EE16B, Spring 2018
UC Berkeley EECS

Maharbiz and Roychowdhury

Lectures 4B & 5A: Overview Slides

Linearization and Stability

Linearization

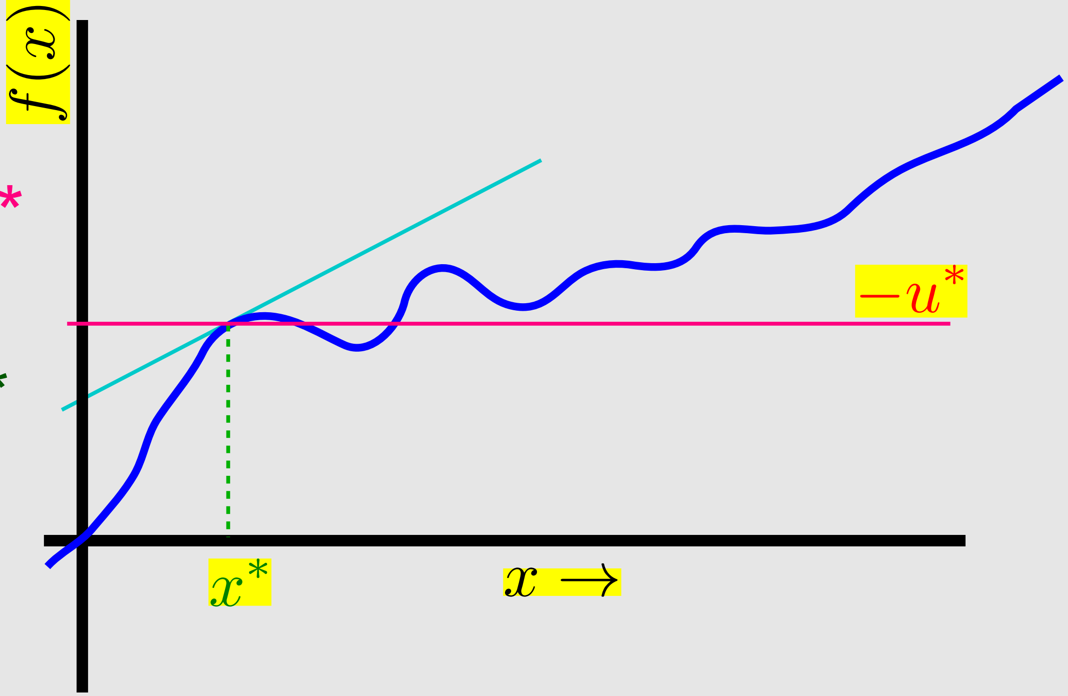
- Approximate a nonlinear system by a linear one
  - (unless it's linear to start with)
  - then apply powerful linear analysis tools
    - gain precise understanding → insight and intuition
- Consider a scalar system first
  - in state space form with additive input (for simplicity)

$$\frac{d}{dt}x(t) = f(x(t)) + u(t)$$

  - step 1: choose DC input u*
    - find DC soln. x*

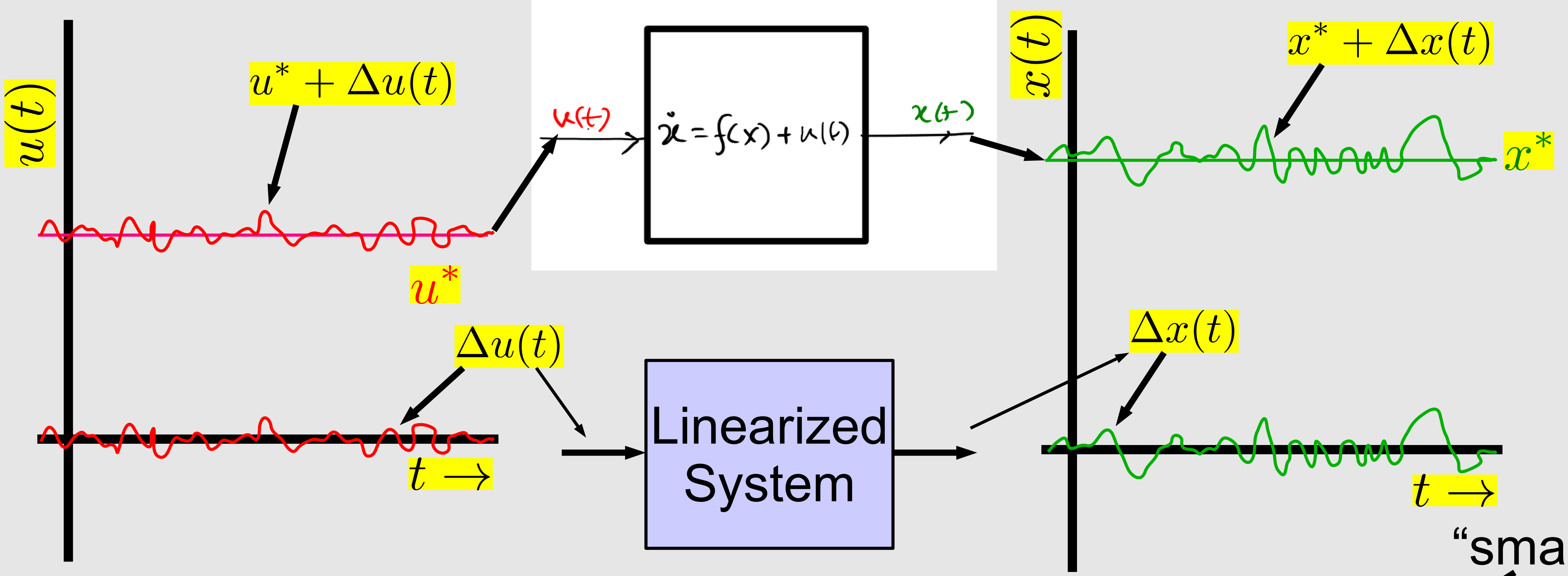
$0 = f(x^*) + u^*$: solve for x*

    - x* is an equilibrium point
      - aka DC operating point
    - for input u*



Linearization (contd. - 2)

- DC operation (equilibrium), viewed in time



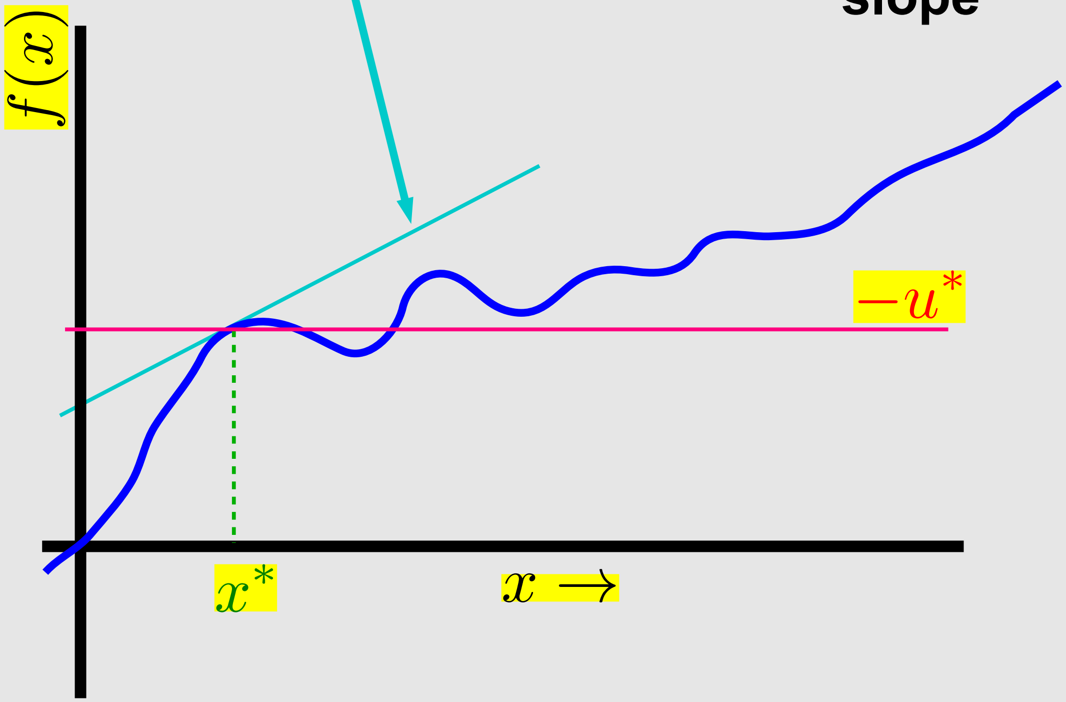
- Now, **perturb** the input a little: $u(t) = u^* + \boxed{\Delta u(t)}$ "small"
- **Suppose** x(t) responds by also changing a little
  - $x(t) = x^* + \boxed{\Delta x(t)}$ ← **ASSUMED** "small"
- Goal: find system relating Δu(t) and Δx(t) directly

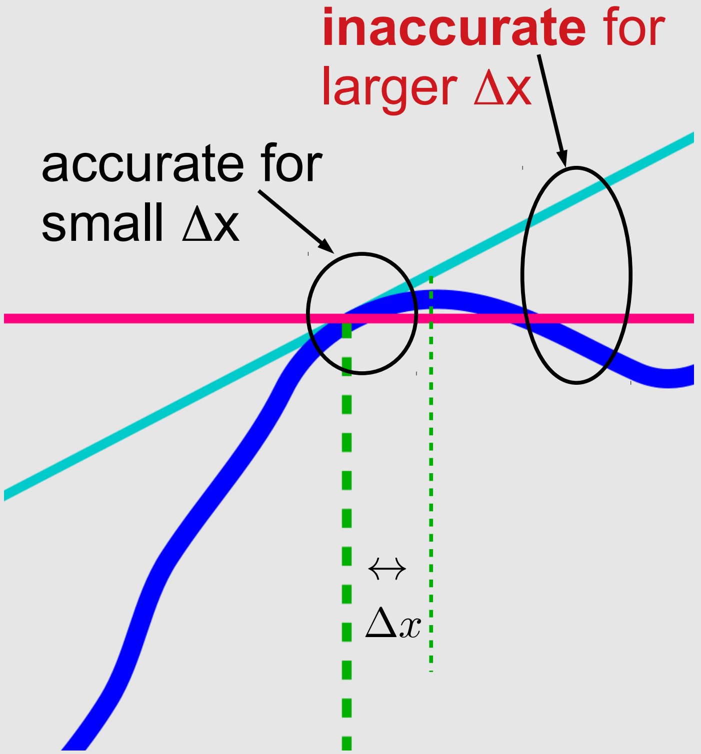
Linearization (contd. - 3)

- Basic notion: replace f(x) by its tangent line at x*
- Mathematically: expand f(x*+Δx) in Taylor Series

$$f(x^* + \Delta x) \simeq f(x^*) + \left.\frac{df}{dx}\right|_{x^*} \Delta x + \frac{1}{2} \left.\frac{d^2 f}{dx^2}\right|_{x^*} \Delta x^2 + \cdots$$

slope





Linearization (contd. - 4)

- applying the Taylor linearization
  - (move to xournal)

$$\xrightarrow{} f(x^*)+u^*=0$$

$$\frac{dx}{dt} = f(x(t)) + u(t), \quad \text{with } 1)\; u = u^* + \Delta u(t) \text{ and } 2)\; x(t) = x^* + \Delta x(t)$$

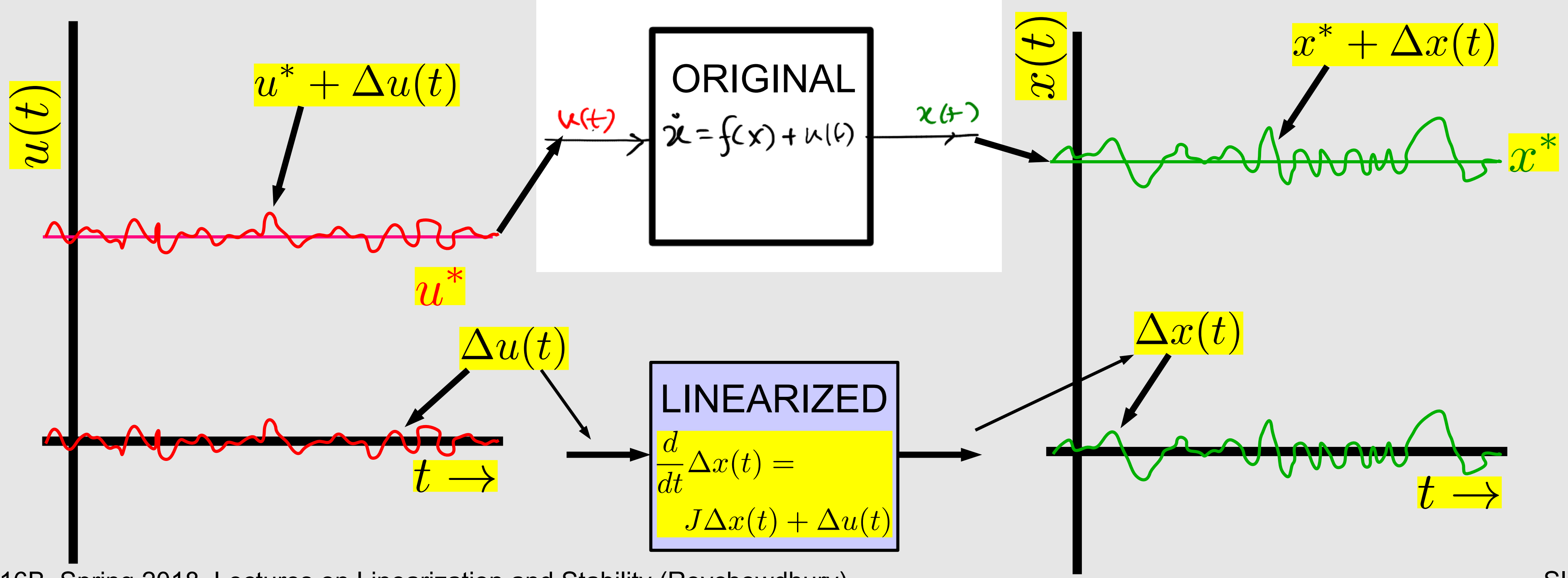
call this J

$$\frac{d}{dt}(x^* + \Delta x(t)) \simeq \cancel{f(x^*)} + \left.\frac{df}{dx}\right|_{x^*} \Delta x(t) + \cancel{u^*} + \Delta u(t)$$

$$J \triangleq \left.\frac{df}{dx}\right|_{x^*}$$

$$\boxed{\frac{d}{dt}\Delta x(t) \simeq J\Delta x(t) + \Delta u(t)}$$

← LINEARIZED SYSTEM



Linearization of Vector S.S. Systems

- Now: the full S.S.R: $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)))$
- step 1: **find a DC. op. pt. (equilibrium pt.)**
  - $\vec{0} = \vec{f}(\vec{x}^*, \vec{u}^*)$ DC input

    DC solution

    Solving for this is often difficult, even using computational methods
- The linearized system is (see handwritten notes for derivation)

$$\frac{d}{dt}\Delta\vec{x}(t) = \boldsymbol{J}_x(\vec{x}^*, \vec{u}^*)\Delta\vec{x}(t) + \boldsymbol{J}_u(\vec{x}^*, \vec{u}^*)\Delta\vec{u}(t))$$

n-vector **nxn matrix** **nxm matrix** m-vector

- What are $\boldsymbol{J}_x$ and $\boldsymbol{J}_u$?
  - called **Jacobian or gradient matrices**

Jacobian (Gradient) Matrices

- If: $\vec{x}(t) = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{u}(t) = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad \vec{f}(\vec{x}, \vec{u}) = \begin{bmatrix} f_1(x_1, \cdots, x_n; u_1, \cdots, u_m) \\ \vdots \\ f_n(x_1, \cdots, x_n; u_1, \cdots, u_m) \end{bmatrix}$, then

nxn matrix

$$\boldsymbol{J}_x(\vec{x}, \vec{u}) = \nabla_x \vec{f}(\vec{x}, \vec{u}) = \left.\frac{\partial \vec{f}}{\partial \vec{x}}\right|_{\vec{x}, \vec{u}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_{n-1}} & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_{n-1}} & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_2} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} & \frac{\partial f_{n-1}}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_{n-1}} & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

nxm matrix

$$\boldsymbol{J}_u(\vec{x}, \vec{u}) = \nabla_u \vec{f}(\vec{x}, \vec{u}) = \left.\frac{\partial \vec{f}}{\partial \vec{u}}\right|_{\vec{x}, \vec{u}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_{m-1}} & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_{m-1}} & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial u_1} & \frac{\partial f_{n-1}}{\partial u_2} & \cdots & \frac{\partial f_{n-1}}{\partial u_{m-1}} & \frac{\partial f_{n-1}}{\partial u_m} \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_{m-1}} & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

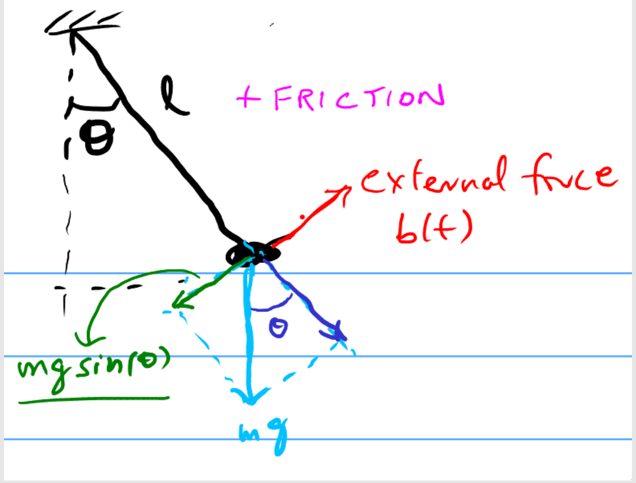
Example: Linearizing the Pendulum

- Pendulum:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -g/\ell \sin(\theta) - k/m\, v_\theta + \frac{b(t)}{m\ell} \end{bmatrix}$$

  - (move to xournal)

$$\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}, \quad \vec{u} = [b(t)]$$



— $n=2$, $m=1$.

— DC input: $u(t) \equiv 0 = u^*$ (no force)

— DC solution: $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{x}^*$ $(\theta^*=0,\ v_\theta^*=0)$: at rest

→ Therefore $\vec{\Delta x} \equiv \vec{x}$, $\Delta u \equiv u$ ⟹ $u(t)$ is $\underline{\text{small}}$, $\underline{\text{assume } x(t) \text{ is small}}$

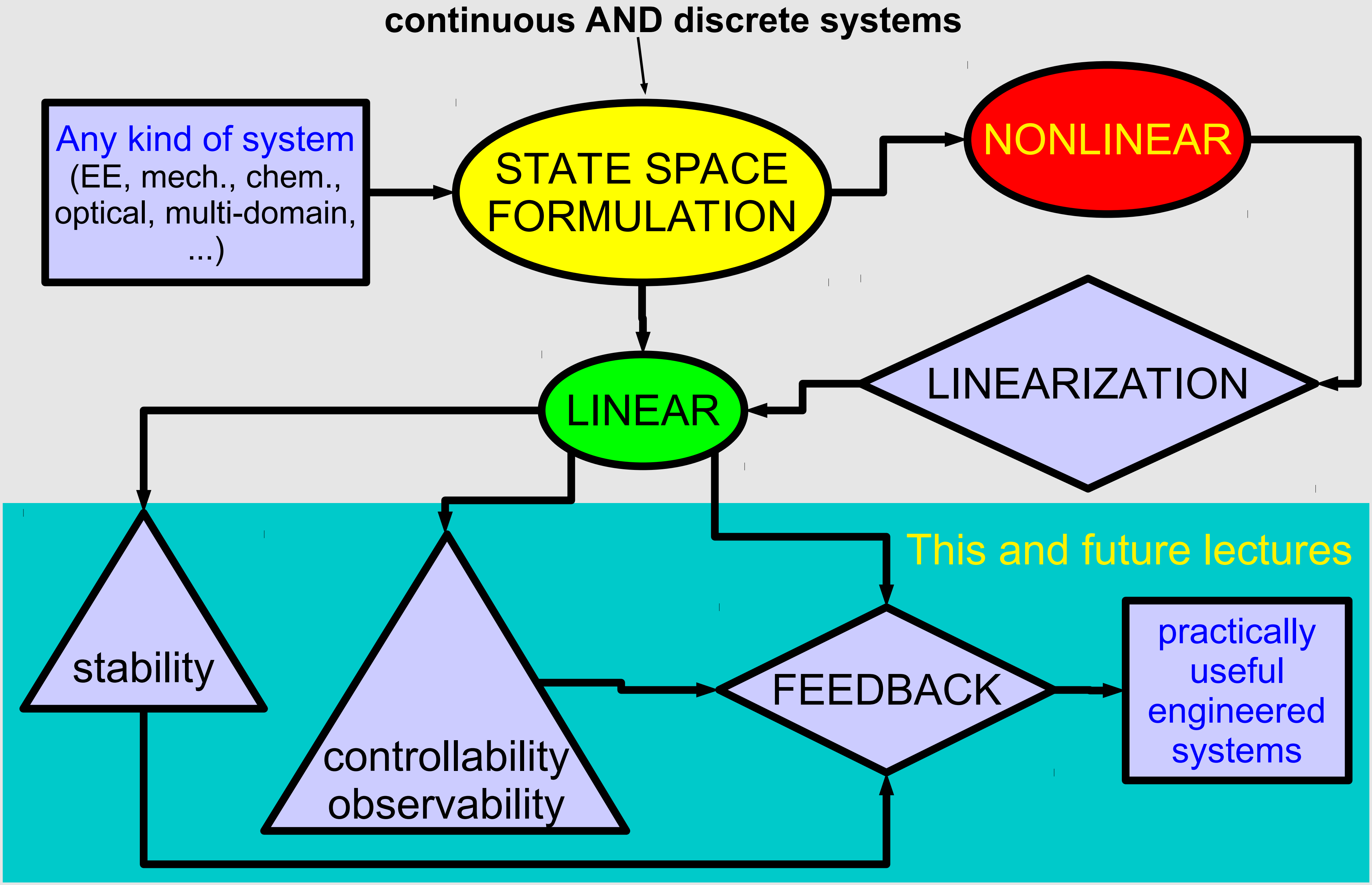
$$\longrightarrow J_{\vec{x}} = \overset{\curvearrowright \text{ 2×2 matrix}}{\begin{bmatrix} 0 & 1 \\ -g/\ell \underbrace{\cos(\theta^*)}_{1} & -k/m \end{bmatrix}}; \quad J_{\vec{u}} = \overset{\text{2×1 "matrix"}}{\begin{bmatrix} 0 \\ +\frac{1}{m\ell} \end{bmatrix}}$$

$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ +\frac{u(t)}{m\ell} \end{bmatrix}}$$

- Compare against sin(θ)≈θ approximation (prev. class)

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -\frac{\alpha}{m} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m\ell} \end{bmatrix} \vec{u}(t)$$

Where We Are Now



continuous AND discrete systems

Any kind of system
(EE, mech., chem., optical, multi-domain, ...)

STATE SPACE FORMULATION

NONLINEAR

LINEARIZATION

LINEAR

This and future lectures

stability

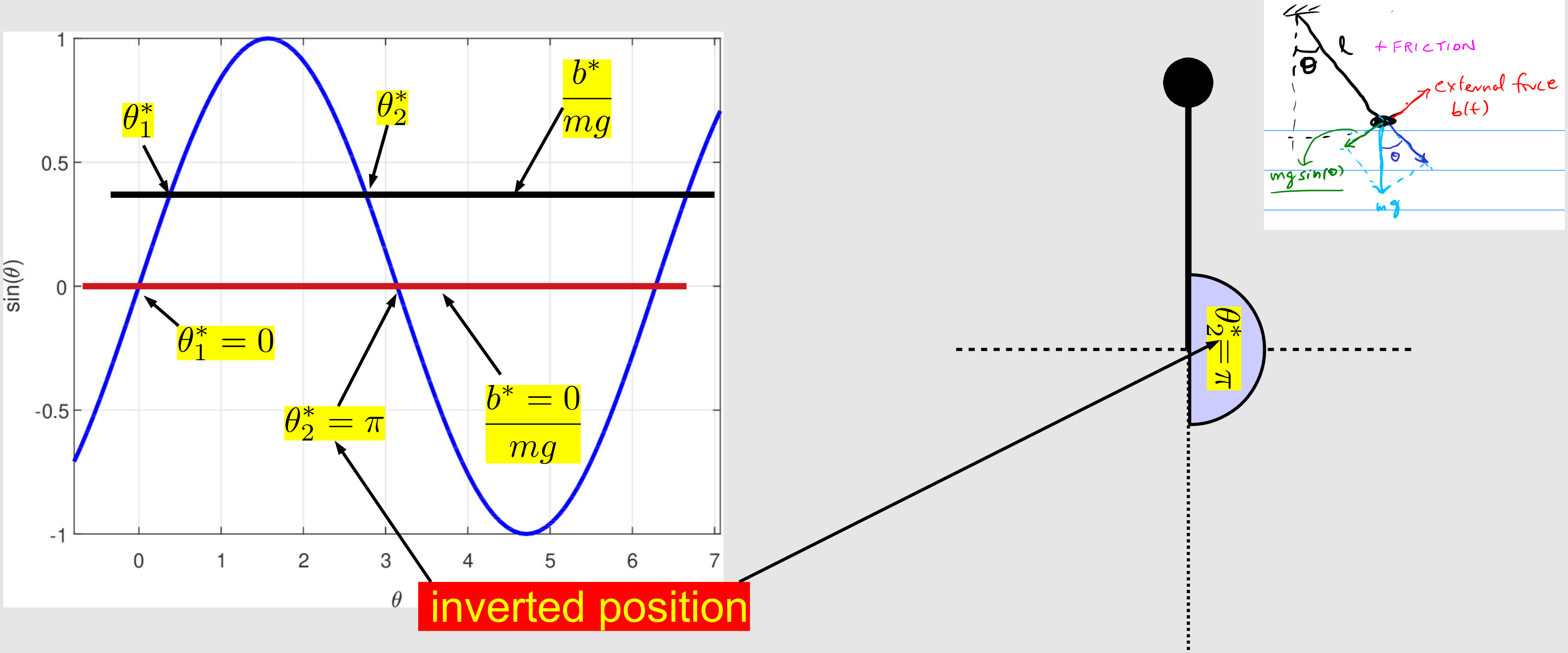
controllability
observability

FEEDBACK

practically useful engineered systems

Pendulum: Inverted Solution

- Pendulum: $$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -g/l \sin(\theta) - k/m\, v_\theta + \frac{b(t)}{ml} \end{bmatrix}$$ $$\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}, \quad \vec{u} = [b(t)]$$
- DC input: b(t) = b*
- DC solution: dx/dt = 0 $\longrightarrow$ $v_\theta = 0$, $\frac{g}{l}\sin(\theta) = \frac{b^*}{ml} \Rightarrow \sin(\theta) = \frac{b^*}{mg}$



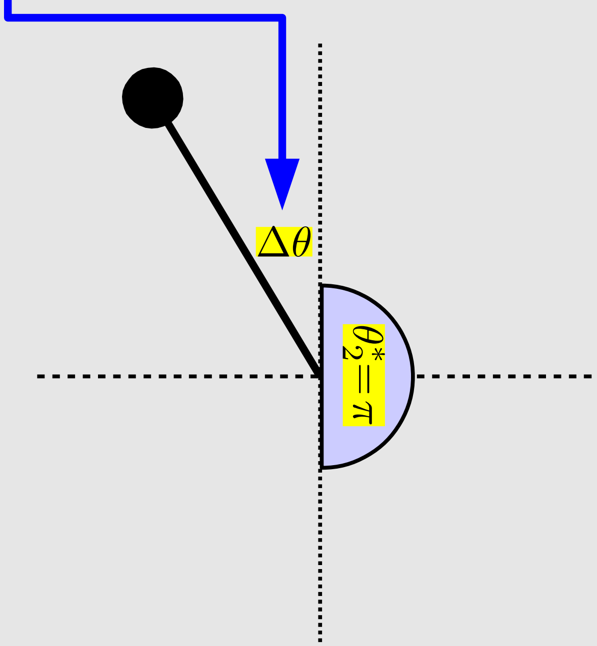
Inverted Pendulum: Linearization

- $$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -g/l \sin(\theta) - k/m \, v_\theta + \frac{b(t)}{ml} \end{bmatrix}$$ $$\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}, \quad \vec{u} = [b(t)]$$

non-inverted

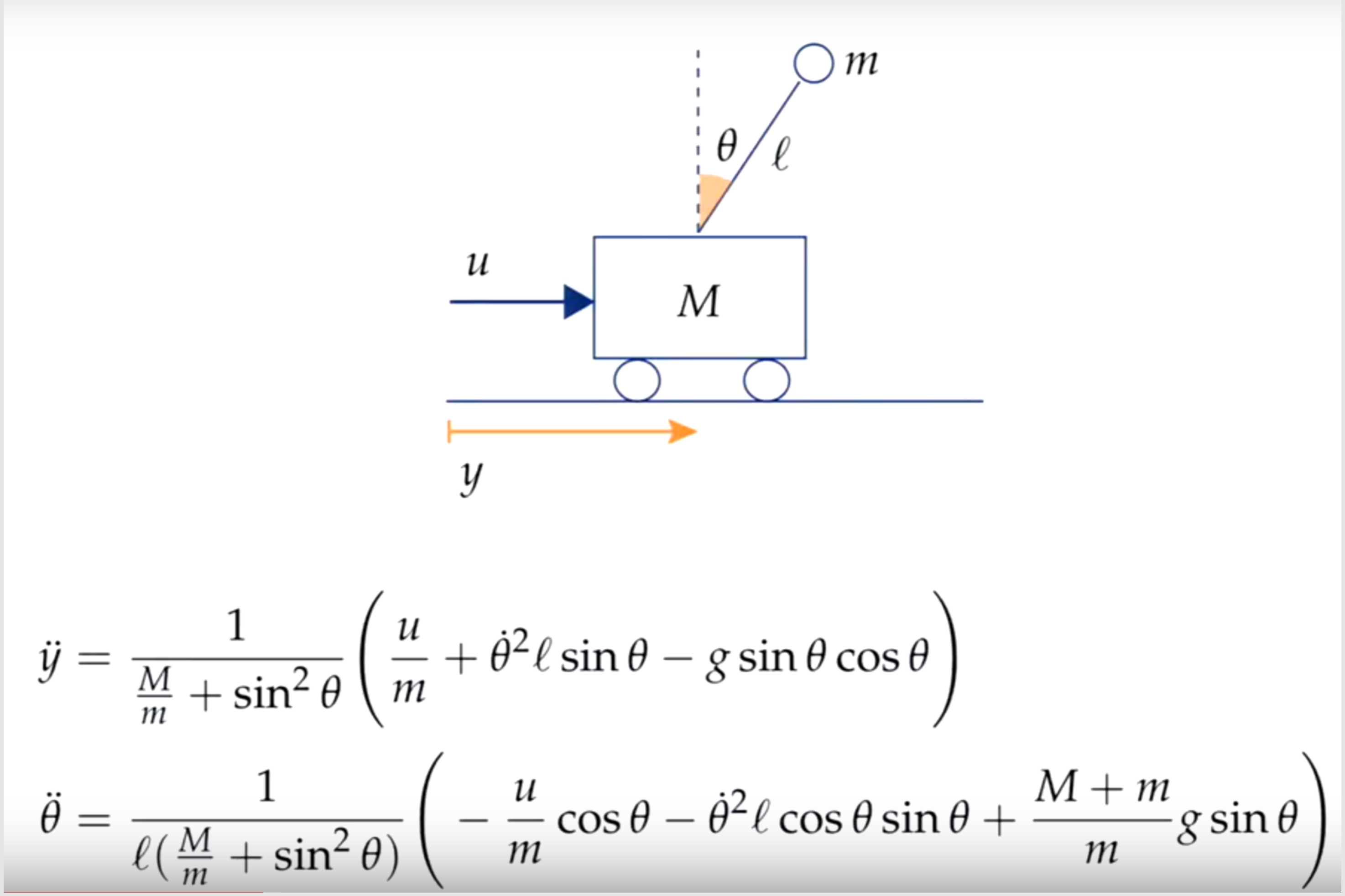
$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/l & -k/m \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ +\frac{u(t)}{ml} \end{bmatrix}$$

- $$J_x(\vec{x}_2^*, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(\theta_2^*) & -\frac{k}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ +\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$
- Linearization: $$\frac{d}{dt}\begin{bmatrix} \Delta\theta(t) \\ \Delta v_\theta(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ +\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} \Delta\theta(t) \\ \Delta v_\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix} \Delta b(t)$$



Pole & Cart (Inverted Pendulum ++)

- Slightly more complicated example



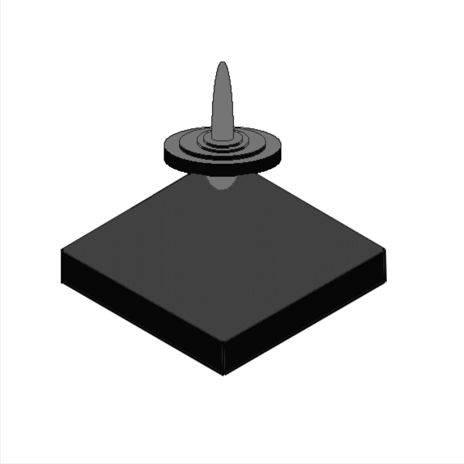
$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 \ell \cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right)$$

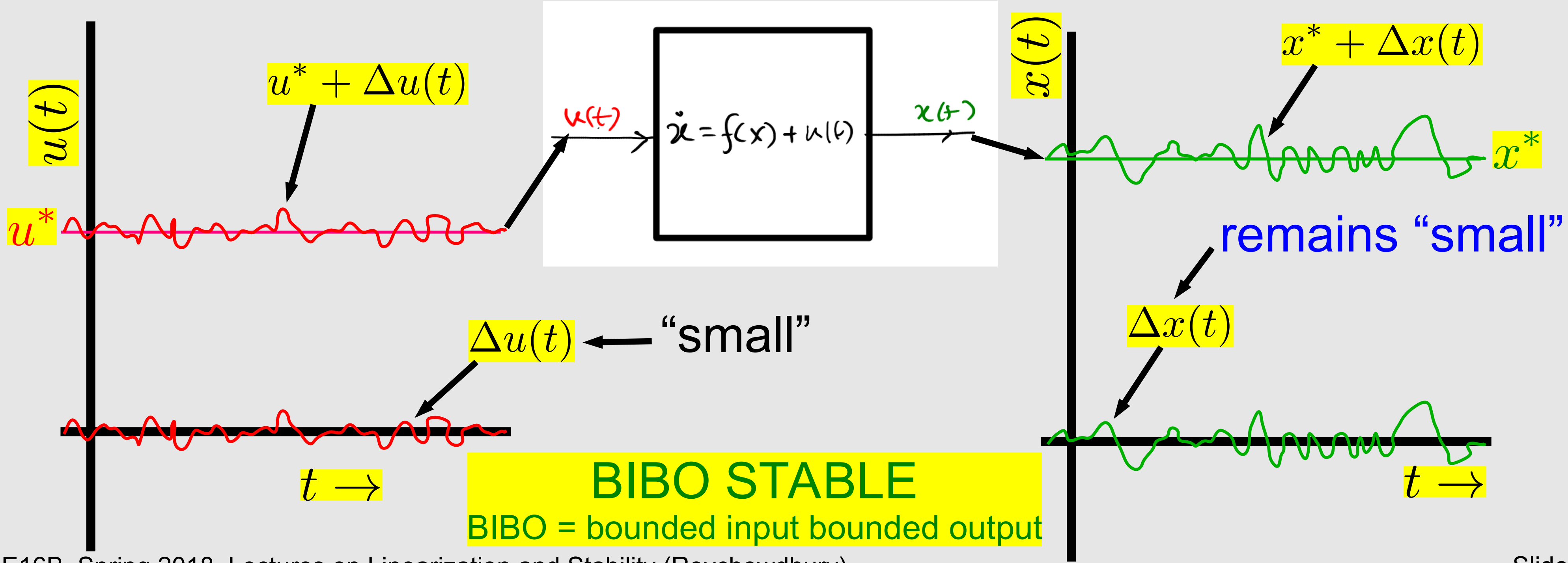
- → discussion / HW

Stability

- Basic idea: **perturb** system **a little** from equilibrium
  - does it come back? yes → STABLE



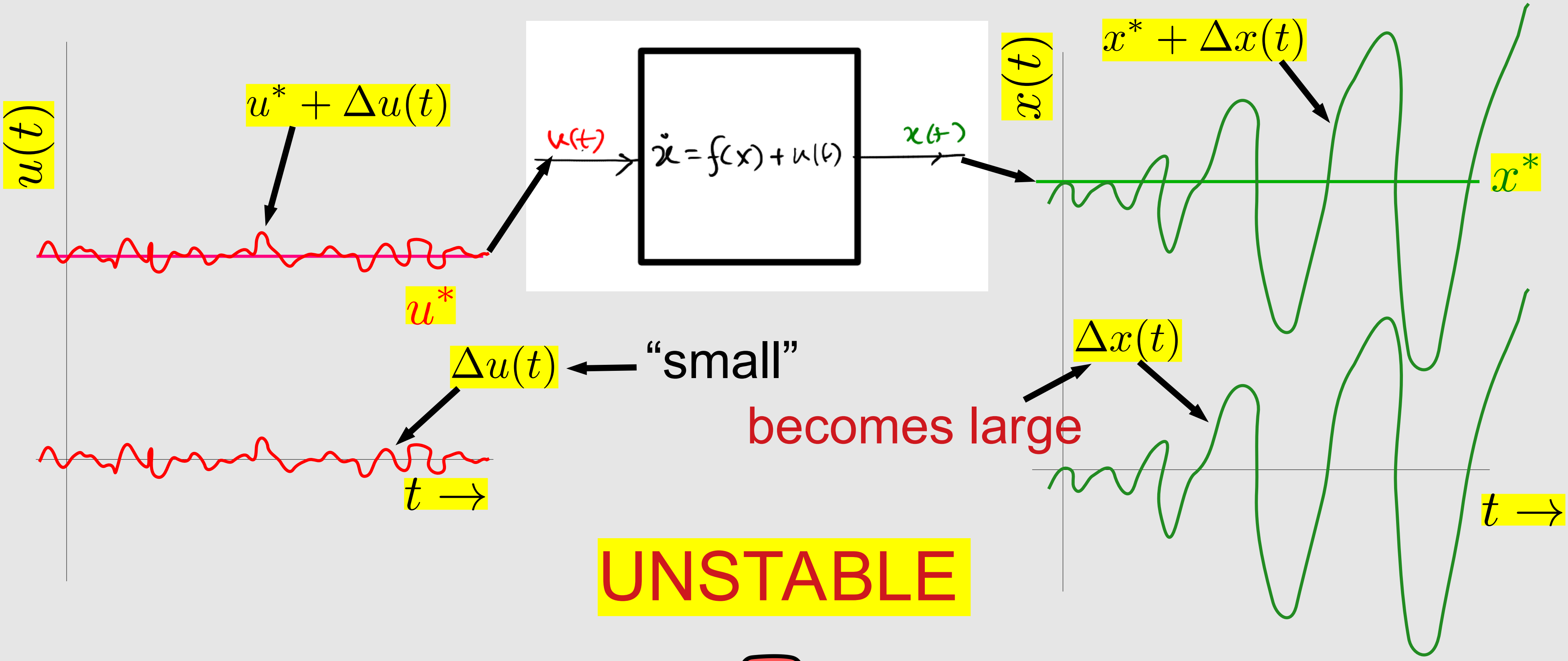
- More precisely:
  - small perturbations → small responses



BIBO STABLE

BIBO = bounded input bounded output

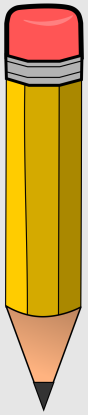
Stability (contd. - 2)



“small”

becomes large

UNSTABLE



Stability: the Scalar Case

real

- Analysis: start w scalar case: $\frac{d}{dt}\Delta x(t) = a\Delta x(t) + b\Delta u(t)$
  - [already linear(ized); everything is real]

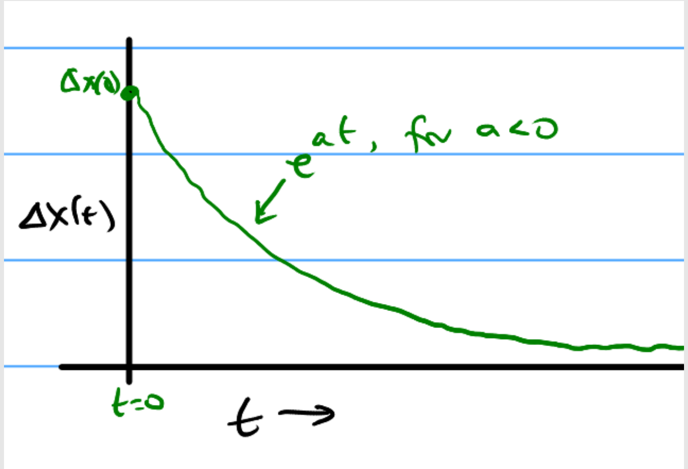
initial condition term

input term **(convolution)**

- Solution: $\Delta x(t) = \Delta x(0)e^{at} + \int_0^t e^{a(t-\tau)}\; b\Delta u(\tau)\, d\tau$ ← $e^{at} * (b\Delta u(t))$
  - [obtained by, eg, the method of integrating factors (Piazza: @88)]
- The initial condition term: $\Delta x(0)e^{at}$. Say $\Delta x(0) \neq 0.$

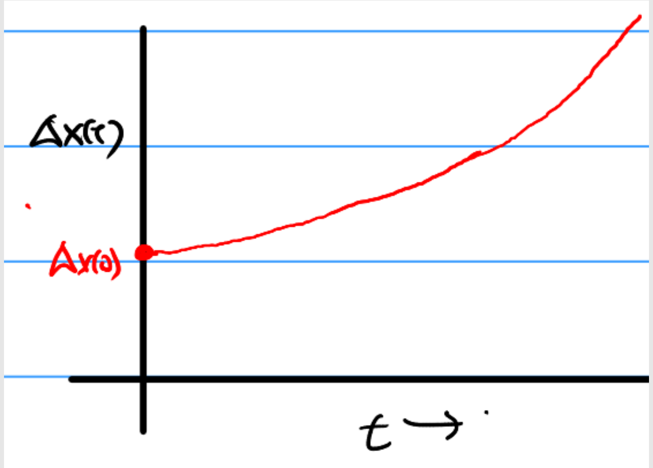
a<0: dies down

STABLE



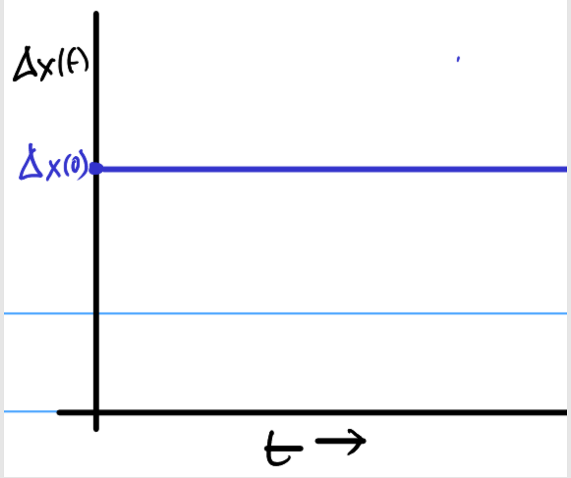
a>0: blows up

UNSTABLE



a=0: stays the same

MARGINALLY STABLE



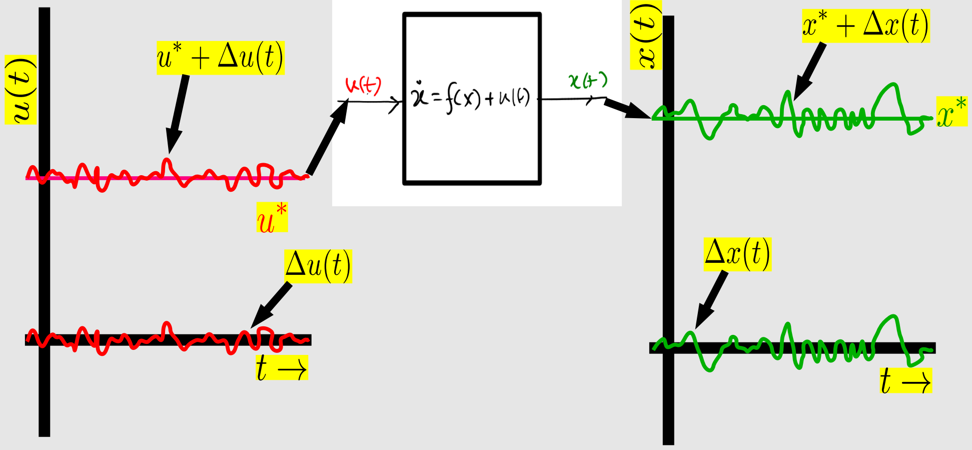
Stability: Scalar Case (contd.)

- Solution: $\Delta x(t) = \Delta x(0)e^{at} + \int_0^t e^{a(t-\tau)}\; b\Delta u(\tau)\, d\tau$ ← $e^{at} * (b\Delta u(t))$

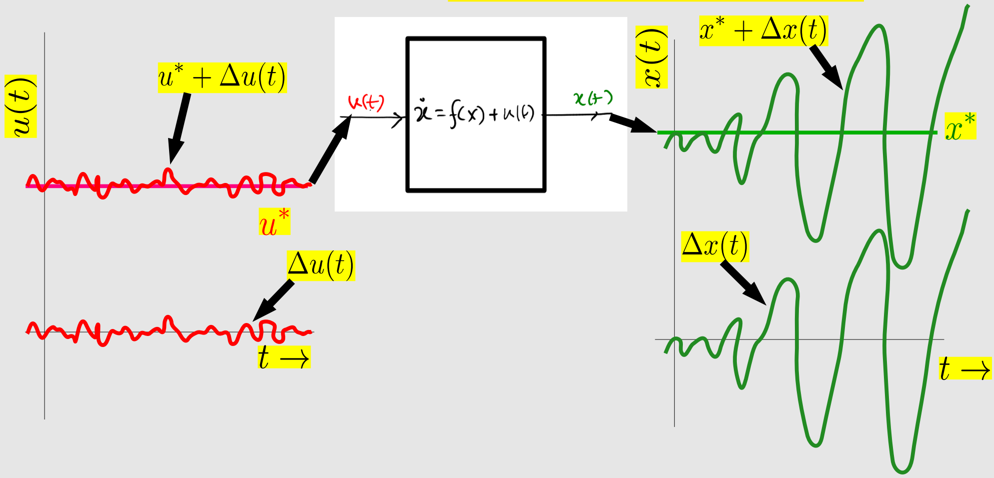
  input term **(convolution)**

- Can show (see handwritten notes):
  - if a<0: $e^{at} * (b\Delta u(t))$ bounded if Δu(t) bounded: **BIBO stable**
  - if a>0: $e^{at} * (b\Delta u(t))$ unbounded even if Δu(t) bounded: **UNSTABLE**
  - if a=0: $e^{at} * (b\Delta u(t))$ unbounded even if Δu(t) bounded: **UNSTABLE**

a<0: BIBO STABLE



a≥0: UNSTABLE



The Vector Case: Eigendecomposition

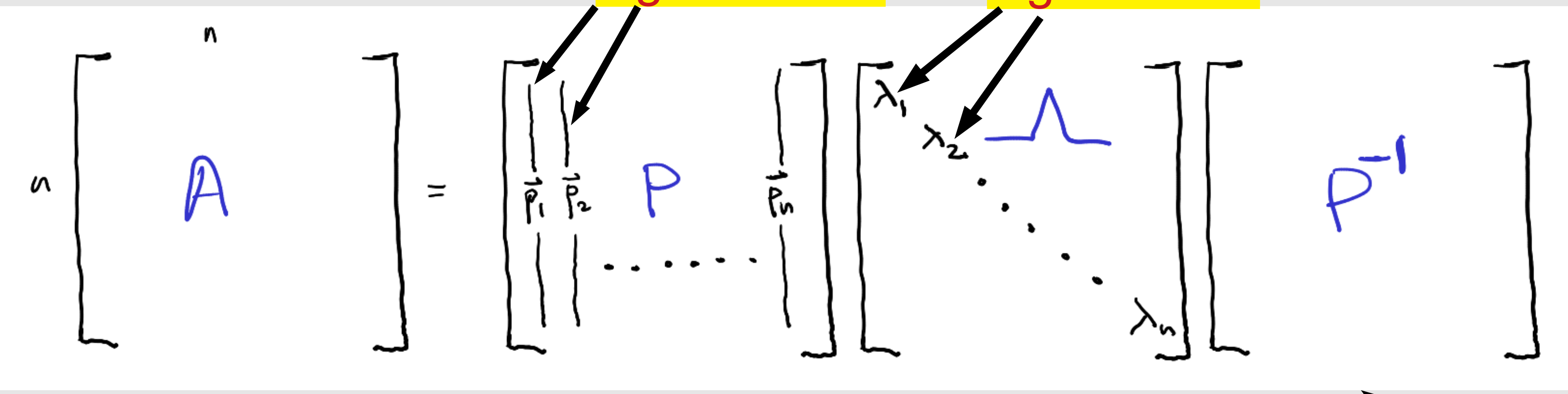
- The vector case: $\frac{d}{dt}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + B\Delta\vec{u}(t)$
  - [already linear(ized); everything is real] real matrices
- Can be “decomposed” into n scalar systems
  - the key idea: to eigendecompose A

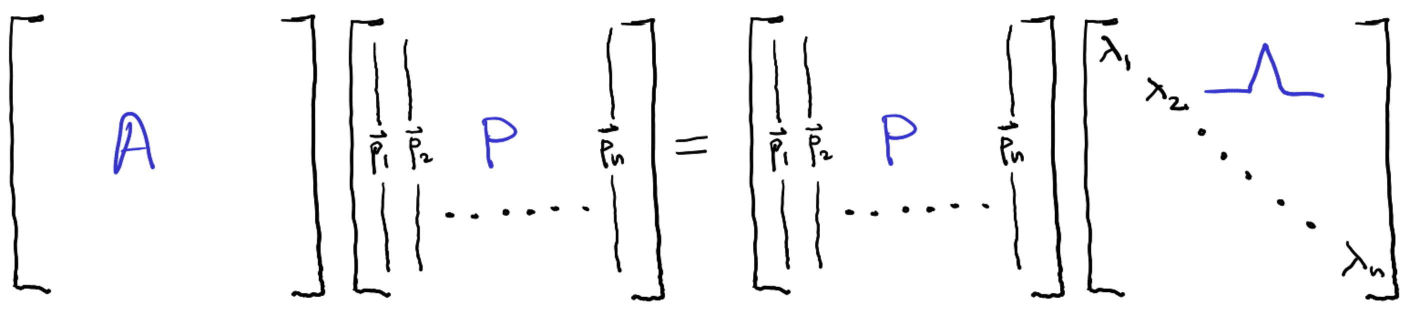
if time, move to xournal

- (recap) eigendecomposition: given an nxn matrix A:*

eigenvectors

eigenvalues





same thing

$$A\vec{p}_i = \lambda_i\vec{p}_i$$

$i = 1, \cdots, n$

---

* diagonalization always possible if all eigenvalues distinct (assumed)

Eigendecomposition (contd.)

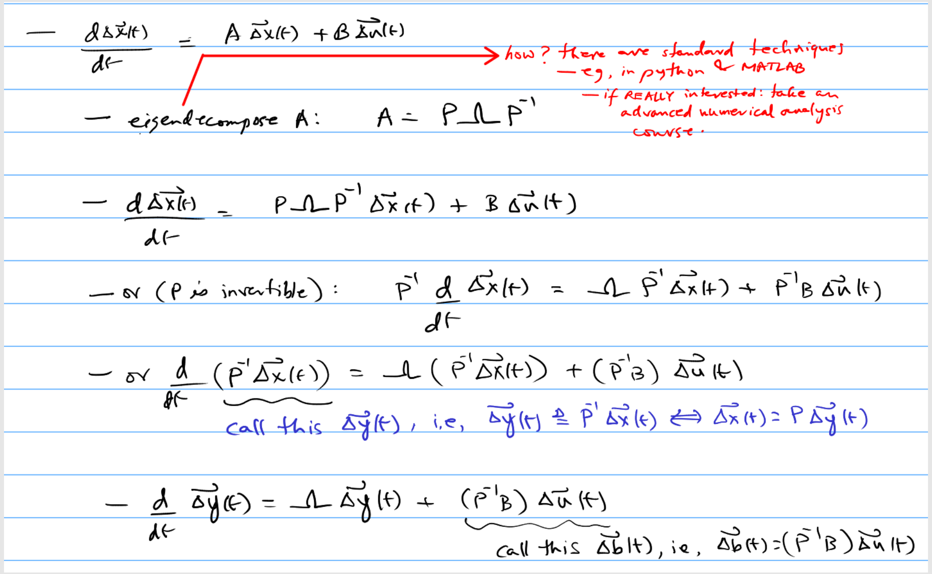
- eigenvalues and determinants
  - $A\vec{p} = \lambda\vec{p} \Leftrightarrow \boxed{(A - \lambda I)}\,\vec{p} = \vec{0}$

    must be singular in order to support a non-zero solution for $\vec{p}$
  - i.e., $det(A - \lambda I) = 0$
- $p_A(\lambda) \triangleq \boxed{det(A - \lambda I) = \lambda^n + c_n\lambda^{n-1} + \cdots + c_2\lambda + c_1}$

  characteristic polynomial of A
- the roots of the char. poly. are the eigenvalues
  - factorized form: $p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0$
  - in general, n roots → n eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

The Vector Case: Diagonalization

- Applying eigendecomposition: diagonalization
  - (move to xournal)



— $\frac{d\vec{\Delta x}(t)}{dt} = A\vec{\Delta x}(t) + B\vec{\Delta u}(t)$

— eigendecompose A: $A = P\Lambda P^{-1}$

how? there are standard techniques
— eg, in python & MATLAB
— if REALLY interested: take an advanced numerical analysis course.

— $\frac{d\vec{\Delta x}(t)}{dt} = P\Lambda P^{-1}\vec{\Delta x}(t) + B\vec{\Delta u}(t)$

— or (P is invertible): $P^{-1}\frac{d}{dt}\vec{\Delta x}(t) = \Lambda P^{-1}\vec{\Delta x}(t) + P^{-1}B\vec{\Delta u}(t)$

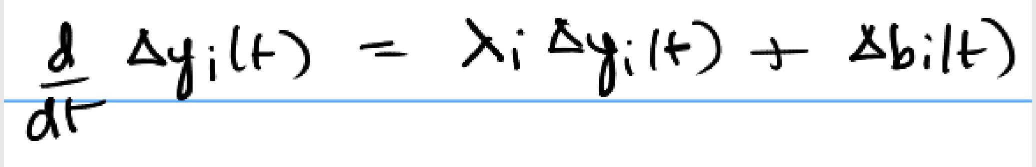
— or $\frac{d}{dt}\underbrace{(P^{-1}\vec{\Delta x}(t))}_{} = \Lambda(P^{-1}\vec{\Delta x}(t)) + (P^{-1}B)\vec{\Delta u}(t)$

call this $\vec{\Delta y}(t)$, i.e, $\vec{\Delta y}(t) \triangleq P^{-1}\vec{\Delta x}(t) \Leftrightarrow \vec{\Delta x}(t) = P\vec{\Delta y}(t)$

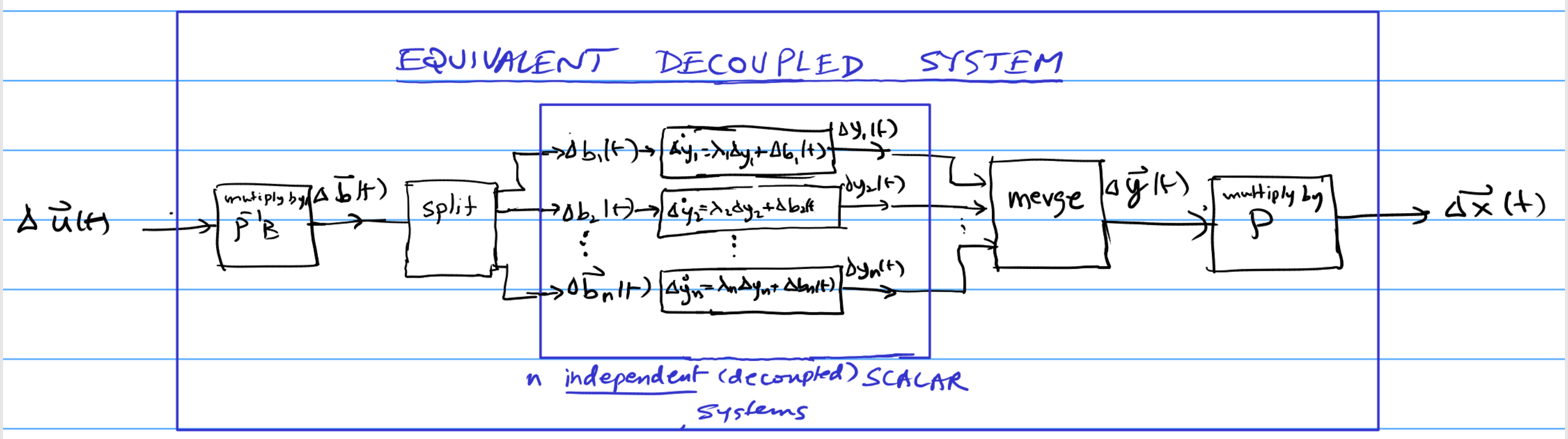
— $\frac{d}{dt}\vec{\Delta y}(t) = \Lambda\vec{\Delta y}(t) + \underbrace{(P^{-1}B)\vec{\Delta u}(t)}_{}$

call this $\vec{\Delta b}(t)$, ie, $\vec{\Delta b}(t) = (P^{-1}B)\vec{\Delta u}(t)$

$$\frac{d}{dt}\begin{bmatrix}\Delta y_1(t)\\ \Delta y_2(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}\Delta y_1(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} + \begin{bmatrix}\Delta b_1(t)\\ \Delta b_2(t)\\ \vdots\\ \Delta b_n(t)\end{bmatrix}$$



$$\frac{d}{dt}\Delta y_i(t) = \lambda_i\Delta y_i(t) + \Delta b_i(t)$$



Stability: the Vector Case

- $$\frac{d}{dt}\Delta y_i(t) = \lambda_i \Delta y_i(t) + \Delta b_i(t) \qquad i = 1, \cdots, n$$

provided λ_i is REAL → $\lambda_i < 0$

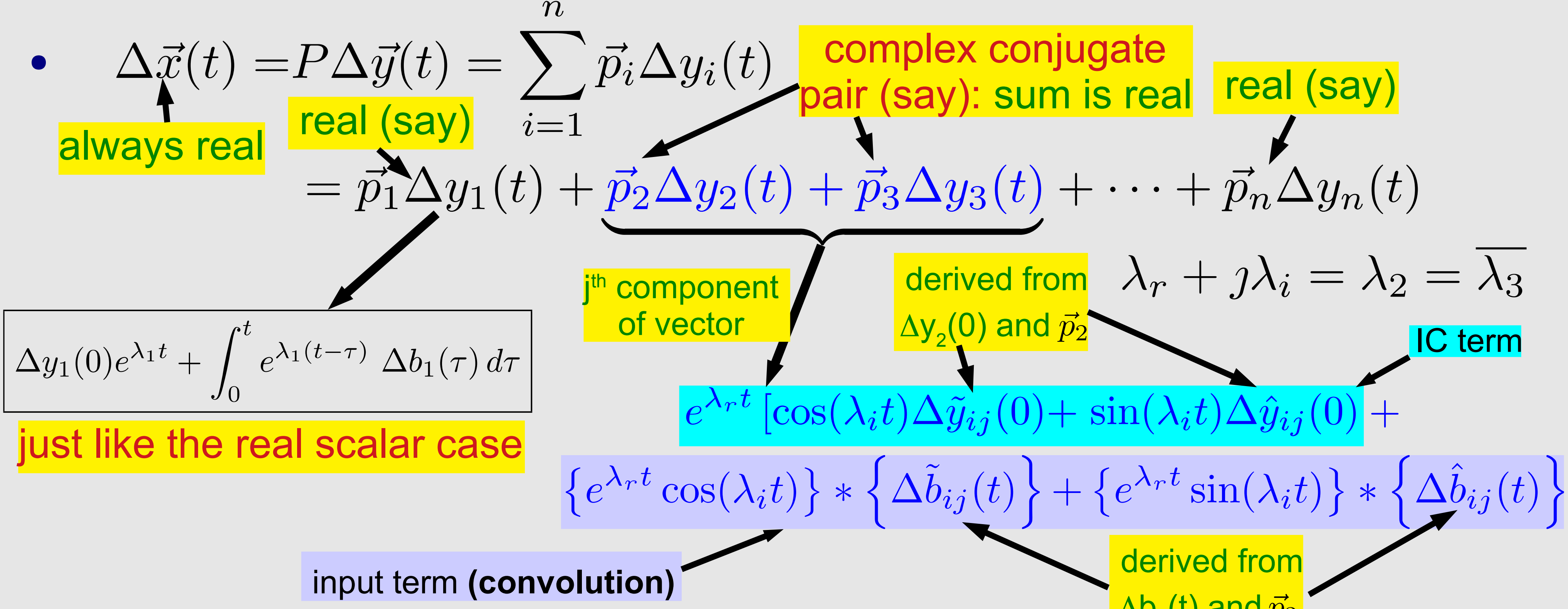
- System stable if **each system is stable**
- **Complication: eigenvalues can be complex**
  - reason: real matrices A can have complex eigen{vals,vecs}
  - examples: (also demo in MATLAB)

➔ $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \overset{P}{\begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} j & \\ & -j \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}} \frac{1}{\sqrt{2}}$$

➔ $$\begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} = \overset{P}{\begin{bmatrix} \frac{1+j}{2} & \frac{1-j}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} \frac{-1+j}{\sqrt{2}} & \\ & \frac{-1-j}{\sqrt{2}} \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} -j & \frac{1+j}{\sqrt{2}} \\ j & \frac{1-j}{\sqrt{2}} \end{bmatrix}}$$

Stability: the Vector Case (contd.)

- If A real, eigen{v,v}s come in **complex conjugate pairs**
  - $A\vec{p}_i = \lambda_i\vec{p}_i \Rightarrow \overline{A}\,\overline{\vec{p}_i} = \overline{\lambda_i}\,\overline{\vec{p}_i} \Rightarrow A\,\overline{\vec{p}_i} = \overline{\lambda_i}\,\overline{\vec{p}_i}$
- Implications (details in handwritten notes)
  - internal quantities in the decomposition come in conjugate pairs
    - the rows of P^{-1}, $\Delta b_i(t)$, the eigenvalues λ_i, $\Delta y_i(t)$, the cols of P $\vec{p}_i$
- $\Delta\vec{x}(t) = P\Delta\vec{y}(t) = \sum_{i=1}^{n}\vec{p}_i\Delta y_i(t)$



always real

real (say)

complex conjugate pair (say): sum is real

real (say)

$$= \vec{p}_1\Delta y_1(t) + \underbrace{\vec{p}_2\Delta y_2(t) + \vec{p}_3\Delta y_3(t)}_{} + \cdots + \vec{p}_n\Delta y_n(t)$$

$$\lambda_r + \jmath\lambda_i = \lambda_2 = \overline{\lambda_3}$$

$$\boxed{\Delta y_1(0)e^{\lambda_1 t} + \int_0^t e^{\lambda_1(t-\tau)}\ \Delta b_1(\tau)\,d\tau}$$

just like the real scalar case

jth component of vector

derived from $\Delta y_2(0)$ and $\vec{p}_2$

IC term

$$e^{\lambda_r t}\left[\cos(\lambda_i t)\Delta\tilde{y}_{ij}(0) + \sin(\lambda_i t)\Delta\hat{y}_{ij}(0)\right. +$$

$$\left\{e^{\lambda_r t}\cos(\lambda_i t)\right\} * \left\{\Delta\tilde{b}_{ij}(t)\right\} + \left\{e^{\lambda_r t}\sin(\lambda_i t)\right\} * \left\{\Delta\hat{b}_{ij}(t)\right\}$$

input term **(convolution)**

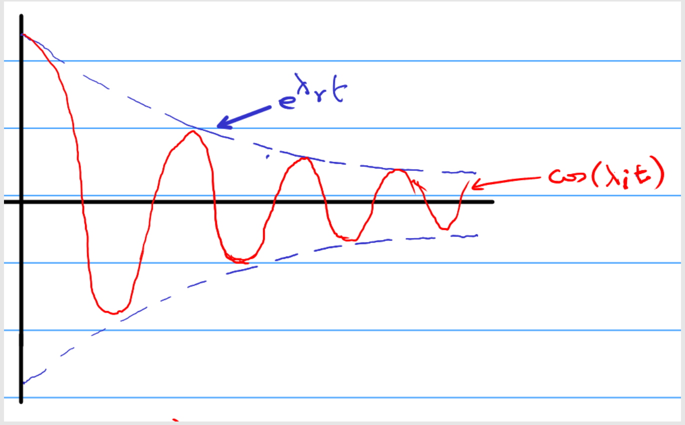
derived from $\Delta b_2(t)$ and $\vec{p}_2$

Stability: the Vector Case (contd. - 2)

- Initial condition terms: $e^{\lambda_r t}\left[\cos(\lambda_i t)\Delta\tilde{y}_{ij}(0) + \sin(\lambda_i t)\Delta\hat{y}_{ij}(0)\right]$

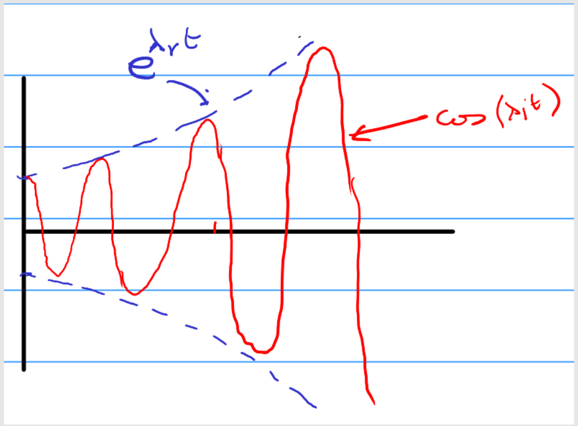
$\lambda_r<0$: envelope dies down

STABLE



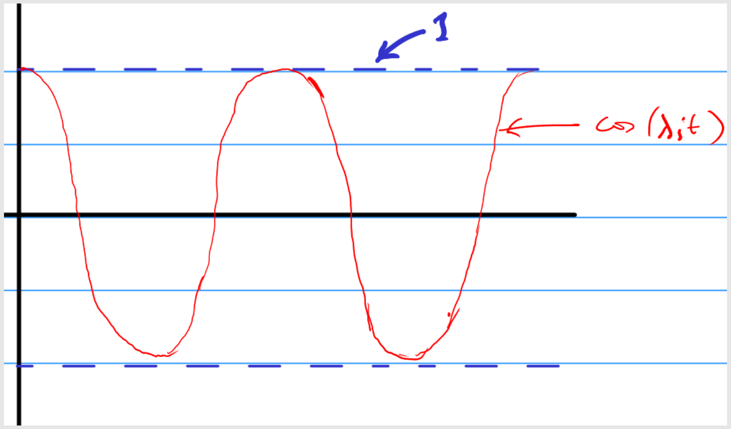
$\lambda_r>0$: envelope blows up

UNSTABLE



$\lambda_r=0$: const. envelope

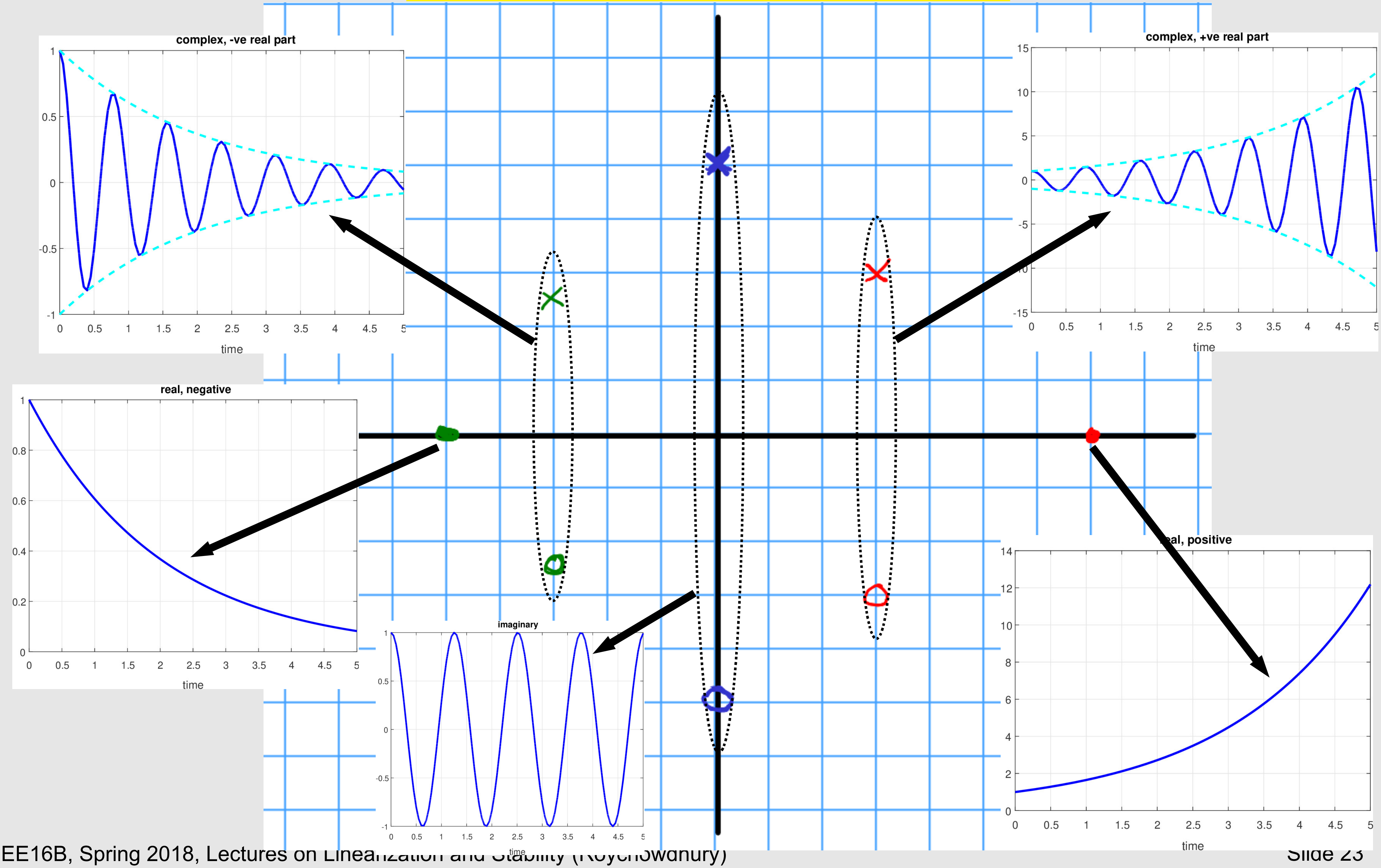
MARGINALLY STABLE



- Input conv. terms: $\{e^{\lambda_r t}\cos(\lambda_i t)\} * \left\{\Delta\tilde{b}_{ij}(t)\right\} + \{e^{\lambda_r t}\sin(\lambda_i t)\} * \left\{\Delta\hat{b}_{ij}(t)\right\}$
  - can show (see notes) that: same as for real eigenvalues, but using the real parts of complex eigenvalues
    - if $\lambda_r<0$: bounded if $\Delta u(t)$ bounded: **BIBO stable**
    - if $\lambda_r>0$: unbounded even if $\Delta u(t)$ bounded: **UNSTABLE**
    - if $\lambda_r=0$: unbounded even if $\Delta u(t)$ bounded: **UNSTABLE**

Eigenvalues and Responses (continuous)

complex plane for plotting eigenvalues



Eigenvalues of Linearized Pendulum

- (move to xournal)

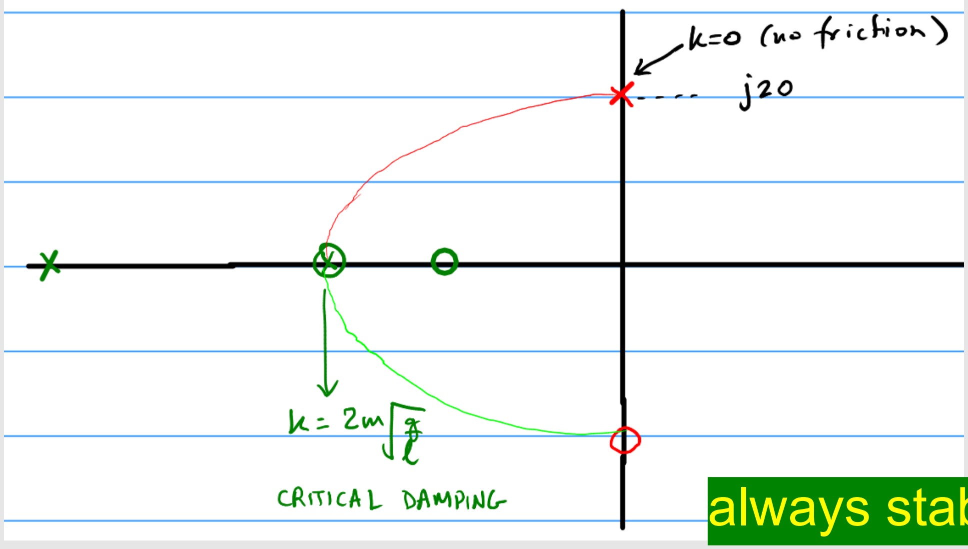
$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ +\frac{u(t)}{m\ell} \end{bmatrix}}$$

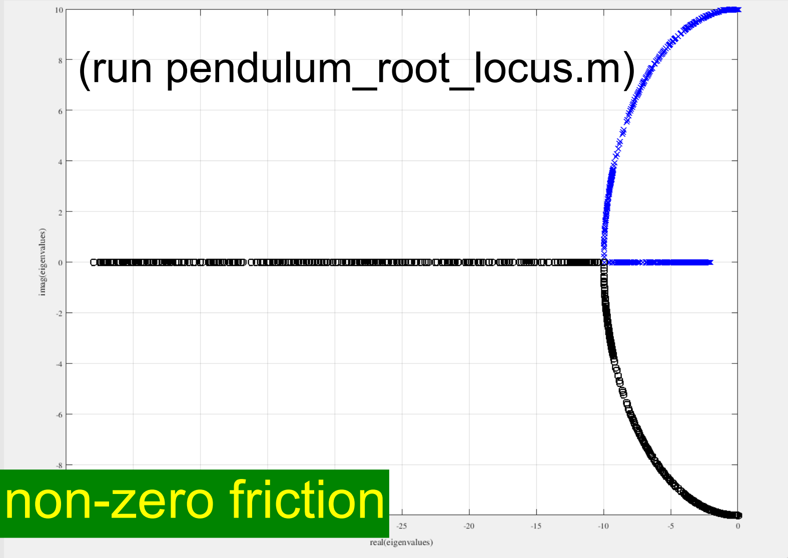
$$\boxed{\lambda_{1,2} = \frac{-k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - \frac{4g}{\ell}}}$$

$$A\vec{p} = \lambda\vec{p} \implies (A - \lambda I)\vec{p} = 0 \implies \underbrace{\begin{bmatrix} -\lambda & 1 \\ -g/\ell & -\frac{k}{m} - \lambda \end{bmatrix}}_{\text{det should} = 0} \overset{\text{want non-zero solution}}{\vec{p}} = 0$$

$$\lambda(\lambda + k/m) + \frac{g}{\ell} = 0 \implies \lambda^2 + \frac{k}{m}\lambda + \frac{g}{\ell} = 0 \implies \boxed{\lambda_{1,2} = \frac{-k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - \frac{4g}{\ell}}}$$

plot eigenvalues as k changes





(run pendulum_root_locus.m)

always stable if non-zero friction

Eigenvalues of Inverted Pendulum

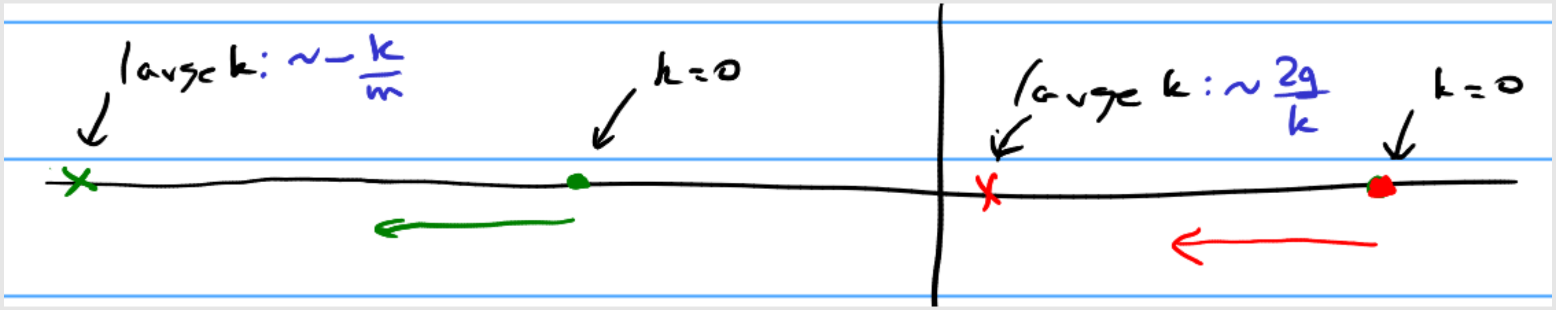
- $A = J_x = \begin{bmatrix} 0 & 1 \\ +\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \boxed{\frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}}$$

always real, always greater than $\frac{k}{2m}$

one eigenvalue always positive!

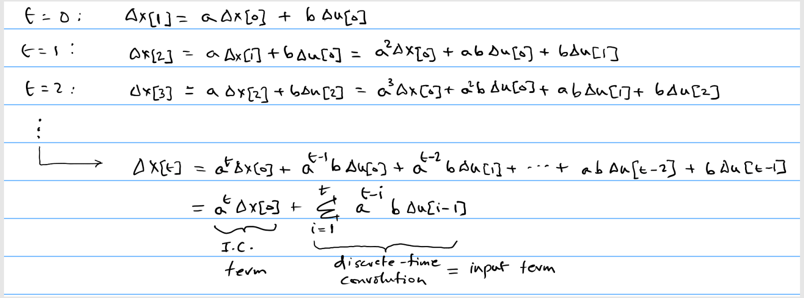
always unstable!



Stability for Discrete Time Systems

real

- The scalar case: $\Delta x[t+1] = a\Delta x[t] + b\Delta u[t]$, IC Δx[0]
  - ➔ [already linear(ized); everything is real]
  - ➔ (move to xournal?)



$t=0$: $\Delta x[1] = a\Delta x[0] + b\Delta u[0]$

$t=1$: $\Delta x[2] = a\Delta x[1] + b\Delta u[0] = a^2\Delta x[0] + ab\Delta u[0] + b\Delta u[1]$

$t=2$: $\Delta x[3] = a\Delta x[2] + b\Delta u[2] = a^3\Delta x[0] + a^2 b\Delta u[0] + ab\Delta u[1] + b\Delta u[2]$

$\vdots$

$$\longrightarrow \Delta x[t] = a^t\Delta x[0] + a^{t-1}b\Delta u[0] + a^{t-2}b\Delta u[1] + \cdots + ab\Delta u[t-2] + b\Delta u[t-1]$$

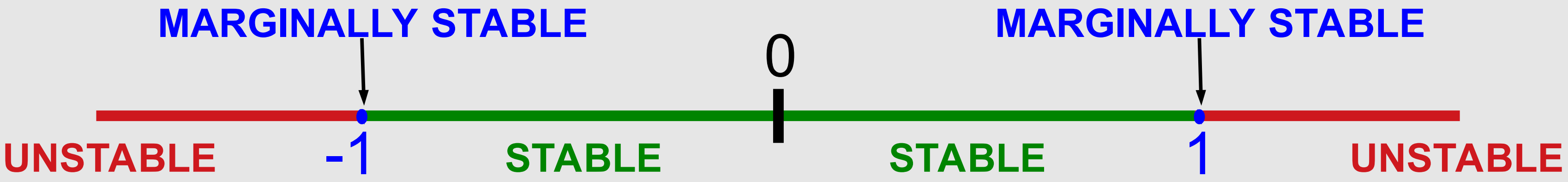
$$= \underbrace{a^t\Delta x[0]}_{\text{I.C. term}} + \underbrace{\sum_{i=1}^{t} a^{t-i} b\Delta u[i-1]}_{\text{discrete-time convolution}} = \text{input term}$$

- $\Delta x[t] = a^t\Delta x[0] + \sum_{i=1}^{t} a^{t-i} b\Delta u[i-1]$

IC term

input term **(discrete convolution)**

- Initial Condition term: $a^t\Delta x[0]$

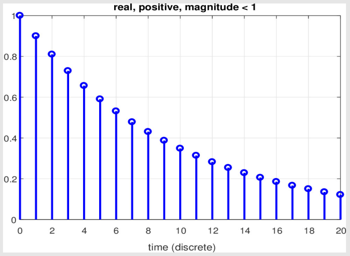


MARGINALLY STABLE **MARGINALLY STABLE**

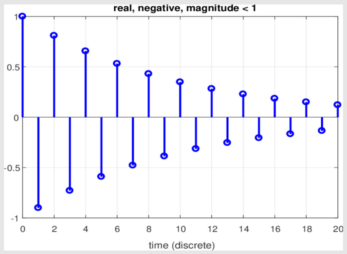
0

UNSTABLE -1 **STABLE** **STABLE** 1 **UNSTABLE**

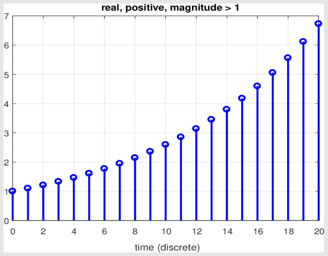
0<a<1: dies down
STABLE



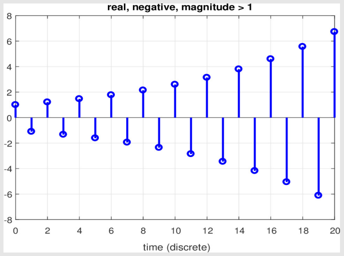
-1<a<0: dies down
STABLE



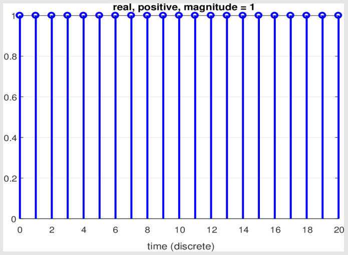
a>1: blows up
UNSTABLE



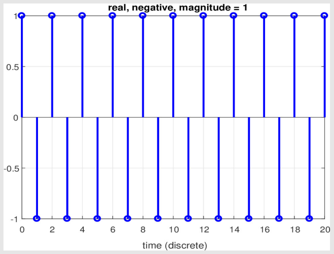
a<-1: blows up
UNSTABLE



a=1: constant
MARGINALLY STABLE



a=-1: constant
MARGINALLY STABLE



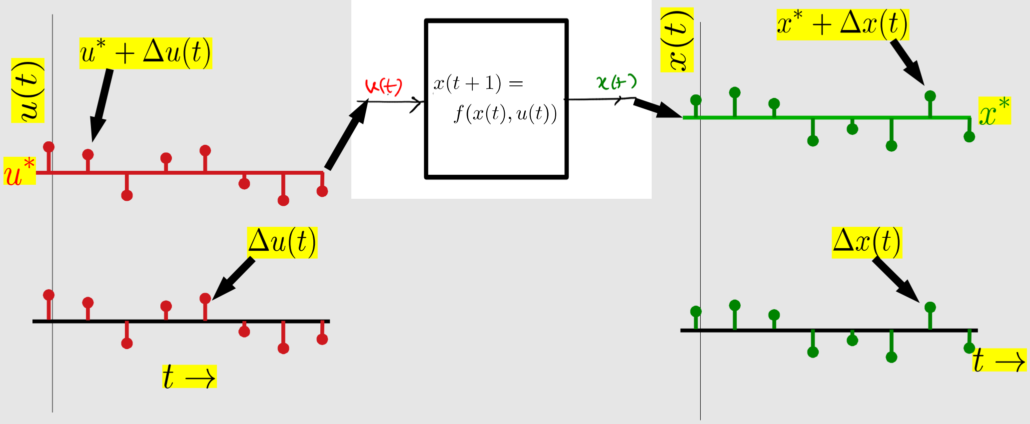
Scalar Discrete-Time Stability (contd.)

- Solution: $\Delta x[t] = a^t \Delta x[0] + \sum_{i=1}^{t} a^{t-i} b \Delta u[i-1]$

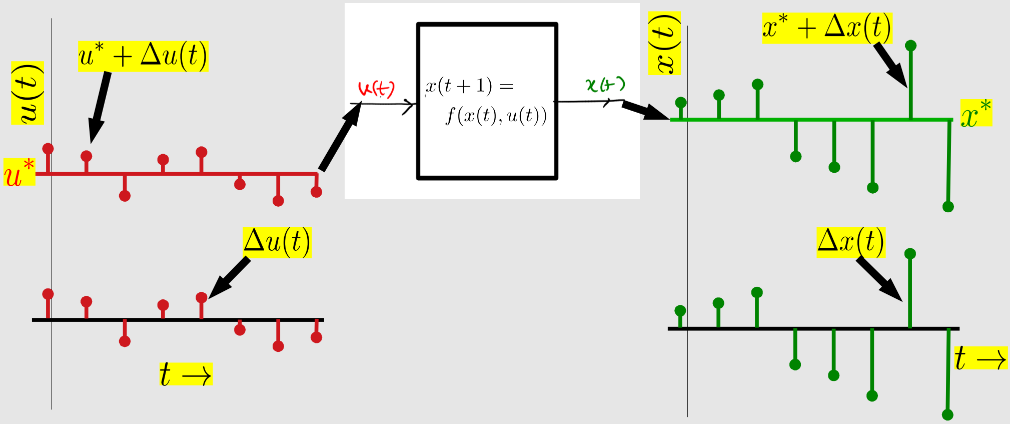
input term **(d. convolution)**

- Can show (see handwritten notes):
  - if |a|<1: bounded if Δu(t) bounded: **BIBO stable**
  - if |a|>1: unbounded even if Δu(t) bounded: **UNSTABLE**
  - if |a|=1: unbounded even if Δu(t) bounded: **UNSTABLE**

|a|<1: BIBO STABLE

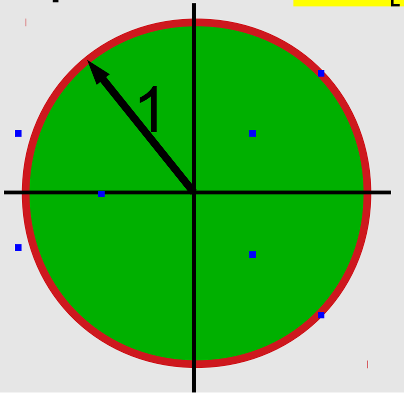


|a|≥1: UNSTABLE



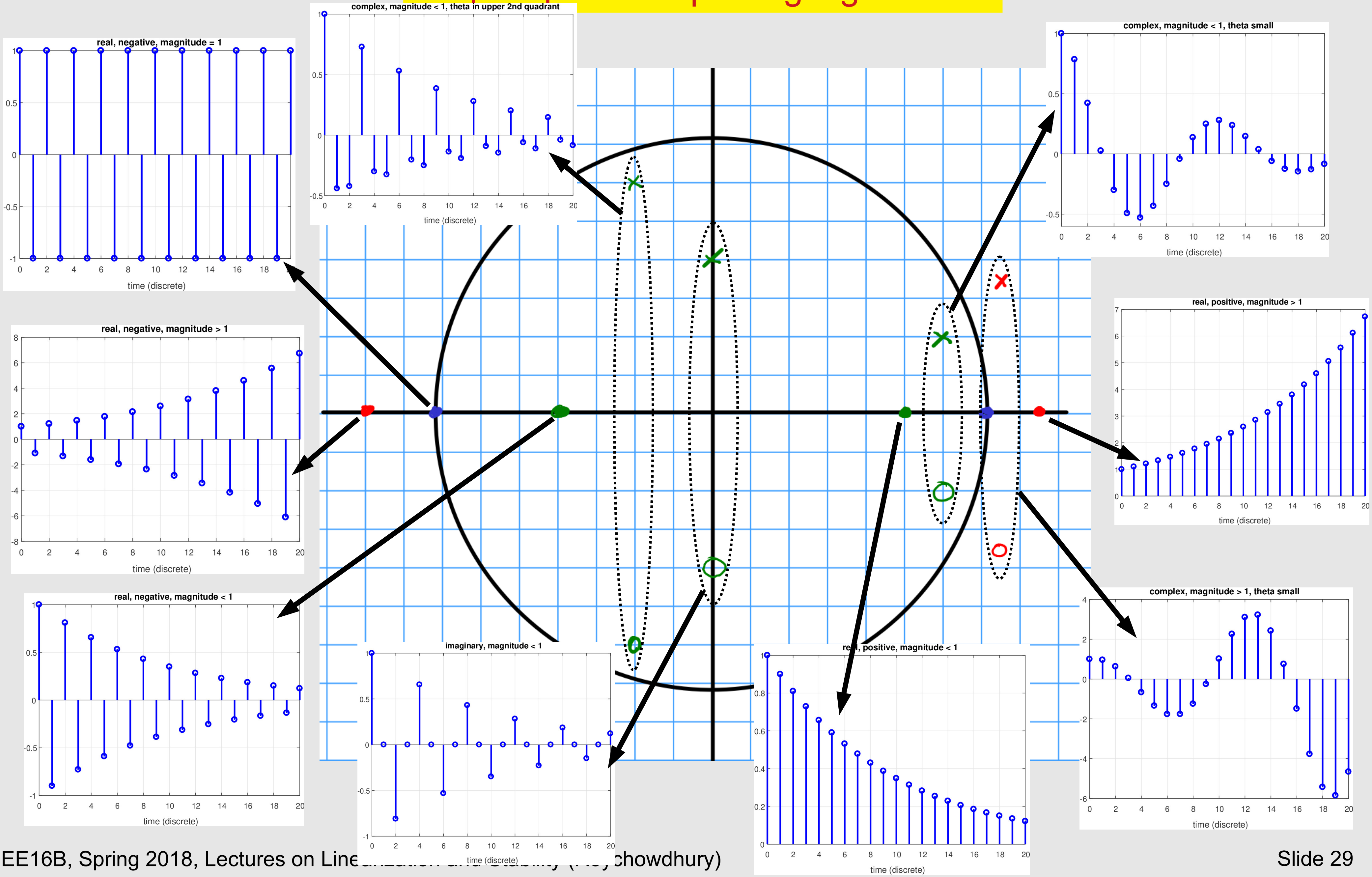
Discrete Time Stability: the Vector Case

- **The vector case:** $\Delta\vec{x}[t+1] = A\Delta\vec{x}[t] + B\Delta\vec{u}[t]$
  - [already linear(ized); everything is real] real matrices
- Eigendecompose A: $A = P\Lambda P^{-1}$
- Define: $\Delta\vec{y}[t] \triangleq P^{-1}\Delta\vec{x}[t] \quad \Leftrightarrow \quad \Delta\vec{x}[t] \triangleq P\Delta\vec{y}[t]$
  - $\Delta\vec{b}[t] \triangleq P^{-1}\Delta\vec{u}[t]$
- Decomposed system: $\Delta\vec{y}_i[t+1] = \underset{\text{scalar}}{\lambda_i}\Delta\vec{y}[t] + \Delta\vec{b}_i[t]$, $i = 1, \cdots, n$
  - same as scalar case, but λ_i now complex
  - same form for $\Delta\vec{x}[t]$ as for the continuous case
    - complex conjugate terms always present in pairs → $\Delta\vec{x}[t]$ real
- Stability:
  - BIBO stable iff $|\lambda_i| < 1$, $i = 1, \cdots, n$



Eigenvalues and IC Responses (discrete)

complex plane for plotting eigenvalues



Summary

- Linearization
  - scalar and vector cases
    - → example: pendulum, (pole-cart)
- Stability
  - scalar and vector cases
    - → continuous: real parts of eigenvalues determine stability
      - pendulum: stable and unstable equilibria
        - eigenvalue vs friction plots (root-locus plots)
    - → discrete: magnitudes of eigenvalues determine stability