EE16B LECS 4B & 5A: LINEARIZATION & STABILITY

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a.k.a operating point

→ LINEARIZATION OF A NONLINEAR SYSTEM AROUND EQUILIBRIUM

— (Why? powerful linear analysis techniques – eg, stability)

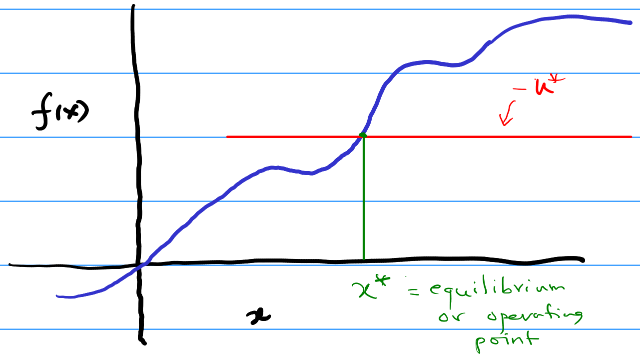
— consider a scalar system: $\frac{dx}{dt} = f(x(t)) + u(t)$

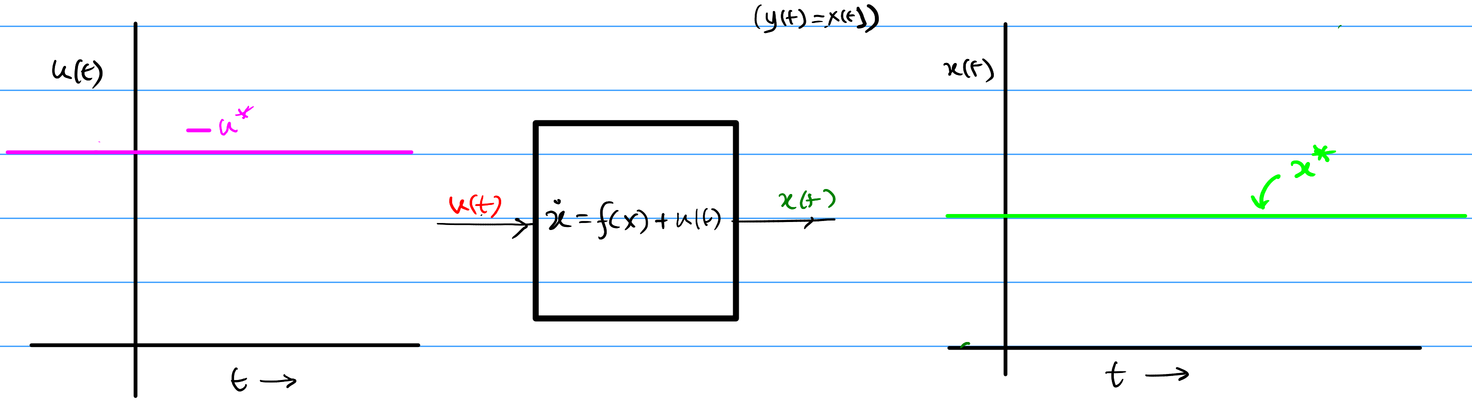
no time variation

— Step 1: Given a DC input $u(t) \equiv u$,

solve the system for a DC solution.

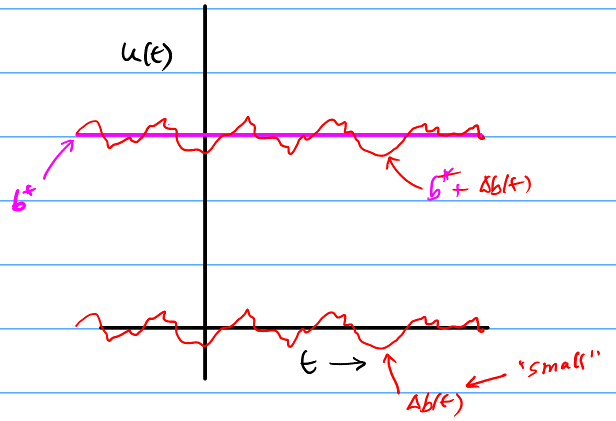
— $0 = f(x^*) + u^* \Rightarrow f(x^*) = -u^*$

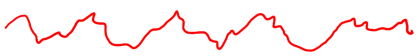




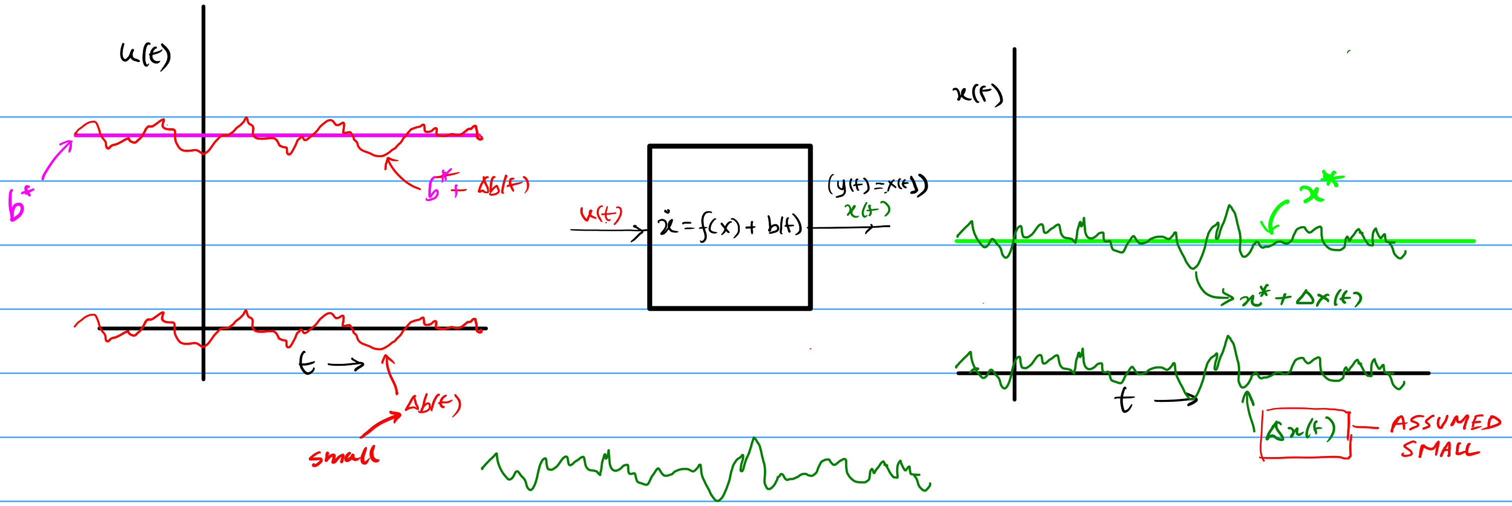
— Step 2: Perturb the input a little

$b(t) = b^* + \Delta b(t)$ ← "small"

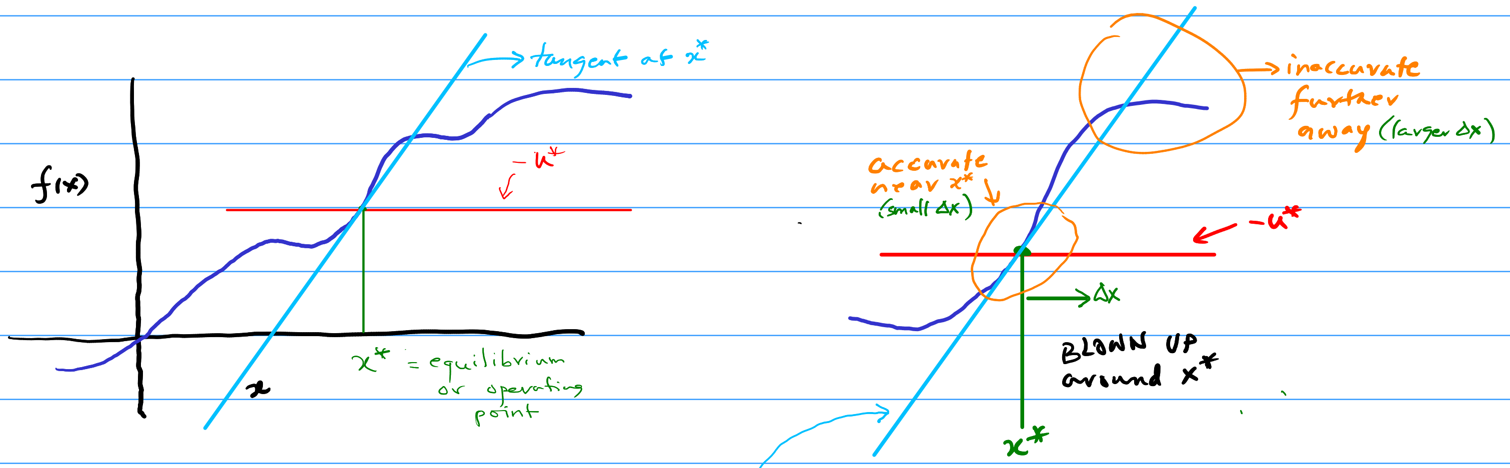




— Step 3: ASSUME $x(t) = x^* + \Delta x(t)$, with $\Delta x(t)$ also "small" ← ASSUMPTION!



– **Step 4:** Approximate f(x) by its tangent at $x = x^*$



KEEP ONLY THIS

– **Mathematically:**

$$\boxed{f(x^* + \Delta x) \simeq \boxed{f(x^*) + \left.\frac{df}{dx}\right|_{x^*} \Delta x} + \frac{1}{2}\left.\frac{d^2 f}{df^2}\right|_{x^*} \Delta x^2 + \frac{1}{2\cdot 3}\frac{d^3 f}{dx_2}\Delta_x^3 + \dots}$$

Taylor series expansion

– **Step 5:** use the approximation

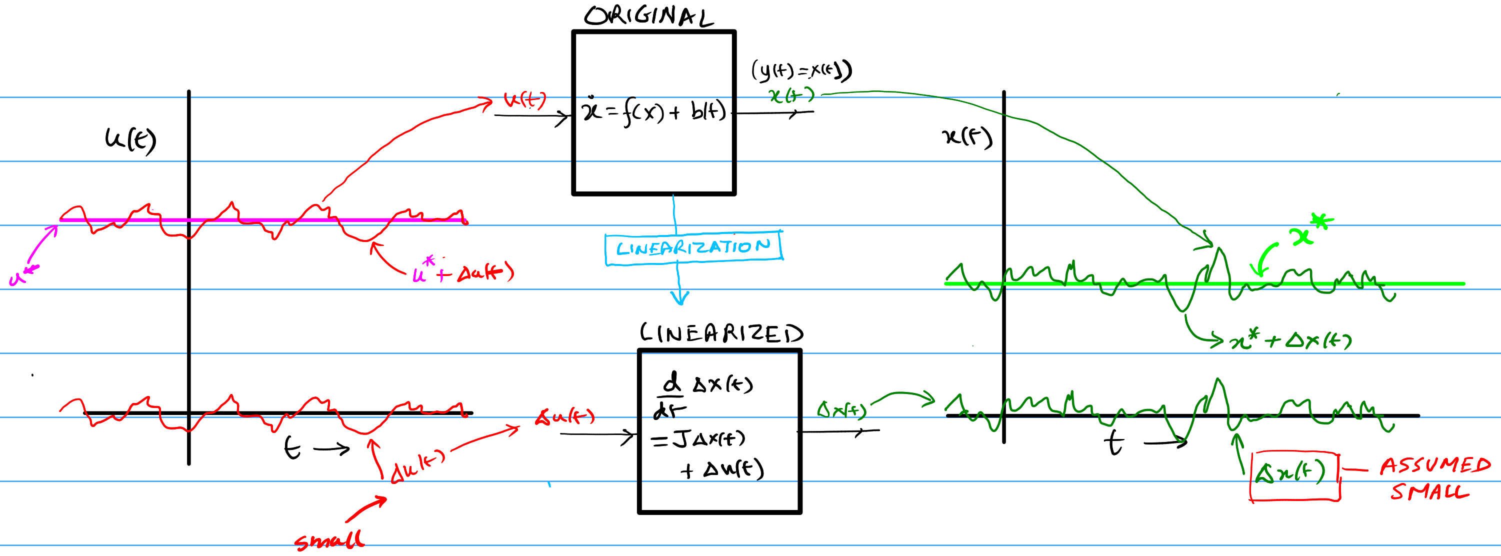
$\rightarrow f(x^*) + u^* = 0$

$$\frac{dx}{dt} = f(x(t)) + u(t), \quad \text{with 1) } u = u^* + \Delta u(t) \text{ and 2) } x(t) = x^* + \Delta x(t)$$

call this J

$$\frac{d}{dt}(x^* + \Delta x(t)) \simeq \cancel{f(x^*)} + \left.\frac{df}{dx}\right|_{x^*} \Delta x(t) + \cancel{u^*} + \Delta u(t)$$

$$\boxed{\frac{d}{dt}\Delta x(t) \simeq J\Delta x(t) + \Delta u(t)} \leftarrow \text{LINEARIZED SYSTEM}$$



— LINEARIZING A VECTOR STATE-SPACE REPRESENTATION

size n vectors

size m vector

— $\frac{d\vec x}{dt} = \vec f(\vec x(t), \vec u(t))$

— Equilibrium (operating) pt: DC input $\vec u^*$, assumed DC state $\vec x^*$

⇒ $\boxed{0 = \vec f(\vec x^*, \vec u^*)}$ → MUST SOLVE FOR $\vec x^*$, given $\vec u^*$

→ CAN BE DIFFICULT!

→ OFTEN DONE COMPUTATIONALLY ("DC OPERATING POINT")

— Perturb: $\vec u(t) = \vec u^* + \underbrace{\Delta\vec u(t)}_{\text{small}}$, $\vec x(t) = \vec x^* + \underbrace{\Delta\vec x(t)}_{\text{assumed small}}$

— State eqn becomes: $\frac{d}{dt}(\vec x^* + \Delta\vec x(t)) = \vec f(x^* + \underbrace{\Delta\vec x(t), \vec u^* + \Delta\vec u(t)}_{\text{small}})$

— Approximate $\vec f(\cdot,\cdot)$ using vector Taylor series

matrix

$$\vec f(\vec x^* + \Delta\vec x, \vec u^* + \Delta\vec u) \simeq \vec f(\vec x^*, \vec u^*) + J_x\,\Delta\vec x + J_u\,\Delta\vec u + \text{higher order terms}$$

what are these?

— WHAT ARE J_x and J_u ?

— Expand $\vec{f}(\cdot,\cdot)$ into its scalar components

$$\vec{f}(\vec{x},\vec{u}) = \begin{bmatrix} f_1(x_1,x_2,x_3,\dots,x_n;u_1,u_2,\dots,u_m) \\ f_2(x_1,x_2,x_3,\dots,x_n;u_1,u_2,\dots,u_m) \\ f_3(x_1,x_2,x_3,\dots,x_n;u_1,u_2,\dots,u_m) \\ \vdots \\ f_n(x_1,x_2,x_3,\dots,x_n;u_1,u_2,\dots,u_m) \end{bmatrix}$$

Jacobian matrix (of $\vec{f}$ wrt $\vec{x}$)

$$J_x(\vec{x},\vec{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1,\dots,u_m) & \frac{\partial f_1}{\partial x_2}(x_1,\dots,u_m) & \dots & \frac{\partial f_1}{\partial x_n}(x_1,\dots,u_m) \\ \frac{\partial f_2}{\partial x_1}(\dots) & \frac{\partial f_2}{\partial x_2}(\dots) & \dots & \frac{\partial f_2}{\partial x_n}(\dots) \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_n}(\dots) & \frac{\partial f_n}{\partial x_2}(\dots) & \dots & \frac{\partial f_n}{\partial x_n}(\dots) \end{bmatrix}$$

SQUARE

$n \times n$ matrix

frequently written as

$\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x},\vec{u})$

or $\nabla_{\vec{x}} \vec{f}(\vec{x},\vec{u}) \rightarrow$ read as "gradient of $\vec{f}$ with respect to $\vec{x}$"

Jacobian matrix (of $\vec{f}$ wrt $\vec{u}$)

$$J_u(\vec{x},\vec{u}) \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(x_1,\dots,x_n;u_1,\dots,u_m) & \frac{\partial f_1}{\partial u_2}(x_1,\dots,x_n;u_1,\dots,u_m) & \dots & \frac{\partial f_1}{\partial u_m}(x_1,\dots,x_n;u_1,\dots,u_m) \\ \partial f_2/\partial u_1(\dots) & \frac{\partial f_2}{\partial u_2}(\dots) & \dots & \frac{\partial f_2}{\partial u_m}(\dots) \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_n}(\dots) & \dots & & \frac{\partial f_n}{\partial u_m}(\dots) \end{bmatrix}$$

$\boxed{n \times m \quad \text{matrix}}$ → may not be square

$\frac{\partial \vec{f}}{\partial \vec{u}}(\vec{x},\vec{u})$ $\quad \nabla_{\vec{u}} \vec{f}(\vec{x},\vec{u})$

— From before, we had:

$$\vec{f}(\vec{x}^*+\vec{\Delta x}, \vec{u}^*+\vec{\Delta u}) \simeq \vec{f}(\vec{x}^*, \vec{u}^*) + \overset{\substack{n\times n \text{ matrix} \\ \nabla_{\vec{x}} \vec{f}}}{J_x} \vec{\Delta x} + \overset{\substack{n\times m \text{ matrix} \\ \nabla_{\vec{u}} \vec{f}}}{J_u} \vec{\Delta u}$$

also: $\frac{d}{dt}(\vec{x}^*+\vec{\Delta x}(t)) = \vec{f}(x^*+\underbrace{\vec{\Delta x}(t), \vec{u}^*+\vec{\Delta u}(t)}_{\text{small}})$

also: $\vec{0} = \vec{f}(x^*, u^*) \rightarrow 0$

$$\frac{d}{dt}\vec{\Delta x}(t) \simeq \cancel{\vec{f}(\vec{x}^*, \vec{u}^*)} + J_x \vec{\Delta x}(t) + J_u \vec{\Delta u}(t)$$

$$\boxed{\frac{d}{dt}\vec{\Delta x}(t) \simeq J_x \vec{\Delta x}(t) + J_u \vec{\Delta u}(t)}$$

← Linearized state space equation [around operating (or equilibrium) point (u^*, x^*)]

— PENDULUM EXAMPLE : $\vec{x}(t) = \begin{bmatrix} \theta(t) \\ v_\theta(t) \end{bmatrix}$, $u(t) = b(t)$, $\dot{\vec{x}} = \begin{bmatrix} v_\theta \\ -g/_\ell \sin(\theta) - \frac{k}{m} v_\theta + \frac{u(t)}{m\ell} \end{bmatrix} = \vec{f}(\vec{x}, u)$

— $n=2$, $m=1$.

— DC input: $u(t) \equiv 0 = u^*$ (no force)

— DC solution: $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{x}^*$ $(\theta^* = 0, v_\theta^* = 0)$: at rest

→ Therefore $\vec{\Delta x} \equiv \vec{x}$, $\Delta u \equiv u$ ⟹ $u(t)$ is small, assume $x(t)$ is small

→ $J_{\vec{x}} = \overset{2\times 2 \text{ matrix}}{\begin{bmatrix} 0 & 1 \\ -g/_\ell \cos(\theta^*) & -k/_m \end{bmatrix}}$; $J_{\vec{u}} = \overset{2\times 1 \text{ "matrix"}}{\begin{bmatrix} 0 \\ \frac{+1}{m\ell} \end{bmatrix}}$

→ Linearized system:

$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/_\ell & -k/_m \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ +\frac{u(t)}{m\ell} \end{bmatrix}}$$

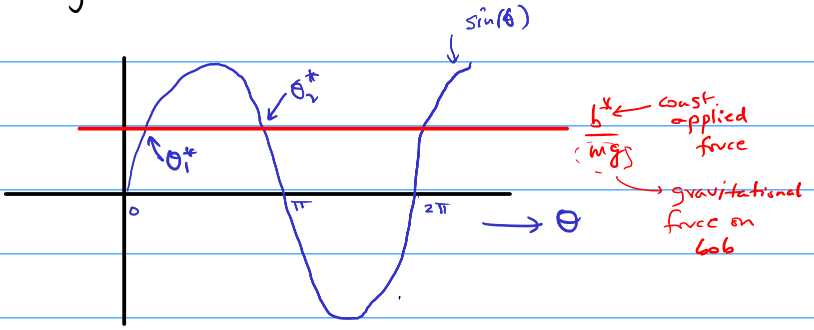
— compare against $\sin(\theta) \simeq 0$ approximation (done previously)

Inverted Pendulum: equilibrium pt. & linearization

Pendulum equations: $$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix} v_\theta(t) \\ -\frac{g}{l}\sin(\theta(t)) - \frac{k}{m}v_\theta(t) + \frac{b(t)}{ml}\end{bmatrix}$$

$\underline{DC} \Rightarrow b(t) = b^*$ (constant wrt time); $\theta(t) = \theta^*$, $v(t) = v_\theta^*$, $\frac{d}{dt}[\cdot] = [0]$

$$\Rightarrow \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix} v_\theta^* \\ -\frac{g}{l}\sin(\theta^*) - \frac{k}{m}v_\theta^* + \frac{b^*}{ml}\end{bmatrix} \Rightarrow \frac{g}{l}\sin(\theta^*) = \frac{b^*}{ml} \Rightarrow \boxed{\sin(\theta^*) = \frac{b^*}{mg}}$$



no applied force

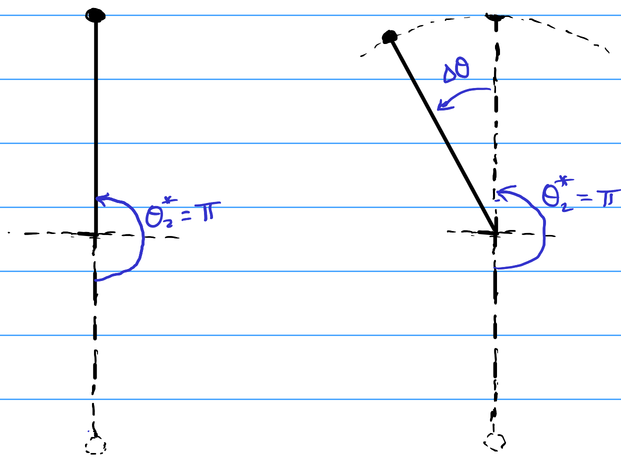
– If $b^* \equiv 0$, then $\theta_1^* = 0$ (we did this before)

OR: $\boxed{\theta_2^* = \pi}$

$$\vec{x}_2^* = \begin{bmatrix}\theta_2^*\\ v_\theta^*\end{bmatrix} = \begin{bmatrix}\pi\\0\end{bmatrix}$$

2nd equilibrium
(DC operating point)

→ $\theta = \pi \Rightarrow$ pendulum is inverted



→ Linearization around $\theta_2^* = \pi$ [and $v_\theta^* = 0$]

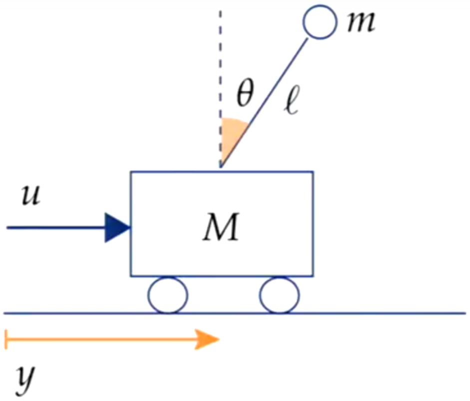
$$\vec{f}(\vec{x}, b) \equiv \begin{bmatrix} v_\theta \\ -\frac{g}{l}\sin(\theta) - \frac{k}{m}v_\theta + \frac{b}{ml}\end{bmatrix};\ \boxed{J_x(x_2^*, 0) = \begin{bmatrix} 0 & 1 \\ -g/l \cos(\overset{\pi}{\theta_2^*}) & -\frac{k}{m}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ +g/l & -\frac{k}{m}\end{bmatrix}}$$

Linearized system: $$\frac{\partial}{\partial t}\begin{bmatrix}\Delta\theta(t)\\ \Delta v_\theta(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ +g/l & -\frac{k}{m}\end{bmatrix}\begin{bmatrix}\Delta\theta(t)\\ \Delta v_\theta(t)\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}}_{J_u}\Delta b(t)$$

— A bit more interesting: POLE ON A CART EXAMPLE

→ discussion/HW

FIG. COURTESY MURAT ARCAK



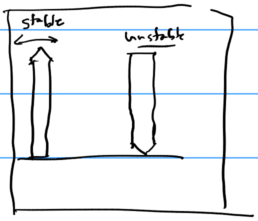
$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right)$$

— STABILITY

— Basic intuitive concepts

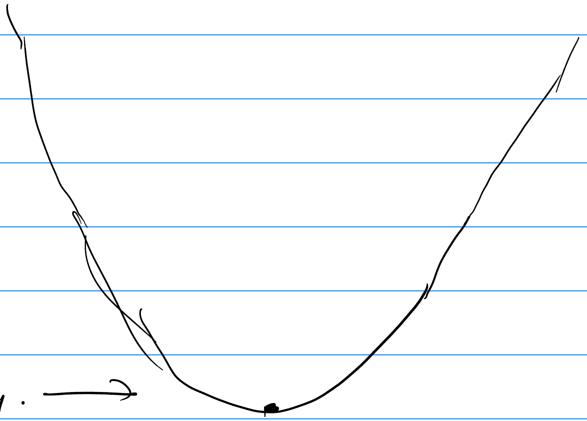
— Live demo of



video

LEVITRON VIDEO

—SPIN STABILIZES SYSTEM



— concepts: local stability, global stability. →

↳ small perturbations

↳ any size perturbation

— to explore stability

— put system in equilibrium (DC operating point) } this is what we did for linearization

— perturb a little

— see (or analyze) what happens next

— does the system move further & further away from equilibrium? If so → unstable

- if deviation from equilibrium remains small → stable

— Mathematical analysis of stability

- Start: $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$

— First: Find an equilibrium point $(\vec{u}^*, \vec{x}^*)$ and linearise about it:

$$— \frac{d}{dt}\Delta\vec{x}(t) = \overset{J_x}{\overset{\downarrow}{A}}\,\Delta\vec{x}(t) + \overset{J_u}{\overset{\downarrow}{B}}\,\Delta\vec{u}(t)$$

— Solve this:

we'll deal with the vector case later

- First, try a scalar example: $\frac{d\Delta x}{dt} = \underset{\text{REAL}}{\underset{\downarrow}{a}}\,\Delta x(t) + \underset{\text{REAL}}{\underset{\downarrow}{b}}\,\Delta u(t)$ (everything real)

→ Solution (by, eg, the method of integrating factors):

$$\boxed{— \Delta x(t) = \underset{\substack{\text{Initial condition}\\ \text{must be given}}}{\underset{\uparrow}{\Delta x(0)}}\, e^{at} + b\underbrace{\int_0^t e^{a(t-\tau)}\Delta u(\tau)\, d\tau}_{\text{convolution}} \xrightarrow{\text{written as}} \text{"}e^{at} \underset{\substack{\uparrow\\ \text{convolution}}}{*} \Delta u(t)\text{"}}$$

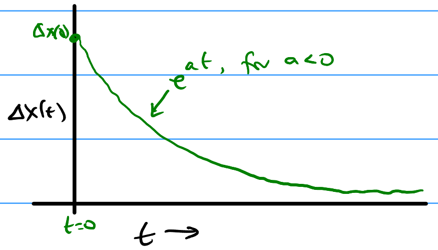
— Suppose $\Delta u(t) \equiv 0$ (no additional external input beyond DC input u^*)

- the perturbation is just the initial condition: $\Delta x(0)$
- then the solution is just $\boxed{\Delta x(t) = \Delta x(0)\, e^{at}}$

- qualitative behaviour:
  - $a < 0$: $\Delta x(t)$ DECREASES exponentially

    STABLE

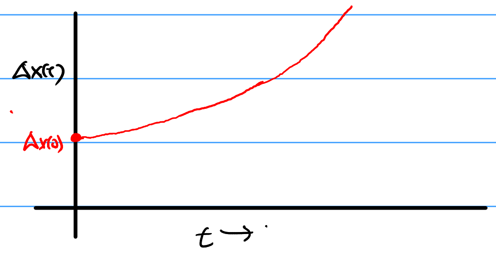
    (small perturbation → returns to equilibrium)



- $a > 0$: $\Delta x(t)$ INCREASES exponentially

  UNSTABLE

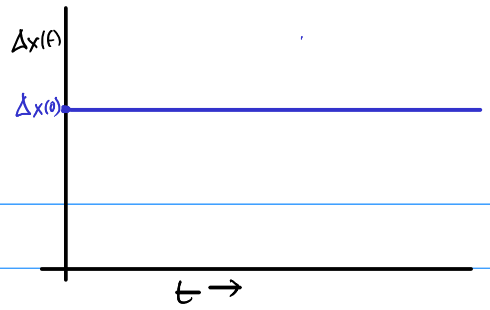
  (small perturbation → goes further away from eq.)



— $a = 0$: $\Delta x(t)$ stays the same

MARGINALLY STABLE

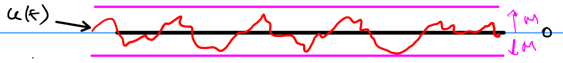
(small perturbation: neither decreases nor increases)



— Now consider the case where $\Delta u(t) \neq 0$ (but is small)

— then, adding to $\Delta x(0) e^{at}$, we have $\boxed{b\int_0^t e^{a(t-\tau)} \Delta u(\tau)\, d\tau}$ → how does this change with time?

— say $\Delta u(t) < M \;\forall t$ ← some small upper bound



— Look at : $\int_0^t \underbrace{e^{a(t-\tau)}}_{\text{always positive for all } a \text{ and } t} \underbrace{\Delta u(\tau)}_{\text{abs. val.} < M} d\tau$

(Δ inequality $|a+b| \leq |a| + |b|$)

invoking

$$\Rightarrow \left| b\int_0^t e^{a(t-\tau)} \Delta u(\tau)\, d\tau \right| = |b| \left| \int_0^t e^{a(t-\tau)} M\, d\tau \right| < |b| \int_0^t |e^{a(t-\tau)}| M\, d\tau$$

→ valid if $a \neq 0$

$$= \left| \frac{bMe^{at}}{-a} \left[e^{-at} - 1 \right] \right| = \frac{|b|M}{|a|} |1 - e^{at}|$$

$$\Rightarrow \text{i.e. } \boxed{|b e^{at} * \Delta u(t)| < \frac{|b|.M}{|a|} |1 - e^{at}| , \quad a \neq 0}$$

— if $a < 0$, $|b e^{at} * \Delta u(t)| < \frac{M|b|}{|a|} \;\forall t$ → bounded (by $\frac{M|b|}{|a|}$) and can be made as small as desired by reducing $\Delta u(t)$.

— Bounded-Input Bounded-Output (BIBO) STABLE

$|\Delta u(t)| < M \;\forall t$ $\qquad$ $\Delta x(t) < \frac{M|b|}{|a|} + \Delta x(0)$

— if $a > 0$, $\frac{|b|M}{a} |1 - e^{at}| \to \infty$ as $t \to \infty$

$\Rightarrow |b e^{at} * \Delta u(t)|$ can be unbounded even if $|\Delta u(t)| > 0$ is made as small as you like! (can always find a small input perturbation that makes this happen)

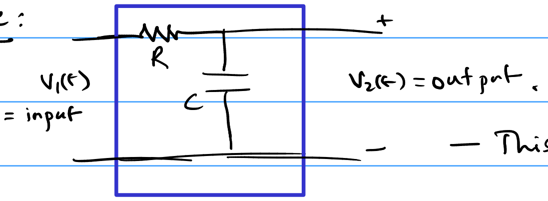
(BIBO) unstable

— if $a=0$, the formula above is not correct!

— instead: $\boxed{|be^{at} * \Delta u(t)| < |b|Mt}$ ← also unbounded as $t \to \infty$

(BIBO) UNSTABLE

— Example:



— This is already linear.

$a = \frac{-1}{RC}$, $b = \frac{1}{RC}$ (why? $C\frac{dv_2}{dt} + \frac{v_2 - v_1}{R} = 0 \Rightarrow \frac{dv_2}{dt} = \frac{-v_2}{RC} + \frac{v_1}{RC}$)

— R & C are always +ve, hence $a<0 \Longrightarrow$ BIBO stable (always)

— THAT WAS FOR SCALAR δx & δu. NOW: the VECTOR CASE.

$$\frac{d\vec{x}}{dt} = \overset{n\times n}{A}\,\Delta\vec{x}(t) + \overset{n\times m}{B}\,\Delta u(t)$$

— Recap: eigendecomposition.

— (almost)* any square matrix A can be written as

* a few matrices cannot — for them: Jordan form (special upper triangular)

— $A = \underset{\text{invertible}}{P}\,\underset{\text{diagonal}}{\Lambda}\,P^{-1}$ or $AP = P\Lambda$

$$\underset{n}{\overset{n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}} = \begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix} \begin{bmatrix} \boxed{\lambda_1} & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} & & \\ & P^{-1} & \\ & & \end{bmatrix}$$

eigenvalues

the columns of P are called eigenvectors

— same thing:

$$\overset{A}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \overset{P}{\begin{bmatrix} | & | & & | \\ \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \\ | & | & & | \end{bmatrix}} = \overset{P}{\begin{bmatrix} | & | & & | \\ \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \\ | & | & & | \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}$$

— EIGENDECOMPOSITION ALWAYS POSSIBLE IF λ_i's are all different

— ASSUME THIS

— Same thing: $A\vec{p}_i = \lambda_i \vec{p}_i$, $i = 1, 2, 3, \cdots, n$

— why is eigendecomposition (immediately) useful?

— $$\frac{d\Delta\vec{x}(t)}{dt} = A\,\Delta\vec{x}(t) + B\,\Delta\vec{u}(t)$$

— eigendecompose A: $A = P\Lambda P^{-1}$ → how? there are standard techniques
— eg, in python & MATLAB
— if REALLY interested: take an advanced numerical analysis course.

— $$\frac{d\Delta\vec{x}(t)}{dt} = P\Lambda P^{-1}\Delta\vec{x}(t) + B\,\Delta\vec{u}(t)$$

— or (P is invertible): $$P^{-1}\frac{d}{dt}\Delta\vec{x}(t) = \Lambda P^{-1}\Delta\vec{x}(t) + P^{-1}B\,\Delta\vec{u}(t)$$

— or $$\frac{d}{dt}\underbrace{\left(P^{-1}\Delta\vec{x}(t)\right)}_{\text{call this } \Delta\vec{y}(t),\ \text{i.e, } \Delta\vec{y}(t) \triangleq P^{-1}\Delta\vec{x}(t) \iff \Delta\vec{x}(t) = P\Delta\vec{y}(t)} = \Lambda\left(P^{-1}\Delta\vec{x}(t)\right) + \left(P^{-1}B\right)\Delta\vec{u}(t)$$

— $$\frac{d}{dt}\Delta\vec{y}(t) = \Lambda\,\Delta\vec{y}(t) + \underbrace{\left(P^{-1}B\right)\Delta\vec{u}(t)}_{\text{call this } \Delta\vec{b}(t),\ \text{ie, } \Delta\vec{b}(t) := \left(P^{-1}B\right)\Delta\vec{u}(t)}$$

— $$\boxed{\frac{d}{dt}\begin{bmatrix}\Delta y_1(t)\\ \Delta y_2(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}\Delta y_1(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} + \begin{bmatrix}\Delta b_1(t)\\ \Delta b_2(t)\\ \vdots\\ \Delta b_n(t)\end{bmatrix}.}$$

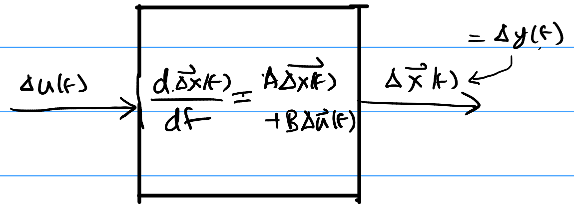
(same thing):

— $$\boxed{\frac{d}{dt}\Delta y_i(t) = \lambda_i\,\Delta y_i(t) + \Delta b_i(t), \quad i = 1, 2, 3, \dots, n}$$

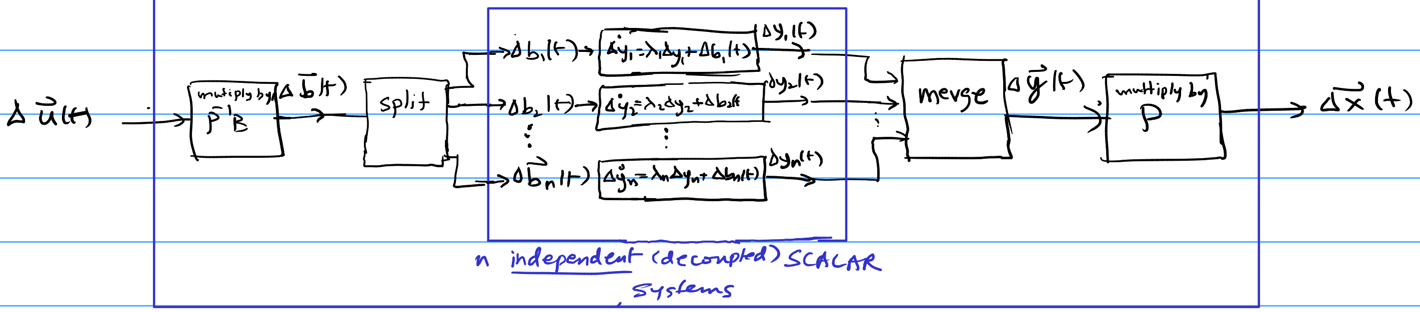
↳ n separate SCALAR (decoupled) SYSTEMS!

SYSTEM EIGENDECOMPOSITION IN PICTURES

ORIGINAL (LINEARIZED) SYSTEM



EQUIVALENT DECOUPLED SYSTEM



— IMPLICATIONS FOR STABILITY

- IF EVEN ONE SCALAR SYSTEM IS UNSTABLE OR MARGINALLY STABLE

  ⇒ SYSTEM UNSTABLE

- IF ALL SCALAR SYSTEMS ARE STABLE ⇒ SYSTEM STABLE.

— A SLIGHT COMPLICATION

- EIGENVALUES / EIGENVECTORS CAN NOW BE COMPLEX
  - WHY? (A is real!)
  - FACT: even if A is real, eigenvalues/vectors can be complex!
- Examples

$$\rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{P}{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} j & \\ & -j \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}\frac{1}{\sqrt{2}}}$$

$$\rightarrow \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} = \overset{P}{\begin{bmatrix} \frac{1+j}{2} & \frac{1-j}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} \frac{-1+j}{\sqrt{2}} & \\ & \frac{-1-j}{\sqrt{2}} \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} -j & \frac{1+j}{\sqrt{2}} \\ j & \frac{1-j}{\sqrt{2}} \end{bmatrix}}$$

— BUT IF A IS REAL

- EIGENVECTORS / VALUES COME IN COMPLEX CONJUGATE PAIRS

- i.e:

$$\boxed{\begin{array}{l} \text{if} \quad A\vec{p}_i = \lambda_i \vec{p}_i \\ \text{then} \quad A\overline{\vec{p}_i} = \overline{\lambda_i}\,\overline{\vec{p}_i} \end{array}}$$

— BECAUSE OF THIS, $\vec{\Delta x}(t)$ will ALWAYS BE REAL

- although INTERNAL QUANTITIES $\Delta b_i(t)$, $\Delta y_i(t)$, etc.) CAN be complex
  - but the internal quantities always come in conjugate pairs

— How can you show this?

- already know that the cols. of P (eigevectors) come in conj. pairs
- next: use $A^T = (P^{-1})^T \Lambda P^T$ to show that the rows of P^{-1} " " " "
- next: $\vec{\Delta b}_i(t)$ = cols. of $\vec{\Delta b}(t) = P^{-1} B \vec{\Delta u}(t)$ come in conj. pairs
- → next: $\Delta y_i(t)$ also come in conj. pairs
- → therefore: $\vec{\Delta x}(t) = P\vec{\Delta y}(t)$ consists of pairwise conj. sums:

$$\left(\vec{p}_i \vec{\Delta y}_i(t) + \overline{\vec{p}_i}\,\overline{\vec{\Delta y}_i(t)}\right),$$ which are real

— THE UPSHOT:

- Solution of each scalar system
  - $\Delta y_i(t) = \Delta y_i(0)\, e^{\lambda_i t} + e^{\lambda_i t} * \Delta b_i(t)$
  - if λ_i is complex, then there will also be: $\Delta y_j(t) = \overline{\Delta y_i(t)}$ for some $j \neq i$

$$— \quad \vec{\Delta x}(t) = \sum_{i=1}^{n} \vec{p}_i \Delta y_i(t) = \underbrace{\vec{p}_1 \Delta y_1(t)}_{\text{real (say)}} + \underbrace{\vec{p}_2 \Delta y_2(t)} + \underbrace{\vec{p}_3 \Delta y_3(t)} + \cdots + \underbrace{\vec{p}_n \Delta y_n(t)}_{\text{real (say)}}$$

SUPPOSE THESE ARE COMPLEX CONJUGATES

— THEIR SUM IS REAL

— HOW COMPLEX EIGENVALUES/VECTORS AFFECT STABILITY

- Each complex conjugate pair contributes (to $\overrightarrow{\Delta x}(t)$)
  - $\vec{p}_i\, \Delta y_i(t) + \overline{\vec{p}_i}\ \overline{\Delta y_i(t)}$, where $\Delta y_i(t) = \Delta y_i(0)\, e^{\lambda_i t} + e^{\lambda_i t} * \Delta b_i(t)$

    $\underbrace{\vec{q}\ z(t) + \overline{\vec{q}}\ \overline{z}(t)}_{2\,\mathrm{Re}\{\vec{q}\, z(t)\}}$, $\qquad z(t) = z(0)\, e^{\gamma t} + e^{\gamma t} * c(t)$

- expand complex quantities into real & imag. parts:
  - $\gamma = \gamma_r + j\gamma_i$; $z(0) = z_r(0) + j\, z_i(0)$
  - $\vec{q} = \vec{q}_r + j\,\vec{q}_i$; $c(t) = c_r(t) + j\, c_i(t)$

- $$\vec{q}\, z(t) = (\vec{q}_r + j\,\vec{q}_i)\Big[\big(z_r(0) + j\, z_i(0)\big)\, e^{\gamma_r t}(\cos\gamma_i t + j \sin\gamma_i t) + $$

$$+ \left\{e^{\gamma_r t}(\cos\gamma_i t + j\sin\gamma_i t)\right\} * \left\{c_r(t) + j\, c_i(t)\right\}\Big]$$

$$= (\vec{q}_r + j\,\vec{q}_i)\left[\begin{array}{l} e^{\gamma_r t}\left[\begin{array}{l}(z_r(0)\cos(\gamma_i t) - z_i(0)\sin(\gamma_i t)) \\ + j\,(z_r(0)\sin(\gamma_i t) + z_i(0)\cos(\gamma_i t))\end{array}\right] \\ + \left[\{e^{\gamma_r t}\cos(\gamma_i t)\} * \{c_r(t)\} - \{e^{\gamma_r(t)}\sin(\gamma_i t)\} * \{c_i(t)\}\right] \\ + j\left[\{e^{\gamma_r(t)}\sin(\gamma_i t)\} * \{c_r(t)\} + \{e^{\gamma_r(t)}\cos(\gamma_i t)\} * \{c_i(t)\}\right]\end{array}\right]$$

$$= \boxed{\begin{array}{l} \vec{q}_r\left[\begin{array}{l} e^{\gamma_r t}(z_r(0)\cos(\gamma_i t) - z_i(0)\sin(\gamma_i t)) \\ + \left[\{e^{\gamma_r t}\cos(\gamma_i t)\} * \{c_r(t)\} - \{e^{\gamma_r(t)}\sin(\gamma_i t)\} * \{c_i(t)\}\right]\end{array}\right] \\ - \vec{q}_i\left[\begin{array}{l} e^{\gamma_r t}(z_r(0)\sin(\gamma_i t) + z_i(0)\cos(\gamma_i t)) \\ + \left[\{e^{\gamma_r(t)}\sin(\gamma_i t)\} * \{c_r(t)\} + \{e^{\gamma_r(t)}\cos(\gamma_i t)\} * \{c_i(t)\}\right]\end{array}\right]\end{array}}$$

+ imaginary terms $\qquad \hookrightarrow \mathrm{Re}\{\vec{q}\, z(t)\}$

$$\boxed{\begin{aligned}&\vec{q}_r\left[e^{\gamma_r t}\left(z_r(0)\cos(\gamma_i t)-z_i(0)\sin(\gamma_i t)\right)\right.\\&\qquad\left.+\left[\{e^{\gamma_r t}\cos(\gamma_i t)\}*\{c_r(t)\}-\{e^{\gamma_r(t)}\sin(\gamma_i t)\}*\{c_i(t)\}\right]\right]\\-&\vec{q}_i\left[e^{\gamma_r t}\left(z_r(0)\sin(\gamma_i t)+z_i(0)\cos(\gamma_i t)\right)\right.\\&\qquad\left.+\left[\{e^{\gamma_r(t)}\sin(\gamma_i t)\}*\{c_r(t)\}+\{e^{\gamma_r(t)}\cos(\gamma_i t)\}*\{c_i(t)\}\right]\right]\end{aligned}}$$

$$\rightarrow Re\{\vec{q}\,z(t)\} = e^{\gamma_r t}\cos(\gamma_i t)\overbrace{\left[\vec{q}_r z_r(0)-\vec{q}_i z_i(0)\right]}^{\text{some constant vector}} - e^{\gamma_r t}\sin(\gamma_i t)\overbrace{\left[\vec{q}_r z_i(0)+\vec{q}_i z_r(0)\right]}^{\text{some constant vector}}$$

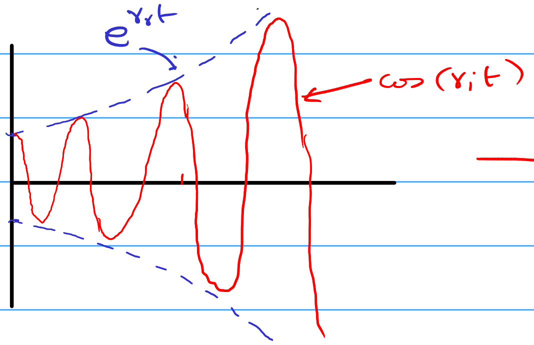
$$+\ \{e^{\gamma_r t}\cos(\gamma_i t)\}*\{\vec{q}_r c_r(t)-\vec{q}_i c_i(t)\}$$

$$-\ \{e^{\gamma_r t}\sin(\gamma_i t)\}*\underbrace{\{\vec{q}_r c_i(t)+\vec{q}_i c_r(t)\}}_{\text{some vector function}}$$

→ STABILITY WITH COMPLEX TERMS

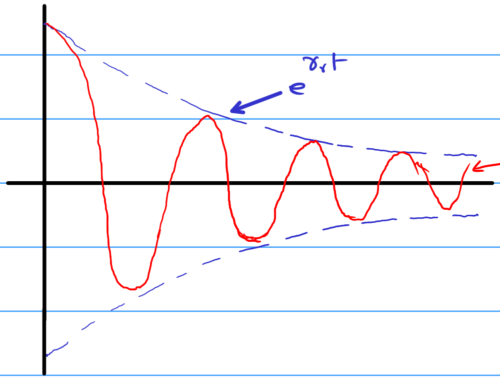
— each component of $e^{\gamma_r t}\cos(\gamma_i t)\overbrace{\left[\vec{q}_r z_r(0)-\vec{q}_i z_i(0)\right]}^{\text{a constant vector} \leftarrow \text{depends only on the initial condition}}$ looks like $k\,e^{\gamma_r t}\cos(\gamma_i t)$

— blows up if $\gamma_r > 0$

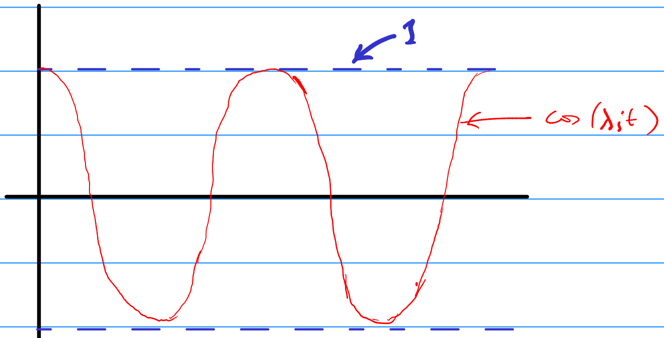


→ UNSTABLE

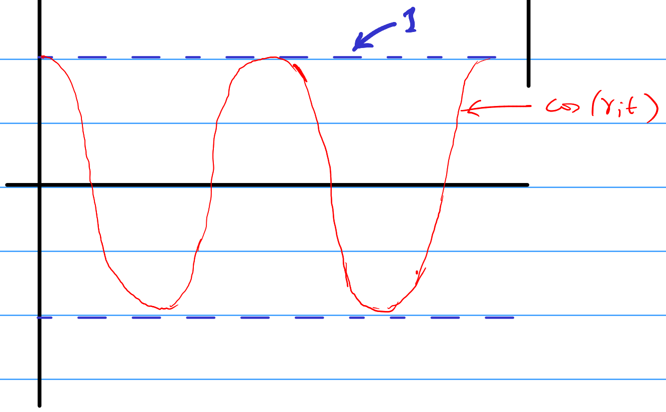
— dies down if $\gamma_r < 0$



→ STABLE



— envelope stays the same if $\gamma_r = 0$



→ MARGINALLY STABLE

- For the complex terms of $\Delta\vec{x}(t)$ involving the initial condition

  - if $\gamma_r = \mathrm{Re}\{\text{complex eigenvalue}\} < 0 \longrightarrow$ STABLE

    $> 0 \longrightarrow$ UNSTABLE

    $= 0 \longrightarrow$ MARGINALLY STABLE

  - $\gamma_i = \mathrm{Im}\{\text{complex eigenvalue}\}$ does not affect stability
    - just determines the frequency of oscillation – via $\cos(\gamma_i t)$ and $\sin(\gamma_i t)$

- For the convolutional terms involving the input $\vec{\Delta b}(t)$:

$$\longrightarrow \{e^{\gamma_r t}\cos(\gamma_i t)\} * \underbrace{\{\vec{q}_r\, c_r(t) - \vec{q}_i\, c_i(t)\}}_{\substack{\text{some vector function of time} \\ \rightarrow \text{ Call it } \vec{q}_c(t)}} = \int_0^t \underbrace{e^{\gamma_r(t-\tau)} \cos(\gamma_i(t-\tau))}_{\hookrightarrow \text{magnitude} \le |e^{\gamma_r(t-\tau)}|} \vec{q}_c(\tau)\, d\tau$$

- can easily show that: [if $\|\vec{q}_c(t)\| < M$ – i.e. input is bounded/small]

- if $\gamma_r < 0 \longrightarrow$ BIBO STABLE

  $\gamma_r > 0 \longrightarrow$ BIBO UNSTABLE

  $\gamma_r = 0 \longrightarrow$ also BIBO UNSTABLE

  (γ_i does not matter for stability)

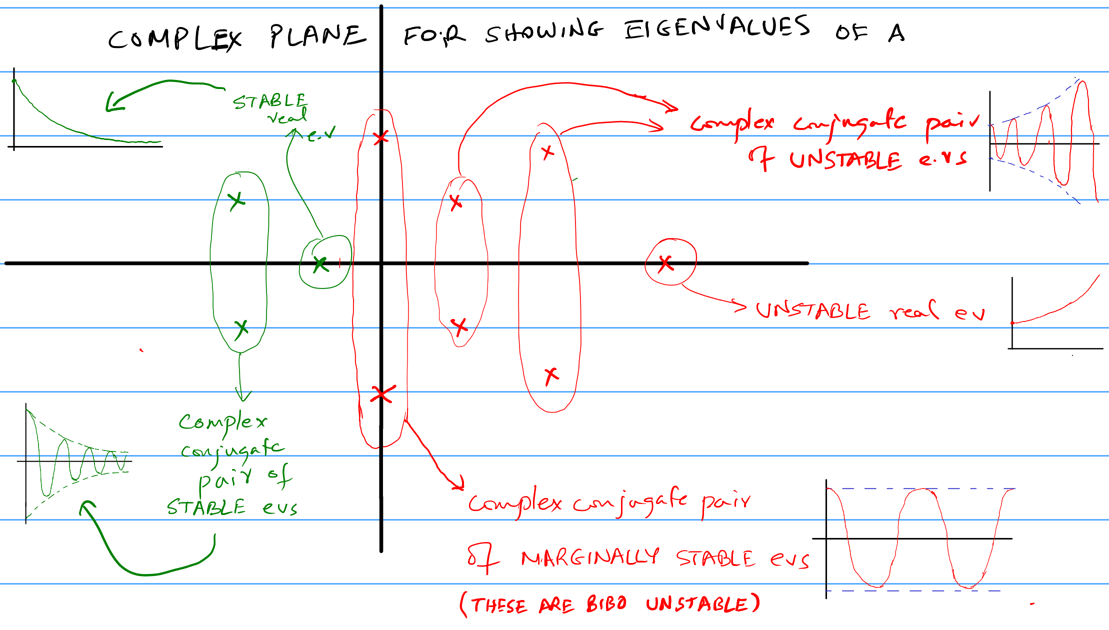
→ SUMMARY OF STABILITY (OF VECTOR STATE SPACE REPRESENTATIONS)

→ FIND THE EIGENVALUES OF A: $\lambda_1, \lambda_2, \ldots, \lambda_2$

→ an $\boxed{\text{eigenvalue } \lambda_i \text{ called STABLE if } \mathrm{Re}\{\lambda_i\} < 0}$

- the system is BIBO stable if ALL λ_i, $i = 1, \cdots, n$ ARE STABLE
- otherwise, it is BIBO UNSTABLE

LINEARIZED S.S.R: $\frac{d\Delta\vec{x}}{dt} = A\Delta\vec{x}(t) + B\Delta\vec{u}(t)$



— LINEARIZED PENDULUM EXAMPLE

$$\boxed{\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -g/l & -k/m \end{bmatrix}}_{A}\vec{x} + \underbrace{\begin{bmatrix} 0 \\ -\frac{u(t)}{ml} \end{bmatrix}}_{B}}$$

— Find the eigenvalues of A:

$$A\vec{p} = \lambda\vec{p} \Rightarrow (A-\lambda I)\vec{p} = 0 \Rightarrow \underbrace{\begin{bmatrix} -\lambda & 1 \\ -g/l & -\frac{k}{m}-\lambda \end{bmatrix}}_{\text{det should} = 0}\overset{\text{want non-zero solution}}{\vec{p}} = 0$$

$$\lambda(\lambda + k/m) + \frac{g}{l} = 0 \Rightarrow \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} = 0 \Rightarrow \boxed{\lambda_{1,2} = \frac{-k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - \frac{4g}{l}}}$$

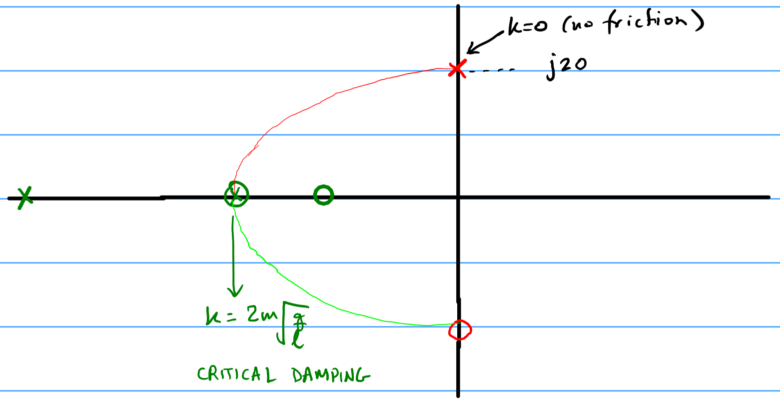
— recall: k is the friction coefficient : always > 0 (unless: negative friction?)

m is the mass : always > 0

$\Rightarrow \frac{k}{m} > 0$

$\Rightarrow g = 9.8\ m/s^2$

$\Rightarrow$ take $l = 10\ cm = 0.1\ m \Rightarrow \frac{g}{l} \sim 100 \Rightarrow \frac{4g}{l} = 400$



$$\boxed{\lambda_{1,2} = \frac{-k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - \frac{4g}{l}}}$$

— Run the interactive MATLAB script (slider controlling k)

Inverted pendulum: stability

$$A = J_x = \begin{bmatrix} 0 & 1 \\ +g/l & -k/m \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{-k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + \frac{4g}{l}}$$

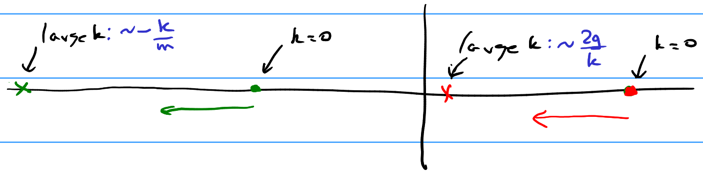
$$\to \left(\frac{k^2}{m^2} + \frac{4g}{l}\right)^{1/2}$$

$$= \left(\frac{k^2}{m^2}\right)^{1/2} + \underbrace{\frac{1}{2}\left(\frac{k^2}{m^2}\right)^{-1/2}\frac{4g}{m}}_{\frac{2g}{m}\cdot\frac{m}{k} = \frac{2g}{k}} + \cdots$$

at $k = 0$: $\lambda_{1,2} = \pm\sqrt{\frac{g}{l}} \Rightarrow$ real eigenvalues, 1 is +ve → UNSTABLE

$\downarrow$
no friction

friction
$\uparrow$
as k increases:



— the positive one $\to 0$

— the negative one $\to -\infty$

STABILITY OF DISCRETE-TIME SYSTEMS

— State space repr. of a discrete-time system: discrete: $\vec{x}[t+1] = \vec{f}(\vec{x}[t], \vec{u}[t])$

— LINEARIZATION:

— CHOOSE "DC" INPUT: $\vec{u}(t) \equiv u^*$, ASSUME DC STATE $= \vec{x}^*$

SOLVE: $\vec{x}^* = \vec{f}(\vec{x}^*, \vec{u}^*)$

— DEFINE $\Delta\vec{u}[t] = \vec{u}[t] - \vec{u}^*$, "small"

— DEFINE $\Delta\vec{x}[t] = \vec{x}[t] - \vec{x}^*$, assumed "small"

— then the linearized system is: $\Delta\vec{x}[t+1] = \underbrace{J_x(\vec{x}^*,\vec{u}^*)}_{\left.\frac{d\vec{f}}{d\vec{x}}\right|_{x^*,u^*}} \Delta\vec{x}[t] + \underbrace{J_u(\vec{x}^*,\vec{u}^*)}_{\left.\frac{d\vec{f}}{d\vec{u}}\right|_{\vec{x}^*,\vec{u}^*}} \Delta\vec{u}[t]$

$\downarrow$

$$\boxed{\Delta\vec{x}[t+1] = A\,\Delta\vec{x}[t] + B\,\Delta\vec{u}[t]}$$

— SCALAR CASE: STABILITY

— scalar case $\Delta x[t] = a\Delta x[t] + b\Delta u[t]$; init. cond. $\Delta x(0)$ given.

$t=0$: $\Delta x[1] = a\Delta x[0] + b\Delta u[0]$

$t=1$: $\Delta x[2] = a\Delta x[1] + b\Delta u[1] = a^2\Delta x[0] + ab\Delta u[0] + b\Delta u[1]$

$t=2$: $\Delta x[3] = a\Delta x[2] + b\Delta u[2] = a^3\Delta x[0] + a^2 b\Delta u[0] + ab\Delta u[1] + b\Delta u[2]$

$\vdots$

$\longrightarrow$ $\Delta x[t] = a^t\Delta x[0] + a^{t-1}b\Delta u[0] + a^{t-2}b\Delta u[1] + \cdots + ab\Delta u[t-2] + b\Delta u[t-1]$

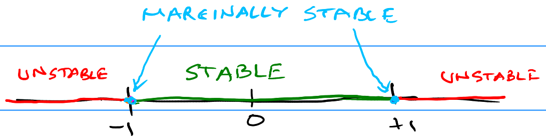
$$= \underbrace{a^t\Delta x[0]}_{\text{I.C. term}} + \underbrace{\sum_{i=1}^{t} a^{t-i}\, b\,\Delta u[i-1]}_{\text{discrete-time convolution}} = \text{input term}$$

Stability:

— IC term: if $|a|<1 \Rightarrow$ stable

if $|a|>1 \Rightarrow$ unstable

if $|a|=1 \Rightarrow$ marginally stable



- input (conv. term)

suppose bounded by M

i.e., $|\Delta u[t-1]| < M, \forall t$

$$\left|\sum_{i=1}^{t} a^{t-i} b \Delta u[i-1]\right| \leq \sum_{i=1}^{t} |a|^{t-i} |b| |\Delta u[t-1]| < M|b| \sum_{i=1}^{t} |a|^{t-i}$$

$$= M(1+|a|+|a|^2+\cdots|a|^{t-1}) = \frac{M|b|(|a|^t-1)}{|a|-1}$$

$$\Longrightarrow \left|\sum_{i=1}^{t} a^{t-i}|b| \Delta u[i-1]\right| < \begin{cases} M|b|\left(\frac{a^t-1}{a-1}\right) & \text{if } |a|\neq 1 \\ M|b|t & \text{if } |a|=1 \end{cases}$$

$$S = 1+a+a^2+\cdots+a^{n-1}$$
$$aS = a+a^2+ \quad +a^{n-1}+a^n$$
$$aS-S = a^n-1$$
$$S(a-1) = a^n-1$$
$$S = \frac{a^n-1}{a-1}$$

→ if $|a| \geq 1$: blows up unboundedly as t increases

→ if $|a|<1$: bounded by $\frac{M|b|}{1-|a|}$ ⇒ BIBO stable

- the VECTOR CASE : $\Delta \vec{X}[t+1] = \overset{n\times n}{A} \Delta \vec{x}[t] + \overset{n\times m}{B} \Delta \vec{u}[t]$

→ eigendecompose $A = P \Lambda P^{-1}$

$$\Delta \vec{x}[t+1] = P \Lambda P^{-1} \Delta \vec{x}[t] + B \Delta \vec{u}[t]$$

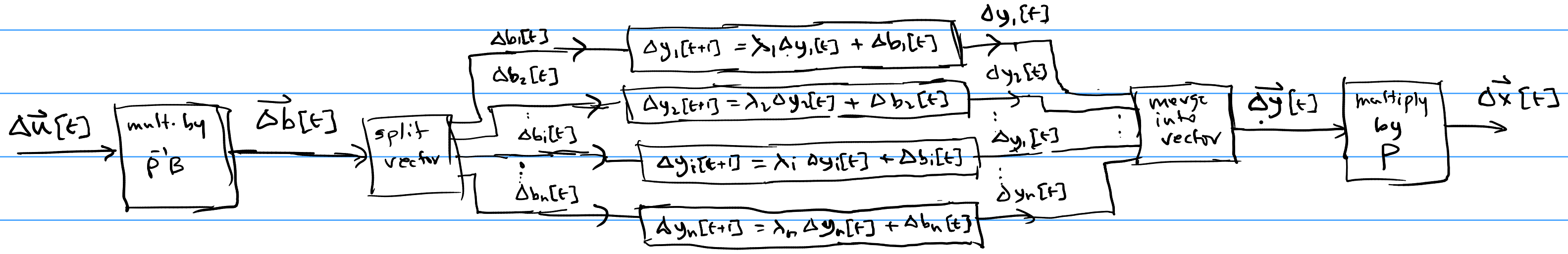
$$\Longrightarrow \underbrace{P^{-1} \Delta \vec{x}[t+1]}_{\Delta \vec{y}[t+1]} = \Lambda \underbrace{P^{-1} \Delta \vec{x}[t]}_{\Delta \vec{y}[t]} + \underbrace{P^{-1} B \Delta \vec{u}[t]}_{\Delta \vec{b}[t]}$$

$$\boxed{\Delta \vec{y} = P^{-1} \Delta \vec{x} \iff \Delta \vec{x} = P \Delta \vec{y}}$$

$$\Longrightarrow \Delta \vec{y}[t+1] = \Lambda \Delta \vec{y}[t] + \Delta \vec{b}[t]$$

$$\Longrightarrow \boxed{\Delta y_i[t+1] = \lambda_i \Delta y_i[t] + \Delta b_i[t]} \longleftarrow \text{decoupled}$$

BLOCK DIAGRAM OF THE DECOUPLED SYSTEM



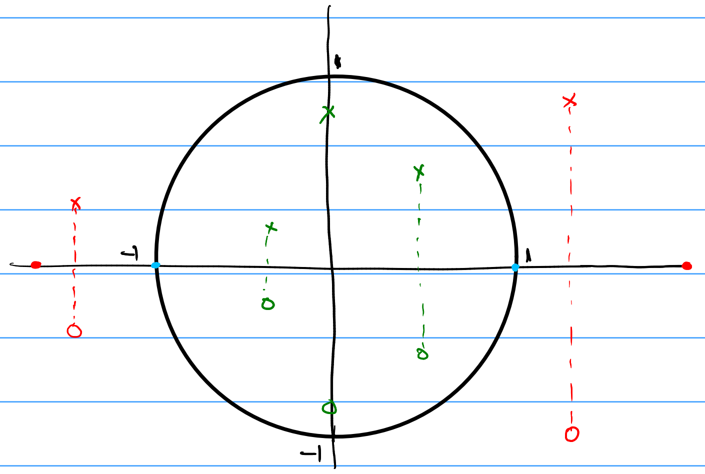
→ just as in the continuous case (the derivation is analogous):

- → internal quantities come in conjugate pairs
- → $\vec{\Delta x}(t)$ always real
- → stability determined by decomposed scalar equations:
  - → $\Delta y_i[t+1] = \lambda_i \Delta y_i[t] + \Delta b_i[t]$
    - → but now, λ_i can be complex
      - → (w/ the understanding that there will be a conjugate eqn)

→ UPSHOT:

CONTINUOUS	DISCRETE
– all $Re\{\lambda_i\}<0$ $\Longrightarrow$ BIBO stable → otherwise: unstable	– all $\lvert\lambda_i\rvert<1$ $\Longrightarrow$ BIBO stable – otherwise: unstable

→ WHAT DO THE I.C. TERMS LOOK LIKE?



$\Delta y_i[t] = \lambda_i^t \Delta y_i[0]$

→ polar $\Upsilon e^{j\phi}$

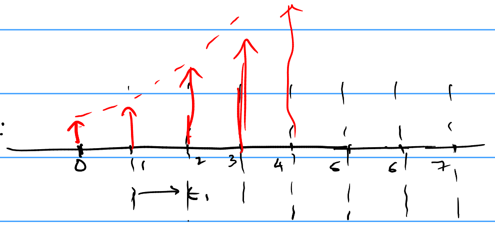
→ polar $Re^{j\theta}$

$\Delta y_i[t] = R^t e^{jt} \; \Upsilon e^{j\phi}$

real part: $\Upsilon R^t \cos(t\theta+\phi)$

imag part: $\Upsilon R^t \sin(t\theta+\phi)$

$R>1, \theta=0$:



$R>1, \theta=180°$: