(\cdot)$ will be different
- – not just circuits, but mech., chemical, etc., systems can also be written in the same form
- – multi-domain systems, too – eg, electromechanical.

→ SINGLE UNIFIED MATHEMATICAL FORM!

ANY KIND OF SYSTEM (EE, mech., chemical, etc.) → STATE SPACE REPRESENTATION ← POWERFUL ANALYSIS METHODS

POWERFUL ANALYSIS METHODS:
- PHASORS
- LINEARIZATION
- STABILITY
- CONTROLLABILITY & OBSERVABILITY
- NONLINEAR ANALYSIS METHODS
- COMPUTATIONAL METHODS

Some fine print:
– a more general state space representation is:

$$\frac{d}{dt}\vec{q}(\vec{x}(t)) = \vec{f}(\vec{x}(t), \vec{u}(t))$$

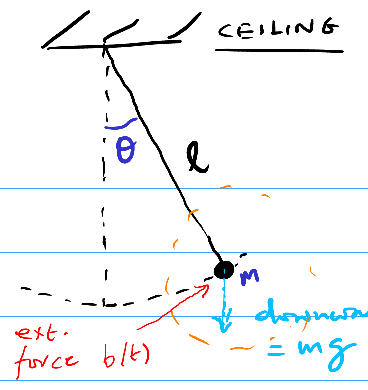
called a DAE (Differential-Algebraic Eqn)

[WON'T BE NEEDED IN THIS CLASS]

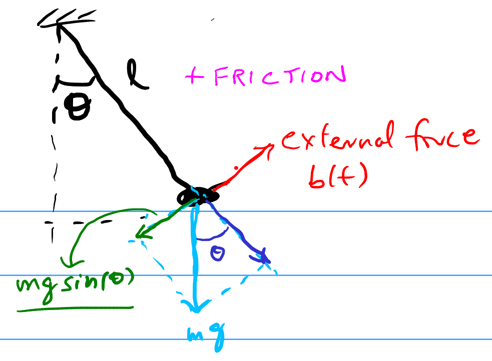
– NOTE: If the output $\vec{y}(t)$ is not explicitly specified, then it is the entire state vector, i.e., $\vec{y}(t) = \vec{x}(t)$.

– A MECHANICAL SYSTEM

— PENDULUM



tangential force = mg sin(θ)



→ Newton's eqn. of motion: $F = ma$ or $a = \frac{F}{m}$

→ total tangential force = (P1)

force due to gravity : $-mg\sin(\theta)$

\+ force " " friction : $-k \cdot$ velocity

\+ externally applied force : $b(t)$

→ arc-length (from bottom) $= y = l\theta$ ← θ in radians

→ velocity $= \frac{dy}{dt} = l \underbrace{d\theta/dt}_{v_\theta}$

→ acceleration $= \frac{d^2y}{dt^2} = l\frac{d^2\theta}{dt^2} = l\frac{dv_\theta}{dt}$

→ total force : $-mg\sin(\theta) - kl\frac{d\theta}{dt} + b(t)$

→ $a = F/m \implies l\frac{d^2\theta}{dt^2} = -g\sin(\theta) - \frac{kl}{m}\frac{d\theta}{dt} + \frac{b(t)}{m}$

$$\implies \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\frac{d\theta}{dt} + \frac{b(t)}{ml}} \quad \text{(P1)}$$

→ STATE SPACE FORM

— Recall : $v_\theta \stackrel{\Delta}{=} \frac{d\theta}{dt}$ (angular velocity) (P2)

— (P1) becomes: $\frac{dv_\theta}{dt} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\frac{d\theta}{dt} + \frac{b(t)}{ml}$

— (P1) & (P2) together are the state space equations:

$$\left.\begin{aligned} \frac{dv_\theta}{dt} &= -\frac{g}{l}\sin(\theta) - \frac{k}{m}v_\theta + \frac{b(t)}{ml} \\ \frac{d\theta}{dt} &= v_\theta \end{aligned}\right\} \quad \text{(P3)}$$

— write in state space form : $\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}$, $\vec{u} = [b(t)]$

$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -g/l\sin(\theta) - k/m\, v_\theta + \frac{b(t)}{ml} \end{bmatrix}} \quad \text{(P4)}$$

nonlinear
(because sin(θ) is nonlinear)
(we'll see later)

— if θ is small (check in MATLAB)

→ $\sin(\theta) \simeq \theta$ (in radians)

→ example of linearization

— then (P4) becomes:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/l & -\frac{k}{m} \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix}\vec{u}(t)$$

DOES IT LOOK FAMILIAR?

DISCRETE-TIME SYSTEMS (+ STATE SPACE REPRESENTATIONS)

→ Discrete-time systems

— Example : compound interest

— Principal P

— Annual rate of interest r, compounded monthly ← DISCRETE (integer, not real number)

— Savings $S[t] = S[t-1] + \frac{r}{12}S[t-1]$ ($t = 1, 2, 3, \ldots$)

with $S[0] = P$ ← INITIAL CONDITION

— Additions/withdrawals each month : $u[t]$

— $S[t] = S[t-1] + \frac{r}{12}S[t-1] + u[t]$ ($t = 1,2,3$)

— with $S[0] = P$

DISCRETE TIME

— general form $\boxed{\vec{x}[t+1] = \vec{f}(\vec{x}[t], \vec{u}[t]), \; t = 1, 2, 3, \dots \quad \text{with I.C.: } \vec{x}[0] \text{ given.}}$

STATE SPACE REPRESENTATION (DISCRETE)

— Another example : Professors and Graduates

— $p[t]$: no. of profs. in the US, year t $(t = 1, 2, 3, \dots)$

— $r[t]$: no. of PhDs in year t

— γ : fraction of PhDs who become professors

— δ : fraction in each profession retiring

— $u[t]$: average number of PhD students graduated per prof. per year

can be manipulated by the professor (controlled by, eg, funding)

— Q: how do $p(t)$ and $r(t)$ evolve with time?

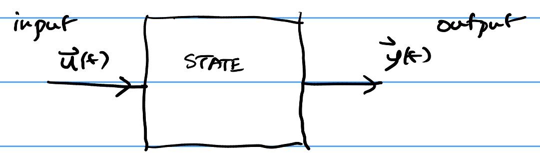
— $p[t+1] = p[t] - \delta p[t] + \gamma r[t]$

— $r[t+1] = r[t] - \delta r[t] - \gamma r[t] + p[t] u[t]$

— State space repr. ?

$$\vec{x}(t) \triangleq \begin{bmatrix} p[t] \\ r[t] \end{bmatrix}; \quad \vec{f}(\vec{x}) = \begin{bmatrix} p(1-\delta) + \gamma r \\ r(1-\delta-\gamma) + pu \end{bmatrix}$$

LINEARITY : A SYSTEMS CONCEPT



⓪ Clearly identify the input & output

① Superposition : if $\vec{u}_1(t) \mapsto \vec{y}_1(t)$, and $\vec{u}_2(t) \mapsto \vec{y}_2(t)$

then $(u_1(t) + u_2(t)) \mapsto (\vec{y}_1(t) + \vec{y}_2(t))$ $\quad \forall\, u_1(t) \;\&\; u_2(t)$

② Scaling : $(\alpha \vec{u}(t)) \mapsto (\alpha \vec{y}(t))$ $\quad \forall\, u(t), \; \forall\, \alpha \in \mathbb{R}$

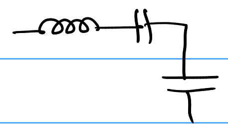
WHEN APPLIED TO STATE SPACE FORMULATIONS

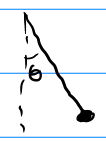
Ⓐ CONTINUOUS : $\dfrac{d\vec{x}}{dt} = \vec{f}(\vec{x}, \vec{u})$ MUST ITSELF BE LINEAR IN x & u

$\boxed{\vec{f}(\vec{x}, \vec{u}) \equiv A\vec{x} + B\vec{u}}$ THIS FORM IS LINEAR (assuming $\vec{y}(t) \equiv \vec{x}(t)$)

Ⓑ DISCRETE : $\vec{x}[t+1] = \vec{f}(\vec{x}[t], \vec{u}[t])$

EXAMPLES : ARE THEY LINEAR?

— 

— 

— Compound interest : $S_{t+1} = (1 + r/12) S_t + u_t$

— professors and PhDs : $\vec{x} = \begin{bmatrix} p(t) \\ r(t) \end{bmatrix}; \quad \vec{f}(\vec{x}) = \begin{bmatrix} p(1-\delta) + \gamma r \\ r(1-\delta-\gamma) + pu \end{bmatrix}$