EE16B, Spring 2018
UC Berkeley EECS

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Lectures 5B & 6A: Overview Slides

Controllability and Feedback

Controllability

nxn matrix nxm matrix

- Given (linearized) S.S.R: $\Delta\vec{x}[t+1] = A\Delta\vec{x}[t] + B\Delta\vec{u}[t]$
  - can you drive $\Delta\vec{x}[t]$ to any value you want (using $\Delta\vec{u}[t]$)?
    - ie, can you **control** $\Delta\vec{x}[t]$ completely?
  - (move to xournal)
  - say $\Delta\vec{x}[0] = 0$ (w.l.o.g, see notes)

$\vec{\Delta x}[t] = A\vec{\Delta x}[t] + B\Delta\vec{u}[t]$, with $\Delta x[0]$ IC

$\vec{\Delta x}[1] = A\vec{\Delta x}[0] + B\Delta\vec{u}[0]$

$\vec{\Delta x}[2] = A\vec{\Delta x}[1] + B\vec{\Delta u}[1] = A^2\vec{\Delta x}[0] + AB\,\Delta u[0] + B\Delta u[1]$

$\vec{\Delta x}[t] = A^t\vec{\Delta x}[0] + \sum_{i=1}^{t} A^{t-i} B\,\vec{\Delta u}[i-1]$

$$\vec{\Delta x}[t] \qquad + \qquad n\updownarrow \underbrace{\left[\overbrace{A^{t-1}B}^{m} \;\vdots\; A^{t-2}B \;\vdots\; \cdots\cdots \;\vdots\; AB \;\vdots\; B\right]}_{mt} \overbrace{\begin{bmatrix}\Delta\vec{u}[0]\\ \Delta\vec{u}[1]\\ \Delta u[2]\\ \vdots\\ \vec{\Delta u}[t-1]\end{bmatrix}}^{\leftarrow 1 \rightarrow} \updownarrow mt$$

must be full rank (ie, rank = n)

- would like to make $\Delta\vec{x}[t]$ anything I like in $\mathcal{R}^n$
- rank: number of lin. indep. columns (= # of lin. indep. rows)

Controllability: simple example

- $\text{span}([A^{t-1}B \,|\, A^{t-2}B \,|\, \cdots \,|\, AB \,|\, B]) = \text{span}([B \,|\, AB \,|\, \cdots \,|\, A^{t-1}B])$
- $A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}, \quad B = \vec{b} = \begin{bmatrix}1\\ 0\end{bmatrix} \quad [B \mid AB \mid A^2B \mid \cdots] = \begin{bmatrix}1 & 1 & 1 & \cdots\\ 0 & 0 & 0 & \cdots\end{bmatrix}$

not controllable

rank = 1 < n=2

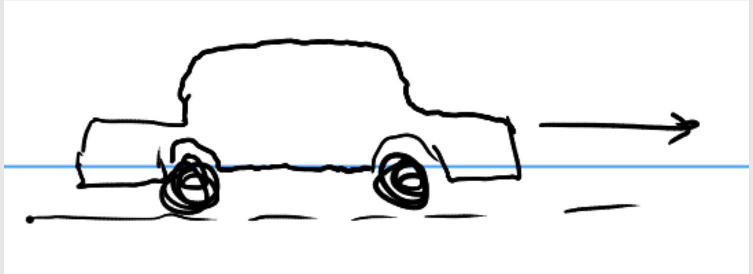
- The system: $\begin{bmatrix}\Delta x_1[t+1]\\ \Delta x_2[t+1]\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\Delta x_1[t]\\ \Delta x_2[t]\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} u(t)$

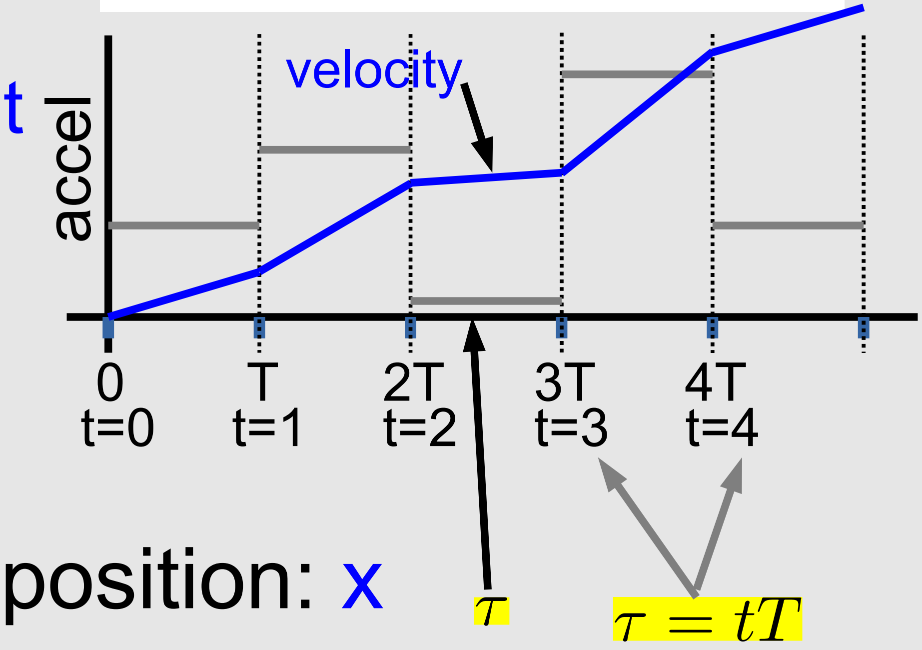
Δu(t) has no influence on $\Delta x_{2[}$t] $\Delta x_2[t+1] = 2\Delta x_2[t]$

- When does A, AB, … run out of lin. indep vectors?
  - every A has a **minimal polynomial** (result from lin. alg.)
    - ie, for some k≤n, $A^k + c_{k-1}A^{k-1} + c_{k-2}A^{k-2} + \cdots + c_1 A + c_0 I = 0$
    - ie, $A^k B = \underbrace{-c_{k-1}A^{k-1}B - c_{k-2}A^{k-2}B - \cdots - c_1 AB - c_0 B}_{\text{linear comb. of } [B,\ AB,\ A^2B,\ \ldots,\ A^{k-1}B]}$
    - ie, A^k, A^{k+1}, … will not contribute new linearly indep. columns

Example: Accelerating Car

- control input: **acceleration**
  - can change only every T secs
    - ➔ stays constant in between
- Q: can we set its **position** AND **velocity** to whatever we want (at time = multiples of T)?
- analysis approach
  - find a discrete SSR for position/vel.
  - analyse its controllability





- acceleration: a; velocity: v; position: x
  - $v(\tau) = \int_0^\tau a(\tau_2) d\tau_2$, $x(\tau) = \int_0^\tau v(\tau_2) d\tau_2$
  - $v(\tau) - v(tT) = \int_{tT}^{\tau} a(\tau_2) d\tau_2 = a(tT) \int_{tT}^{\tau} d\tau_2 = (\tau - tT) a(tT)$

$tT \leq \tau \leq (t+1)T$

Accelerating car (contd.)

- acceleration: a; velocity: v; position: x
  - $v(\tau) = \int_0^\tau a(\tau_2) d\tau_2$, $x(\tau) = \int_0^\tau v(\tau_2) d\tau_2$
  - $v(\tau) - v(tT) = \int_{tT}^\tau a(\tau_2) d\tau_2 = a(tT) \int_{tT}^\tau d\tau_2 = (\tau - tT) a(tT)$

    $tT \leq \tau \leq (t+1)T$
  - $x(\tau) - x(tT) = \int_{tT}^\tau v(\tau_2) d\tau_2 = \int_{tT}^\tau \left[v(tT) + a(tT)(\tau_2 - tT) \right] d\tau_2$

    $= (\tau - tT) v(tT) + a(tT) \frac{(\tau - tT)^2}{2}$ $tT \leq \tau \leq (t+1)T$
  - set $\tau = (t+1)T$; the above become:
  - $$x((t+1)T) = x(tT) + Tv(tT) + \frac{T^2 a(tT)}{2}$$
    $$v((t+1)T) = v(tT) + Ta(tT)$$

Linearization of Vector S.S. Systems

- $x((t+1)T) = x(tT) + Tv(tT) + \dfrac{T^2 a(tT)}{2}$

  $v((t+1)T) = v(tT) + Ta(tT)$

  - S.S.R in matrix-vector form:
  - $$\begin{bmatrix} x((t+1)T) \\ v((t+1)T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(tT) \\ v(tT) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} a(t)$$

- Controllability: $[B \mid AB] = \begin{bmatrix} \frac{T^2}{2} & 3\frac{T^2}{2} \\ T & T \end{bmatrix}$

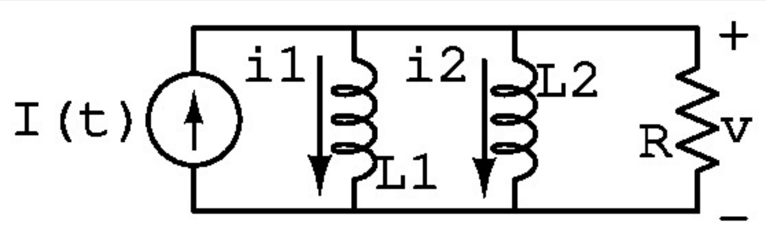
  - $$det\left(\begin{bmatrix} \frac{T^2}{2} & 3\frac{T^2}{2} \\ T & T \end{bmatrix}\right) = \frac{T^3}{2} - 3\frac{T^3}{2} = -T^3 \longleftarrow \text{always nonzero (for } T\neq 0)$$

- A: YES, we **can** drive the car's **position** AND **velocity** to whatever values we want (at every $\tau = tT$ for $t \geq 2$)

Continuous Time Controllability

nxn matrix nxm matrix

- System: $\frac{d}{dt}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + B\Delta\vec{u}(t)$
- Controllability: **same condition as for discrete**
  - $rank\left(\left[B \mid AB \mid \cdots \mid A^{t-1}B\right]\right) = n$

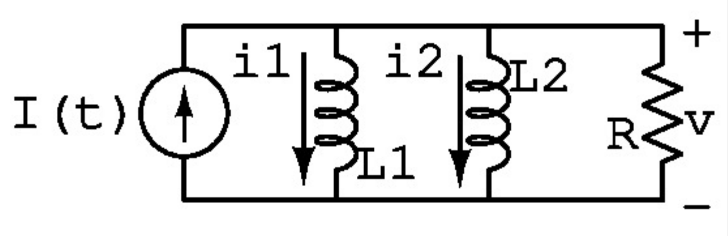


- **Example: RL circuit**
  - $i_1 + i_2 + \frac{v}{R} = I_1(t), \quad \frac{di_1}{dt} = \frac{v}{L_1}, \quad \frac{di_2}{dt} = \frac{v}{L_2}$
  - $\frac{di_1}{dt} = \frac{R(I_1(t) - i_1(t) - i_2(t))}{L_1}, \quad \frac{di_2}{dt} = \frac{R(I_1(t) - i_1(t) - i_2(t))}{L_2}$
  - $\frac{d}{dt}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix} I(t)$

Continuous Controllability (contd.)

- $$\frac{d}{dt}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix} I(t)$$

not controllable



- Controllability:
  - $$[B \mid AB] = \begin{bmatrix} \frac{R}{L_1} & -\frac{R^2}{L_1^2} - \frac{R^2}{L_1 L_2} \\ \frac{R}{L_2} & -\frac{R^2}{L_1 L_2} - \frac{R^2}{L_2^2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{R}{L_1} - \frac{R}{L_2} \\ 1 & -\frac{R}{L_1} - \frac{R}{L_2} \end{bmatrix} \begin{bmatrix} \frac{R}{L_1} & \\ 1 & \frac{R}{L_2} \end{bmatrix}$$

rank = 1 < n=2

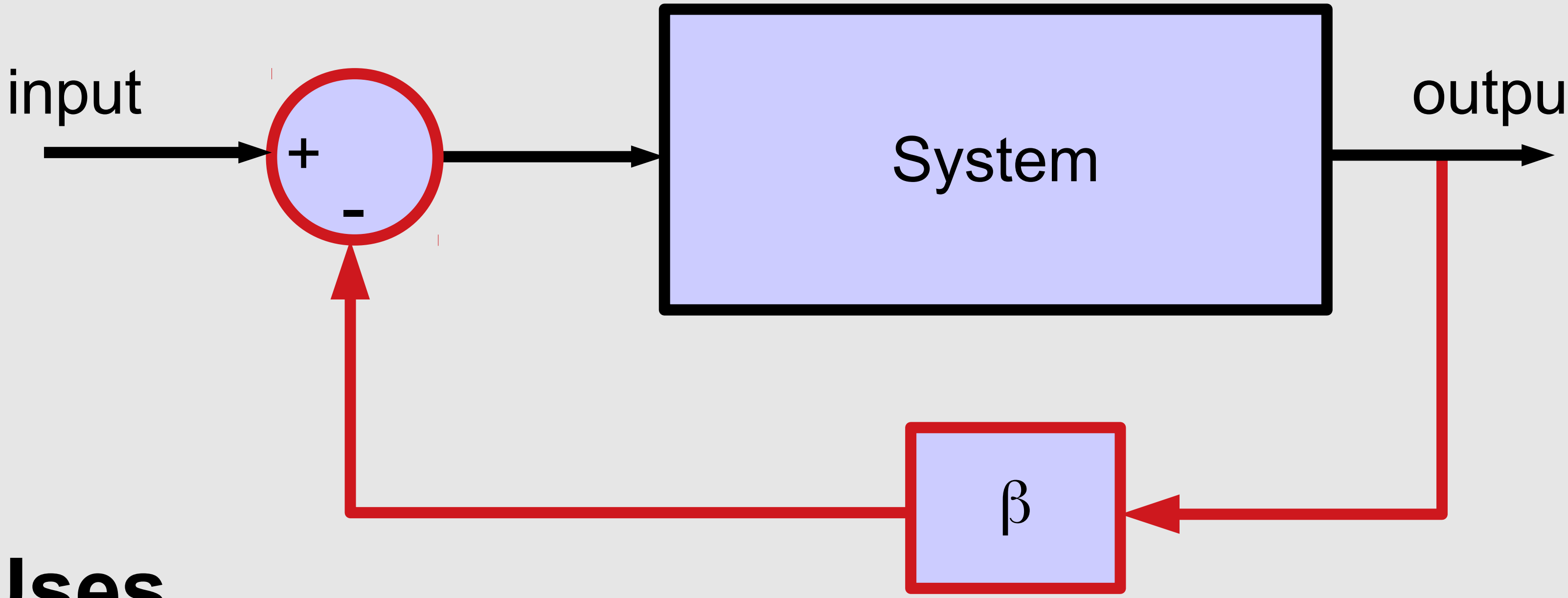
- Intuitive/“physical” way to see it:
  - i_1 and i_2 both **directly determined** by the same v(t)
  - $$\frac{di_1}{dt} = \frac{v}{L_1}, \quad \frac{di_2}{dt} = \frac{v}{L_2}$$

cannot be set independently

  - $$\frac{d}{dt}(L_1 i_1(t) - L_2 i_2(t)) = 0 \;\longrightarrow\; L_1 i_1(t) - L_2 i_2(t) = \text{constant}$$

Feedback

- The concept of **feedback**
  - add/subtract some of the output/state from the input



- **Uses**
  - making systems **less sensitive** to undesired noise and uncertainties (ALWAYS PRESENT in practical systems)
  - **stabilizing** unstable systems (if they are controllable)
    - ➔ thus making them practically usable

The Problem with Open Loop Control

- “open loop” means: no feedback
  - “closed loop” means a system with feedback
- example:* $\dot{x}(t) = a x(t) + u(t), \quad a = 1 > 0 \longleftarrow$ unstable
  - but controllable (why?)

* dropping Δ from Δx and Δx (for convenience)

  - goal: make x(t=10) = 1, starting with I.C. x(0) = 1

try u(τ) = constant

  - $x(10) = 1 \cdot e^{10} + \int_0^{10} e^{10-\tau} u(\tau)\, d\tau = 1 \cdot e^{10} + u(e^{10} - 1)$

-1

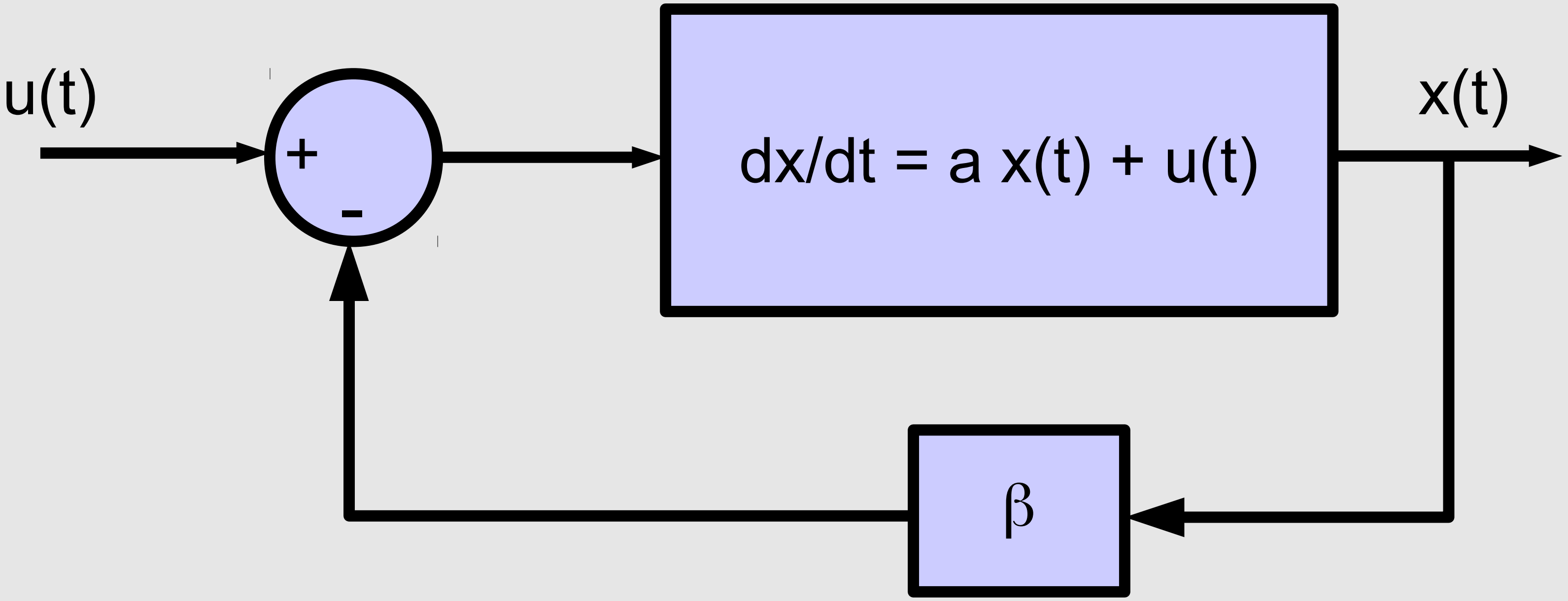
  - want: $1 = x(10) = 1 \cdot e^{10} + u(e^{10} - 1)$
  - suppose there’s a 0.1% error in the IC: 1 → 1.001
  - new $x(10) = 1.001 \cdot e^{10} + u(e^{10} - 1) = 1 + \boxed{10^{-3} e^{10}} \rightarrow$ ~ 22
  - **0.1% error** in IC → **2200% error** in x(10)

$e^{10} \simeq 22026$

- How will this change if a = -1?

if system unstable, control in the presence of errors/noise is impossible in practice

Stabilization via Feedback

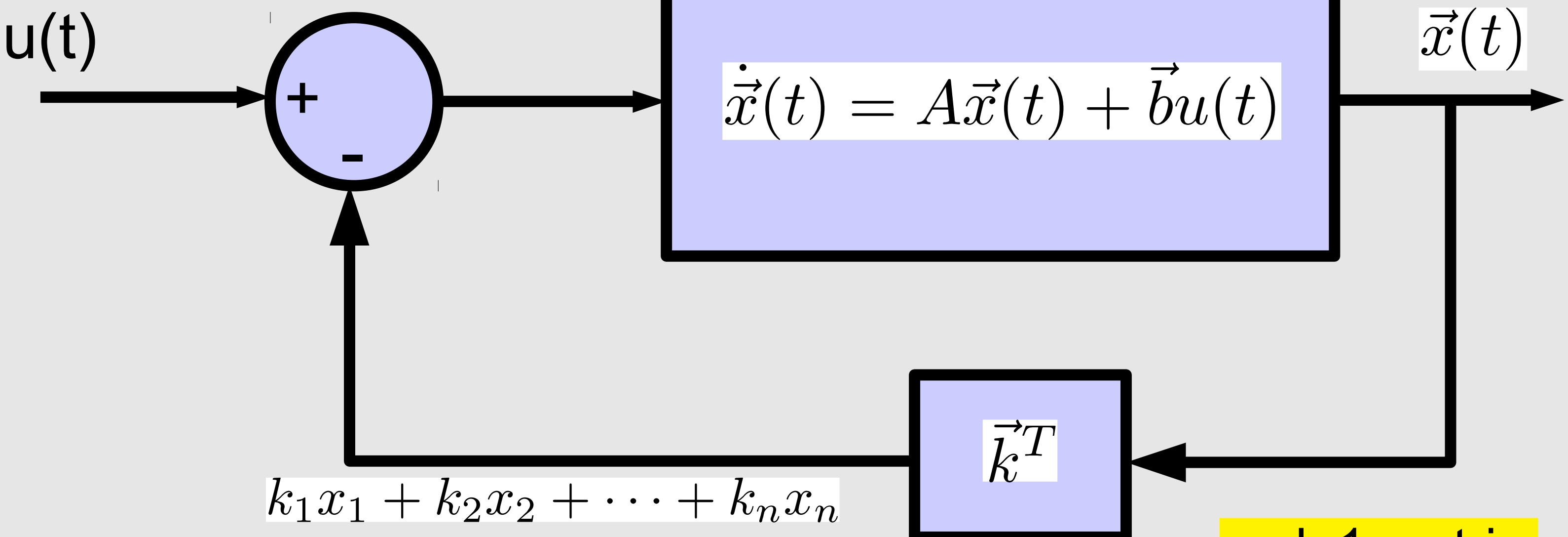


- apply feedback: $u(t) \mapsto u(t) - \beta x(t)$
  - $\dot{x}(t) = ax(t) + u(t) \mapsto \boxed{\dot{x}(t) = ax(t) + u(t) - \beta x(t), \quad a = 1 > 0}$
- $\dot{x}(t) = \boxed{(a - \beta)}x(t) + u(t), \quad a = 1 > 0$

choose β>a → **system is stabilized**

Feedback for Vector S.S. Systems

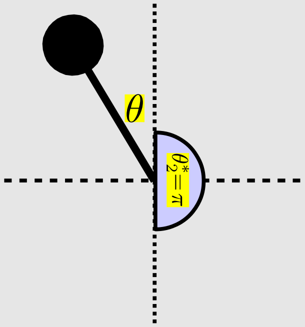
assumption for simplicity:
the input is still scalar



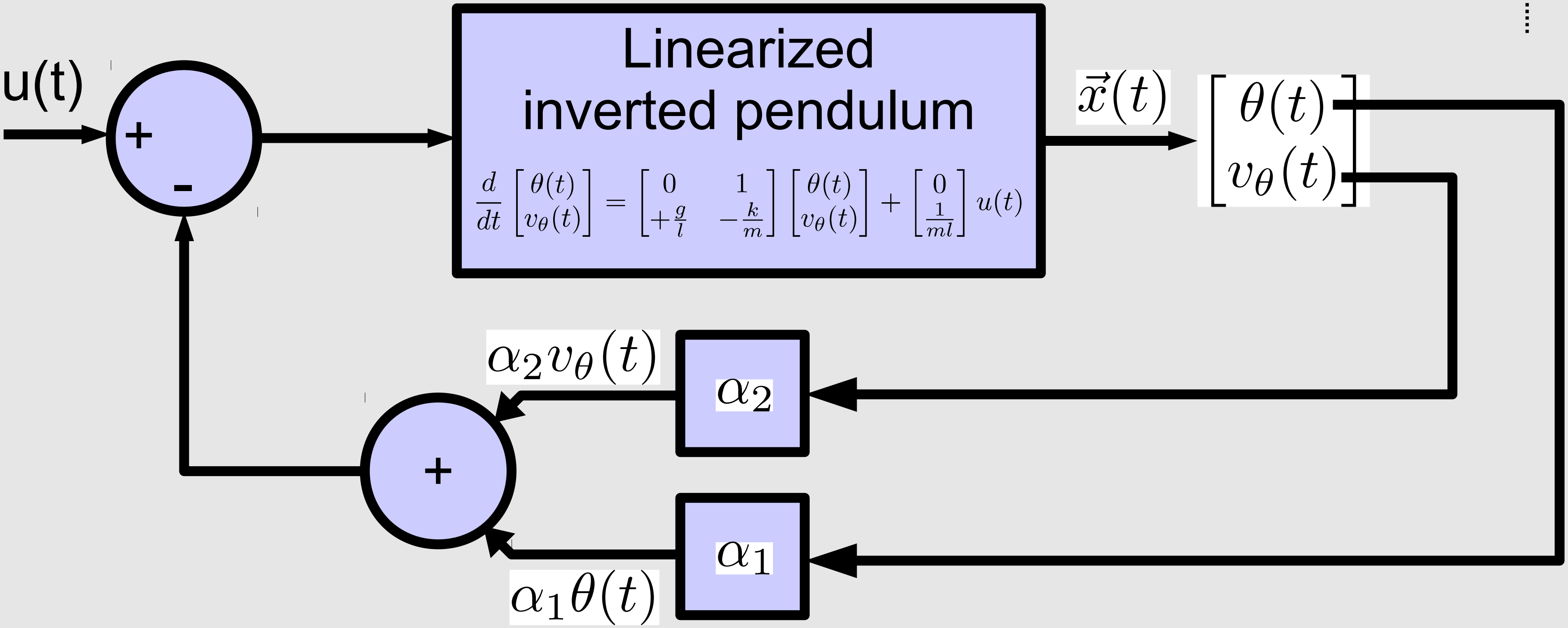
rank 1 matrix

- system w feedback: $\dot{\vec{x}}(t) = (A - \boxed{\vec{b}\vec{k}^T})\vec{x}(t) + \vec{b}u(t)$
  - stability governed by eigenvalues of $A \boxed{- \vec{b}\vec{k}^T}$
- Q: how do the e.values of A change due to
  - very difficult to figure out analytically!
    - ➔ can do simple examples; otherwise, numerically

Example: stabilizing an inverted pendulum using feedback



- i.p.: $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ +\frac{g}{l} & -\frac{k}{m}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)$
- Closed loop system (ie, with feedback)



- $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)$

Stabilizing I.P. via feedback (contd.)

- I.P. w F.: $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)$
  - eigenvalues of this determine stability
    - ➔ $det\left(\begin{bmatrix}-\lambda & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2-ml\lambda}{ml}\end{bmatrix}\right) = 0 \Rightarrow ml\lambda^2 + (kl+\alpha_2)\lambda - (mg-\alpha_1) = 0$

make this negative

make this smaller than $|kl+\alpha_2|$

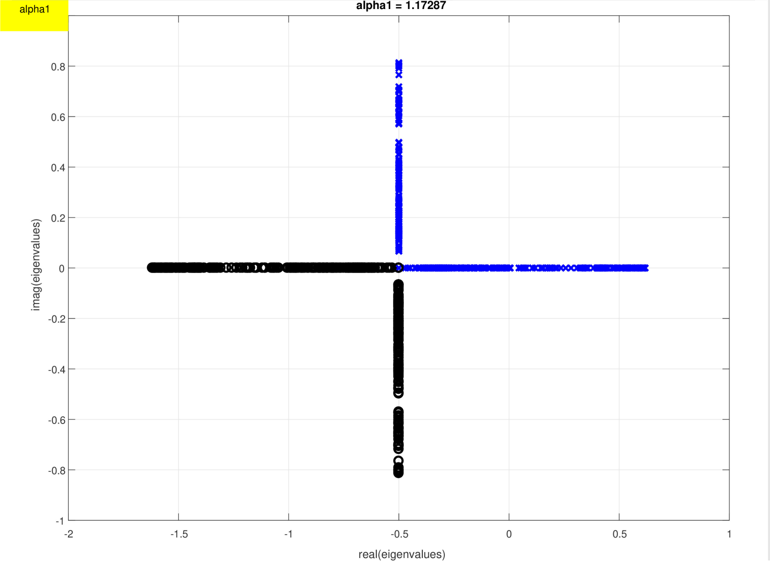
- ➔ $\lambda_{1,2} = \frac{\boxed{-(kl+\alpha_2)}}{2ml} \pm \frac{\boxed{\sqrt{(kl+\alpha_2)^2 + \boxed{4ml(mg-\alpha_1)}}}}{2ml}$

make this negative

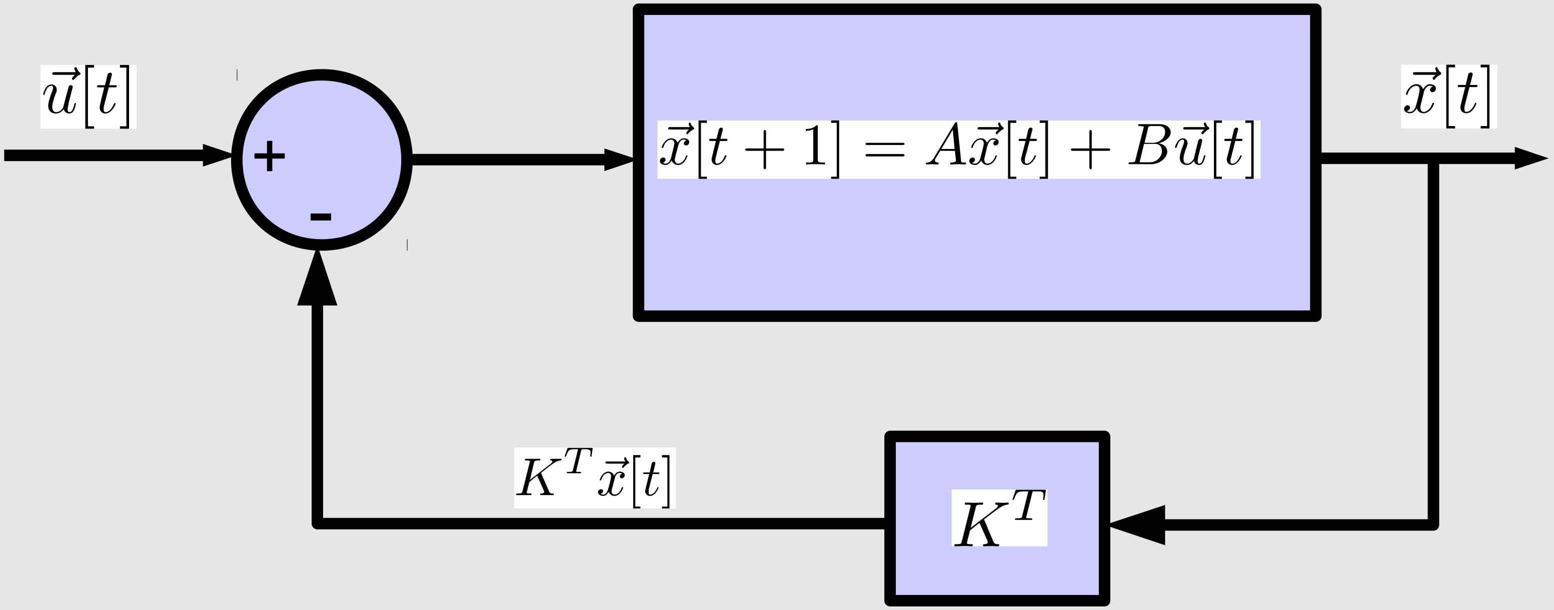
- to stabilize: make both evs -ve (real part)
  - ➔ choose any $\alpha_2 > -kl, \quad \alpha_1 > mg$

run MATLAB demo

inverted_pendulum_w_feedback_root_locus.m



Feedback for Discrete-Time S.S.Rs

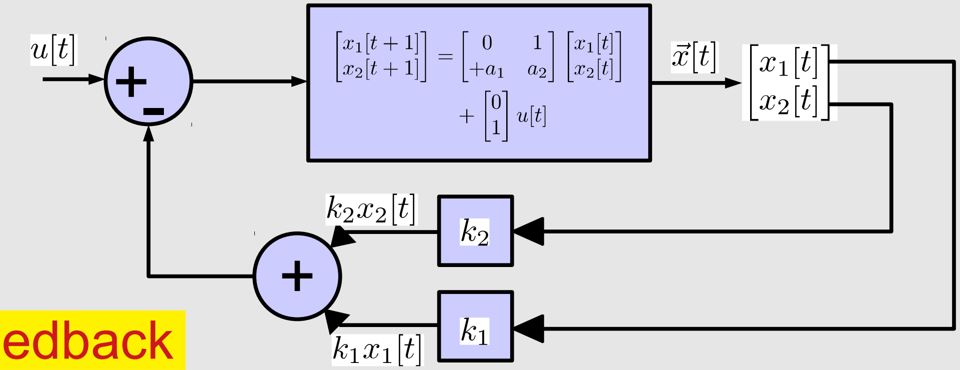


- system w feedback: $\vec{x}[t+1] = (A - B\vec{K}^T)\vec{x}[t] + B\vec{u}[t]$
  - stability still governed by the eigenvalues of $A - BK^T$
- stability (discr.) → magnitude of eigenvalues < 1
  - different from the continuous case

Example: Discrete-Time Feedback

- $\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$

$$\boxed{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}} \longleftarrow \text{w feedback}$$



- char. poly.: $\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$
  - roots: $\lambda_{1,2} = \frac{a_2 - k_2}{2} \pm \frac{1}{2}\sqrt{(a_2 - k_2)^2 + 4(a_1 - k_1)}$
- easy to express k_1, k_2 in terms of λ_1, λ_2:
  - $k_1 = \lambda_1 \lambda_2 - a_1$
 $k_2 = a_2 - \lambda_1 - \lambda_2$ ← choose any λ_1 and λ_2 (eg, stable ones); set k_1 and k_2
  - **if** λ_1 is complex: **make sure** λ_2 is the conjugate of λ_1!
    - ➔ otherwise, k_1/k_2/x_1/x_2 will have imaginary components
      - which would be physically meaningless

Another D-T. Feedback Example

- $\vec{x}[t+1] = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\vec{x}[t] + \begin{bmatrix}1\\ 0\end{bmatrix}u[t]$ ← not controllable

$$\boxed{\begin{bmatrix}1-k_1 & 1-k_2\\ 0 & 2\end{bmatrix}}$$ ← w feedback

- char. poly.: $(1-k_1-\lambda)(2-\lambda)=0$
  - roots: $\lambda_1 = 1-k_1,\quad \boxed{\lambda_2 = 2}$ ← does not depend on k_1 or k_2; ie, cannot be altered via feedback
- suspicions (based on a few examples)
  - controllable → can place all eigenvalues via careful feedback
  - not controllable → might not be able to place all evs