NOTES FOR LECTURES 5B & 6A: CONTROLLABILITY AND FEEDBACK

Controllability: DISCRETE-TIME SYSTEMS

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$$\Delta\vec{x}[t] = A\,\Delta\vec{x}[t] + B\,\Delta\vec{u}[t], \quad \text{with } \Delta x[0] \text{ IC}$$

$$\Delta\vec{x}[1] = A\,\Delta\vec{x}[0] + B\,\Delta\vec{u}[0]$$

$$\Delta\vec{x}[2] = A\,\Delta\vec{x}[1] + B\,\Delta\vec{u}[1] = A^2\,\Delta\vec{x}[0] + AB\,\Delta u[0] + B\,\Delta u[1]$$

$$\Delta\vec{x}[t] = A^t\,\Delta\vec{x}[0] + \sum_{i=1}^{t} A^{t-i} B\,\Delta\vec{u}[i-1]$$

$$\Delta\vec{x}[t] = A^t\,\Delta\vec{x}[0] + \underbrace{\overbrace{\left[\begin{array}{c:c:c:c:c} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{array}\right]}^{m}}_{mt}\left.\begin{bmatrix} \Delta\vec{u}[0] \\ \Delta\vec{u}[1] \\ \Delta u[2] \\ \vdots \\ \Delta\vec{u}[t-1] \end{bmatrix}\right\} mt$$

— Goal: given some IC $\Delta\vec{x}[0]$, to drive the system (via a sequence of inputs) to anything you like.

$\Rightarrow$ $\Delta x[t]$ can be any vector (say $\vec{z}$)

$\Rightarrow$ $\Delta\vec{x}[t] - A^t\,\Delta\vec{x}[0]$ can be any vector ($\vec{z} + A^t\,\Delta\vec{x}[0]$)

$$\Rightarrow R_t \triangleq \underbrace{\left[\begin{array}{c:c:c:c:c} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{array}\right]}_{mt}\left.\begin{bmatrix} \Delta\vec{u}[0] \\ \Delta\vec{u}[1] \\ \Delta u[2] \\ \vdots \\ \Delta\vec{u}[t-1] \end{bmatrix}\right\} mt$$

this must be capable of taking on any value in $\mathbb{R}^n$

SYSTEM (A, B) CALLED CONTROLLABLE IF THIS IS FULL RANK (= n)

$\Rightarrow$ (result from linear algebra): $\left[\begin{array}{c:c:c:c:c} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{array}\right]$ must be full rank.
(n independent columns)

THE CONTROLLABILITY MATRIX

Q: Just choose t s.t. $mt \geq n$, then there are at least n columns.
Won't it be full rank?

A: not necessarily. Example:

$\rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$A^t B = B \Rightarrow \mathcal{C}(t) = [\overbrace{B \vdots B \vdots B \vdots \cdots \vdots B}^{2t}]$

$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$

$\Rightarrow$ rank $= 1 < 2$.

$\rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow A\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathcal{C}(2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow$ rank 1

- Consider $B, AB, A^2B, \cdots,$ in sequence
  - when does this sequence start running out of linearly indep. vectors?

- **Thm**: For some $m \leq n$ (size of the $\square$ matrix A), $A^m B$ will not add any more lin. indep. vectors to $[B, AB, A^2B, \cdots, A^{m-1}B]$
  - $\rightarrow$ moreover, for all $p > m$, $A^p B$ will not have any indep. vectors, either.

- Consequence of the minimal polynomial theorem: ($\rightarrow$ related to the Cayley–Hamilton theorem.)
  - for some $k \leq n$, $A^k + \sum_{i=0}^{k-1} c_i A^i = 0 \iff$ i.e., $A^k = -(c_{k-1}A^{k-1} + c_{k-2}A^{k-2} + \cdots + c_0 I)$

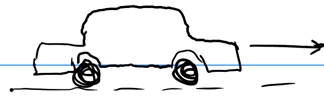
    (c_i: some specific scalars)
  - note $k = n$ if $\lambda_i \neq \lambda_j \; \forall i \neq j$
  - $\rightarrow m \leq k \leq n$
  - Proof: $A^k B = -(\underbrace{c_{k-1}A^{k-1}B + c_{k-2}A^{k-2}B + \cdots + c_0 B}_{\text{linear combination of all previous columns}})$

<u>Example</u>: use the above A & $\vec{b}$ in discrete time system:

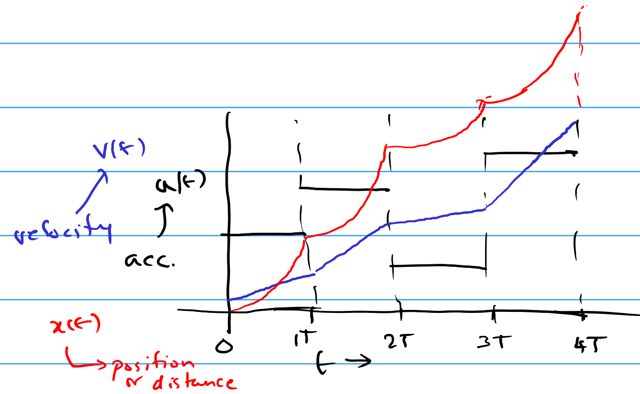
$\rightarrow \quad \vec{x}[t+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$

$\rightarrow$ NOT CONTROLLABLE

$\rightarrow$ Insight: the 2^{nd} equ. is: $x_2[t+1] = 2x_2[t] \leftarrow$ no way $u(t)$ can influence this

→ Another example: 

– pedal changes acceleration – but only every T seconds:



$$a(t) = \frac{dv(t)}{dt} \Rightarrow \int_0^t a(\tau)\,d\tau \quad :$$

$\longrightarrow$ $a(\tau) = a(t) \qquad tT < \tau < (t+1)T$

$\uparrow$ discrete

$$\underset{\text{discrete}}{v(tT+T) - v(tT)} = \int_{tT}^{(t+1)T} a(\tau)\,d\tau = Ta(t)$$

$$\Rightarrow v(tT+T) = v(tT) + Ta(t)$$

– also $\quad v(tT+\tau) = v(tT) + \tau a(t) \qquad 0 \leq \tau \leq T$

— position $\quad x(t) = \int_0^t v(\tau)\,d\tau$

$$x(tT+T) - x(tT) = \int_{tT}^{(t+1)T} v(\tau)\,d\tau = \int_{tT}^{(t+1)T} v(tT)\,d\tau + a(t)\int_0^T \tau\,d\tau$$

$$= Tv(tT) + \frac{a(t)T^2}{2}$$

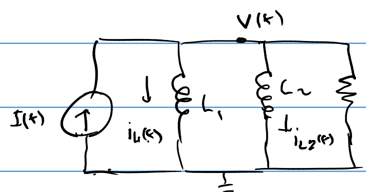
$$\begin{bmatrix} x((t+1)T) \\ v((t+1)T) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x(tT) \\ v(tT) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}}_{b} a(t)$$

Controllability: $[\vec{b}, A\vec{b}] = \begin{bmatrix} \frac{T^2}{2} & \frac{3}{2}T^2 \\ T & T \end{bmatrix} = T\begin{bmatrix} \frac{T}{2} & \frac{3}{2}T \\ 1 & 1 \end{bmatrix}$

$$\det = T\left(T/2 - \tfrac{3}{2}T\right) = -T^2 \neq 0 \;\; \forall T \neq 0$$

Hence controllable: you can devise $a(t)$ to achieve any position/velocity for $t \geq 2$

— RL ckt example



$$\frac{V(t)}{R} + i_{L1}(t) + i_{L2}(t) - I(t) = 0 \Rightarrow V(t) = (I(t) - i_{L1}(t) - i_{L2}(t))R$$

$$\cancel{L_1}\frac{di_{L1}}{dt} = \frac{V(t)}{L_1} = \frac{(I(t) - i_{L1}(t) - i_{L2}(t))R}{L_1}$$

$$\cancel{L_2}\frac{di_{L2}}{dt} = \frac{V(t)}{L_2} = \frac{(I(t) - i_{L1}(t) - i_{L2}(t))R}{L_2}$$

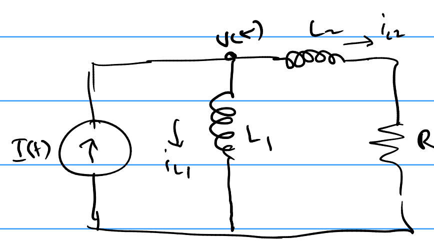
$$\frac{d}{dt}\begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix} = R\underbrace{\begin{bmatrix} -\frac{1}{L_1} & -\frac{1}{L_1} \\ -\frac{1}{L_2} & -\frac{1}{L_2} \end{bmatrix}}_{A}\begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix}}_{\vec{b}} I(t)$$

$$\begin{bmatrix} \underbrace{\begin{matrix} -2/L_1 \\ -\frac{2}{L_2} \end{matrix}}_{A\vec{b}} & \underbrace{\begin{matrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{matrix}}_{\vec{b}} \end{bmatrix} \longleftarrow \text{rank} \Rightarrow \text{not controllable}$$

→ "Physical" reason: same voltage determines both currents, can't make both indep. of each other:

$$-\frac{d}{dt}[L_1 i_1 - L_2 i_2] = 0 \Rightarrow L_1 i_1 = L_2 i_2 + \text{constant}!$$

→ How about



FEEDBACK

→ **CONTROLLABILITY IS NOT OF MUCH USE WITHOUT STABILITY**

→ Suppose you have a controllable, but dynamically unstable, system.

→ Can you really control it, in a practical sense?

→ NO: the slightest error (eg, in the input, or IC), will totally trash your control strategy.

→ <u>Example</u>: $\frac{dx}{dt} = x + u(t)$.

→ system is controllable; but it is dynamically UNSTABLE (eigenvalue $= 1 > 0$)

→ say we want to move $x(t)$ to 1 at $t=10$ (from 0 at 0)

→ $x(10) = \int_0^{10} e^{(10-\tau)} u(\tau)\, d\tau = e^{10} \int_0^{10} e^{-\tau} u(\tau)\, d\tau$ ← don't need to go through this.

→ try $u(\tau) \equiv \text{constant} = b$

$\Rightarrow x(10) = b e^{10} \int_0^{10} e^{-\tau} d\tau = b e^{10} \left[e^{-\tau}\right]_{10}^{0} = b e^{10}\left[1 - e^{-10}\right] = b \underbrace{\left[e^{10} - 1\right]}_{\sim 2.2 \times 10^4}$

$\Rightarrow$ (since $x(10)$ is 1) $b = \frac{1}{e^{10}-1} \approx 4.542 \times 10^{-5}$

→ Suppose your initial condition is not exactly 0, but $= 10^{-3}$

→ what is the error is $x(10)$?

→ it is $10^{-3} e^{10} \simeq 22$ ← totally wipes out desired value of 1.

— Now suppose you have $\dot{x} = -x + u(t)$

— $x(10) = e^{-10} \int_0^{10} e^{+\tau} = $ (for fixed $u(t) = b$) $b\left[e^{-10} - 1\right] \simeq -b$

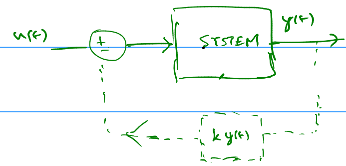
— now if $x(0) = 10^{-3}$, $\Delta x(10) = 10^{-3} e^{-10} \simeq 0$ ← very accurate

→ Ie, an unstable system, even if controllable, is effectively not so in the presence of even tiny errors.

→ For a system to be practically controllable (and useful), IT MUST BE STABLE.

→ CAN ONE SOMEHOW MAKE AN UNSTABLE SYSTEM STABLE?

→ Yes: via **feedback**

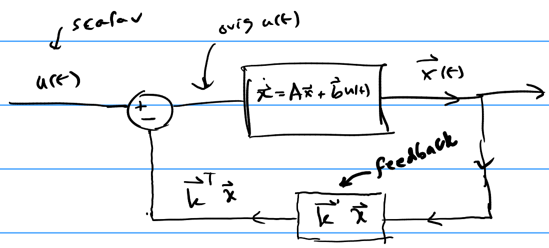


→ what is feedback?

→ Take some of the state and add/subtract it from the input.

→ Uses: less sensitive to errors/variations/noise

→ SYSTEM w scalar input:



$k_1x_1 + k_2x_2 + \dots + k_nx_n$

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}\left(u(t) - \vec{k}^T\vec{x}(t)\right) \Longrightarrow \frac{d\vec{x}}{dt} = \underbrace{\left(A - \vec{b}\vec{k}^T\right)}\vec{x} + \vec{b}u(t)$$

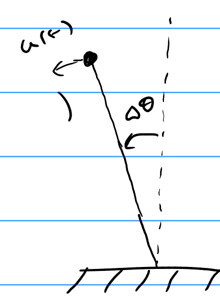
– how do the eigenvalues of this relate to those of A?

these determine stability

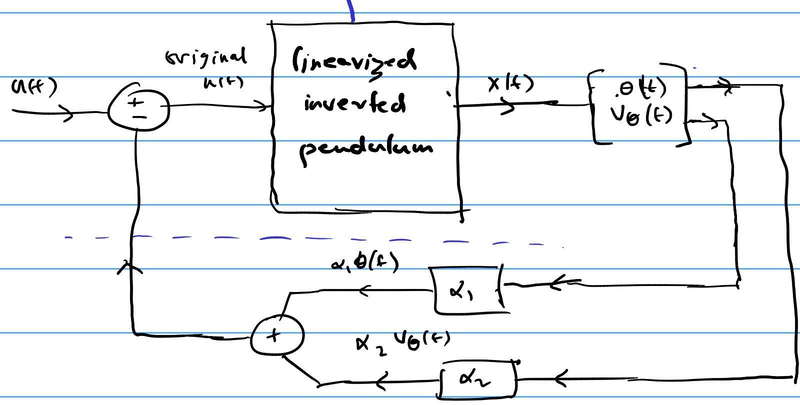
↳ no general analytical formula exists

— FEEDBACK CAN STABILIZE UNSTABLE SYSTEMS

— EXAMPLE: inverted pendulum



$$\underbrace{\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ g/l & -k/m\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)}$$



SYSTEM w/ feedback

due to feedback

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ g/l & -k/m\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}\left(u(t) - \alpha_1\theta(t) - \alpha_2 v_\theta(t)\right)$$

$$+\begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t) - \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}\theta\\ v_\theta\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\ \frac{\alpha_1}{ml} & \frac{\alpha_2}{ml}\end{bmatrix}$$

$$\longrightarrow \frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \underbrace{\left(\begin{bmatrix}0 & 1\\ g/l & -k/m\end{bmatrix} - \boxed{\begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}}\right)}_{\text{new A for system w feedback}}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)$$

$$\begin{bmatrix}0 & 1\\ g/l - \frac{\alpha_1}{ml} & -\frac{k}{m} - \frac{\alpha_2}{ml}\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}$$

$$\Rightarrow \boxed{\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)}$$

↳ inverted pendulum w feedback

$\longrightarrow$ STABILITY: determined by the eigenvalues of $\overbrace{\begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}}^{A}$

$$\Rightarrow \det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix}-\lambda & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2-ml\lambda}{ml}\end{pmatrix} = 0$$

$$\Rightarrow \frac{\lambda(kl+\alpha_2+ml\lambda) - (mg-\alpha_1)}{ml} = 0$$

$$\Rightarrow ml\lambda^2 + (kl+\alpha_2)\lambda - (mg-\alpha_1) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-(\alpha_2+kl) \pm \sqrt{(\alpha_2+kl)^2 + 4ml(mg-\alpha_1)}}{+2ml}$$

$$= \frac{-(kl+\alpha_2)}{2ml} \pm \frac{1}{2ml}\sqrt{(kl+\alpha_2)^2 + 4ml(mg-\alpha_1)}$$

— want both eigenvalues to have —ve real parts

→ $\left|\sqrt{(kl+\alpha_2)^2 + 4ml(mg-\alpha_1)}\right|$ real & $< \left|\underbrace{-(kl+\alpha_2)}\right|$

↳ and this is negative

1. $\boxed{\alpha_2 > -kl}$ $\Rightarrow$ $(kl+\alpha_2)$ positive $\Rightarrow$ $\dfrac{-(kl+\alpha_2)}{2ml}$ is negative

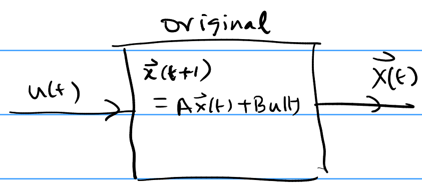
2. $4ml(mg-\alpha_1)$ should be negative $\Rightarrow$ $\boxed{\alpha_1 > mg}$

inverted_pendulum_w_feedback_root_locus.m

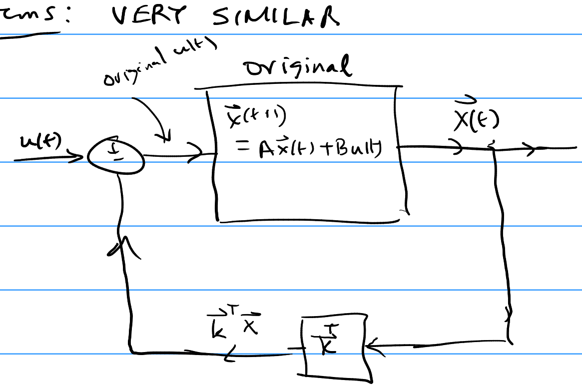
— plot root-locus wrt α_1 & α_2

— see transient time-domain simulations for various α_1 & α_2

→ Feedback for discrete-time systems: VERY SIMILAR



$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$



$\vec{x}(t+1) = (A - B\vec{k}^T)\vec{x}(t) + Bu(t)$

→ Stability: find the eigenvalues of $(A + B\vec{k}^T)$, choose $\vec{k}$ to stabilize as needed

↳ ONLY DIFFERENCE: stability for discrete means $|\lambda| < 1$ (not $Re\{\lambda\} < 0$)

→ **Example:** $\vec{x}[t+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$

→ original eigenvalues given by: $\det(A - \lambda I) = 0 \Rightarrow -\lambda(a_2 - \lambda) - a_1 = 0$

$\Rightarrow \lambda^2 - a_2\lambda - a_1 = 0$

↳ roots of this

→ feedback: $u \mapsto u - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \;\; k_2] \vec{x}$

$\Rightarrow A \mapsto A - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}$

→ new eigenvalues: $\tilde{\lambda}^2 - (a_2 - k_2)\tilde{\lambda} - (a_1 - k_1) = 0$

$$\tilde{\lambda}_{1,2} = \frac{(a_2 - k_2) \pm \sqrt{(a_2 - k_2)^2 + 4(a_1 - k_1)}}{2}$$

$\tilde{\lambda}_1 + \tilde{\lambda}_2 = (a_2 - k_2) \Rightarrow k_2 = a_2 - \tilde{\lambda}_1 - \tilde{\lambda}_2$

$\tilde{\lambda}_1 - \tilde{\lambda}_2 = \sqrt{(\tilde{\lambda}_1 + \tilde{\lambda}_2)^2 + 4(a_1 - k_1)} \Rightarrow (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 = (\tilde{\lambda}_1 + \tilde{\lambda}_2)^2 + 4(a_1 - k_1)$

$\Rightarrow k_1 = \frac{(\tilde{\lambda}_1 + \tilde{\lambda}_2)^2 - (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{4} + a_1$

(using $a^2 - b^2 = (a+b)(a-b)$)

$\Rightarrow$ i.e., $\boxed{\begin{aligned} k_1 &= \tilde{\lambda}_1 \tilde{\lambda}_2 + a_1 \\ k_2 &= a_2 - \tilde{\lambda}_1 - \tilde{\lambda}_2 \end{aligned}}$.

→ choose k_1 & k_2 to place the eigenvalues wherever you like!

→ **Another discrete-time example**

→ $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → Note: not controllable!

→ $A - \vec{b}\vec{k}^T = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix}$

Intuition: 2nd eqn is uncontrollable ⟹ cannot influence it by feedback

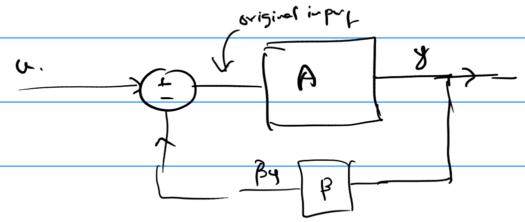
→ eigenvalues: $(1 - k_1 - \lambda)(2 - \lambda) =$

$\Rightarrow (\lambda_1 = 2)$, $\lambda_2 = (1 - k_1)$

↳ does not depend on k_1 or k_2 ⟹ can't change this eigenvalue

— SIMPLER (BUT ILLUMINATING) VIEWS OF FEEDBACK AND ITS USES

— these are often used by circuit designers.



: Orig. xfer fn.: $y = Au$

New " " : $A(u - \beta y) = y$

$\Rightarrow (1+\beta A) y = Au$

$\Rightarrow y = \dfrac{A}{1+\beta A} u$

→ note: if $\beta A \gg 1$

→ then $y \simeq \dfrac{1}{\beta} A$

→ this happens in op-amps: A is very large, but its value can be very "noisy": eg: $10^5 - 10^6$

→ Suppose you want a gain of 10^3, but reasonably precisely.

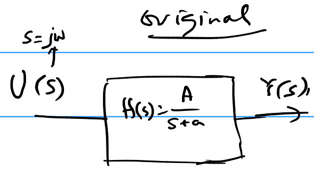
→ set $\beta = 10^{-3}$ precisely (using a resistor capacitor)

→ then if $A = 10^5$, gain $= \dfrac{10^5}{10^2+1} \simeq 10^3$

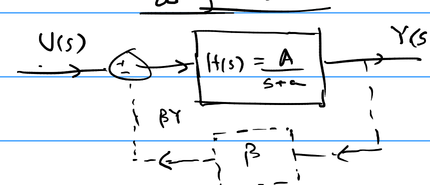
if $A = 10^6$, gain $= \dfrac{10^6}{10^3+1} \simeq 10^3$

→ **Feedback in the phasor domain:**

original



w feedback



$$Y(s) = \frac{\frac{A}{s+a}}{1+\frac{\beta A}{s+a}} U(s) = \frac{A}{s+a+\beta A} U(s)$$

→ now suppose the original pole a was dynamically unstable $\Rightarrow a < 0$

→ you can FIX it by adding βA

— and if $|\beta A| \gg |a|$, then $H(s) \simeq \dfrac{A}{s+\beta A}$

— at DC: $\dfrac{1}{\beta}$

— at $s = j\underbrace{\beta A}_{\omega}$, $= \frac{1}{2}$ of DC gain

THESE CAN BE EASILY DERIVED FROM THE STATE SPACE FORM – eg, BY REPRESETING $\vec{u}(t)$, $\vec{x}(t)$, etc. AS PHASORS