EE 16B, SPRING 2018

NOTES FOR LECTURES 6B & 7A: CONTROLLER CANONICAL FORM & OBSERVABILITY

JAIJEET ROYCHOWDHURY

CONTROLLER CANONICAL FORM (CCF)

→ Recall a previous controllability example: $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$

→ its characteristic polynomial $c(\lambda) \triangleq \det(A - \lambda I)$ had a nice form:

→ $-\lambda(a_2-\lambda) - a_1 \equiv \boxed{\lambda^2 - a_2\lambda - a_1}$

→ A rather elegant observation someone made long ago is this:

→ suppose you have a size n matrix in the same form:

$$\underbrace{A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}; \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

TOGETHER CALLED CONTROLLER CANONICAL FORM (CCF)

→ then, $\boxed{c(\lambda) \triangleq \det(A - \lambda I) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_3\lambda^2 - a_2\lambda - a_1}$

→ proof: easily (though laboriously) obtained by applying the formula for determinants involving expansion using minors to $(A - \lambda I)$. Try it yourself if you have some time on your hands and you like getting to the bottom of things.

→ Now, suppose you have a scalar input to your system ($u(t)$) and the system is: $\underbrace{\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}\,u(t)}$

this (the original system without feedback) is called the OPEN LOOP system

→ Now, if you apply feedback $u(t) \mapsto u(t) - \vec{k}^T \vec{x}(t)$, then the "closed loop system" becomes:

$$\dot{\vec{x}} = (A - \vec{b}\vec{k}^T)\vec{x} + \vec{b}\,u(t),$$

$$\text{with } (A - \vec{b}\vec{k}^T) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ a_1-k_1 & a_2-k_2 & a_3-k_3 & \cdots & a_n-k_n \end{bmatrix}$$

→ with characteristic polynomial: $C_f(\lambda) \triangleq \det(A - \vec{b}\vec{k}^T - \lambda I)$

$$= \lambda^n - (a_n-k_n)\lambda^{n-1} - (a_{n-1}-k_{n-1})\lambda^{n-2} - \cdots - (a_3-k_3)\lambda^2 - (a_2-k_2)\lambda - (a_1-k_1)$$

→ Now, our goal is to choose $k_1, \dots, k_n$ (the feedback) to place the eigenvalues of $(A - \vec{b}\vec{k}^T)$, i.e, the roots of $C_f(\lambda)$, wherever we want.

→ Suppose we want the roots to be $\lambda_1, \lambda_2, \dots, \lambda_n$

↳ remember that if complex, the conjugate must also be present.

→ i.e, the desired char. poly. is:

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\cdots(\lambda-\lambda_n)$$

$$= \lambda^n - \underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_n)}_{b_n}\lambda^{n-1} + \underbrace{\begin{pmatrix} \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots \lambda_1\lambda_n \\ + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \cdots \lambda_2\lambda_n \\ + \lambda_3\lambda_4 + \cdots \quad \lambda_3\lambda_n \\ + \lambda_{n-1}\lambda_n \end{pmatrix}}_{b_{n-1}}\lambda^{n-2} + \cdots + \underbrace{(-1)^n \lambda_1\lambda_2\lambda_3\cdots\lambda_n}_{b_1}$$

→ i.e., $b_1, \dots, b_n$ can be calculated from $\lambda_1, \dots, \lambda_n$, though the process may be extremely tedious.

→ now all you have to do to devise the feedback is equate the coefficients of the 2 char. poly. expressions:

$$a_n - k_n = -b_n$$
$$a_{n-1} - k_{n-1} = -b_{n-1}$$
$$\vdots$$
$$a_1 - k_1 = -b_1$$

→ I.e, if $(A, \vec{b})$ is in controllability canonical form, then it is always possible to devise feedback to place the eigenvalues wherever you like.

→ Size 3 example:

$$\rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} ; \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$\rightarrow A - \vec{b}\vec{k}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-k_1 & 2-k_2 & 3-k_3 \end{bmatrix}$$

→ char. poly.: $\lambda^3 - (3-k_3)\lambda^2 - (2-k_2)\lambda - (1-k_1)$

→ Suppose we want the 3 new eigenvalues to be:

$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$

$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3$

→ Therefore we should set: $k_3 = 3, k_2 = 2, k_1 = 1$

$\rightarrow$ Suppose we want $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

$$\Rightarrow (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \lambda(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) - \lambda_1\lambda_2\lambda_3$$

$$= \lambda^3 + 6\lambda^2 + \lambda\underbrace{(2+6+3)}_{11} + 6$$

$$\Rightarrow -(3-k_3) = -6 \Rightarrow k_3 = -3$$

$$\Rightarrow -(2-k_2) = -11 \Rightarrow k_2 = -9$$

$$\Rightarrow -(1-k_1) = -6 \Rightarrow k_1 = -5$$

$\rightarrow$ But can any system be put in controller canonical form?

$\rightarrow$ A: yes, if it is controllable!

$\rightarrow$ Here is how you put any controllable system in c.c.f.:

1. Form the controllability matrix:

— $R_n \triangleq \left[\vec{b}, A\vec{b}, A^2\vec{b}, \ldots, A^{n-1}\vec{b}\right] \rightarrow$ a square matrix

— if the system is controllable, then R_n is full rank, i.e., invertible.

2. Calculate R_n^{-1}

3. Take the last row of R_n^{-1} — call this last row $\vec{q}^T$ ($\vec{q}$ is a col. vector; $\vec{q}^T$ a row vector)

4. Form the following matrix, row by row:

$$T \triangleq \begin{bmatrix} \leftarrow & \vec{q}^T & \rightarrow \\ \leftarrow & \vec{q}^T A & \rightarrow \\ \leftarrow & \vec{q}^T A^2 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{q}^T A^{n-1} & \rightarrow \end{bmatrix}$$

$\leftarrow$ this is an $n \times n$ matrix

$\leftarrow$ it will be full rank and invertible

5. Define $\vec{z} = T\vec{x} \Longleftrightarrow \vec{x} = T^{-1}\vec{z}$ (this is called a basis transformation from $\vec{x}$ to $\vec{z}$, using T)

6. Using the definition of $\vec{z}$ in $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}u(t)$, you get, equivalently,

$$\frac{d\vec{z}}{dt} = \overbrace{TAT^{-1}}^{\hat{A}}\vec{z} + \overbrace{T\vec{b}}^{\hat{b}}\, u(t)$$

7. then:

$\longrightarrow$ $\hat{A} \triangleq TAT^{-1}$, with $\hat{b} \triangleq T\vec{b}$ will be in c.c.f!

$\rightarrow$ <u>Proof of the above procedure and claims:</u>

$\rightarrow$ from definition, $R_n^{-1} R_n = I$, or

$$\longrightarrow \begin{array}{cccc} R_n^{-1} & R_n & & I \\ \begin{bmatrix} \text{———} \\ \text{———} \\ \vdots \\ \text{———} \\ \leftarrow \vec{q}^{\,T} \rightarrow \end{bmatrix} & \begin{bmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & | & & | \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & & 0 & 0 \\ 0 & 1 & & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ & \vdots & & 1 & \vdots \\ 0 & 0 & & 0 & 1 \end{bmatrix} \end{array}$$

$$\longrightarrow \text{Hence: } \vec{q}^{\,T} \begin{array}{c} R_n \\ \begin{bmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & | & & | \end{bmatrix} \end{array} = [0, 0, \ldots, 0, 1]$$

$$\Longrightarrow \vec{q}^{\,T}\vec{b} = 0, \quad \vec{q}^{\,T}A\vec{b} = 0, \ldots, \vec{q}^{\,T}A^{n-2}\vec{b} = 0$$

and $\vec{q}^{\,T}A^{n-1}\vec{b} = 1$

$$\rightarrow \text{also from definition, } T \triangleq \begin{bmatrix} \leftarrow \vec{q}^{\,T} \rightarrow \\ \leftarrow \vec{q}^{\,T}A \rightarrow \\ \leftarrow \vec{q}^{\,T}A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^{\,T}A^{n-1} \rightarrow \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{q}^{\,T}\vec{b} \\ \vec{q}^{\,T}A\vec{b} \\ \vec{q}^{\,T}A^2\vec{b} \\ \vdots \\ \vec{q}^{\,T}A^{n-1}\vec{b} \end{bmatrix} = \overset{\hat{b}}{\widetilde{T\vec{b}}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\longrightarrow$ Consider TA:

$$\longrightarrow TA: \begin{bmatrix} \leftarrow \vec{q}^{\,T} \rightarrow \\ \leftarrow \vec{q}^{\,T}A \rightarrow \\ \leftarrow \vec{q}^{\,T}A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^{\,T}A^{n-1} \rightarrow \end{bmatrix} A = \begin{bmatrix} \leftarrow \vec{q}^{\,T}A \rightarrow \\ \leftarrow \vec{q}^{\,T}A^2 \rightarrow \\ \leftarrow \vec{q}^{\,T}A^3 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^{\,T}A^{n-1} \rightarrow \\ \leftarrow \vec{q}^{\,T}A^{n} \rightarrow \end{bmatrix}$$

→ notice that the top $(n-1)$ rows of are just the last $(n-1)$ rows of T,

i.e., the latter are shifted up by 1.

→ The last entry, $\vec{q}^T A^n$, can be expressed as a linear combination of the others, using the Cayley-Hamilton Theorem

→ C-H. Thm: A satisfies its own characteristic polynomial.

→ i.e., if $c(\lambda) = \lambda^n + a_n \lambda^{n-1} + \cdots + a_1$ is the char. poly. of A,

then $A^n + a_n A^{n-1} + \cdots + a_1 I = 0$, or

$$A^n = -a_n A^{n-1} - a_{n-1} A^{n-2} - \cdots - a_1 I$$

$$\rightarrow \vec{q}^T A^n = -a_n \vec{q}^T A^{n-1} - a_{n-1} \vec{q}^T A^{n-2} - \cdots - a_2 \vec{q}^T A - a_1 \vec{q}^T$$

→ These two observations can be encapsulated in matrix form as:

$$\rightarrow TA = \overset{TA}{\begin{bmatrix} \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \leftarrow \vec{q}^T A^3 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \\ \leftarrow \vec{q}^T A^n \rightarrow \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & a_{n-1} & a_n \end{bmatrix}}_{\text{call this } \hat{A}} \overset{T}{\begin{bmatrix} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{bmatrix}}$$

→ thus we have $TA = \hat{A} T$ or $\hat{A} = TAT^{-1}$

→ and $(\hat{A}, \hat{b})$ are in c.c.f.

→ A note about a fallacious proof:

→ in some places, a procedure is proposed that:

1. starts from some $(\hat{A}, \hat{b})$ in c.c.f. (with $a_1, \ldots, a_n$ arbitrary)
2. builds $\hat{R}_n \triangleq [\hat{A}^{n-1}\hat{b}, \hat{A}^{n-2}\hat{b}, \ldots, \hat{b}]$ and shows that it is lower triangular, with 1s on the diagonal,
3. defines $R_n \triangleq [A^{n-1}\vec{b}, A^{n-2}\vec{b}, \ldots, \vec{b}]$
4. defines $T = \hat{R}_n R_n^{-1}$

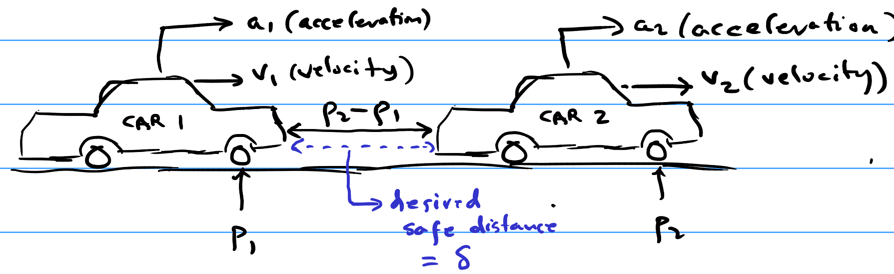
and 5. claims that $\hat{A} = TAT^{-1}$

because step 5 is NOT TRUE IN GENERAL.

→ This procedure DOES NOT PROVE that controllability of $(A, \vec{b}) \Rightarrow (TAT^{-1}, T\vec{b})$ is in c.c.f.

- Why are the above results useful?
  - Because they tell us that if a system is controllable, we can always devise feedback to place its eigenvalues wherever we want.
  - but we don't necessarily have to put it in c.c.f first to do eigenvalue placement.
    - we can do it directly – ie, writing out $c(\lambda) \triangleq \det(A - \vec{b}\vec{k}^T - \lambda I)$ and matching its coefficients w those of $(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$
      ↳ leads to a linear system of eqns in the unknowns $k_1, \dots, k_n$, which can be solved (eg.) numerically.

— Example: co-operative adaptive cruise control



CAR 1

$$v_1 = \frac{dp_1}{dt}$$

$$a_1 = \frac{dv_1}{dt}$$

CAR 2

$$v_2 = \frac{dp_2}{dt}$$

$$a_2 = \frac{dv_2}{dt}$$

$$\frac{d}{dt}(p_2 - p_1) = v_2 - v_1$$

call this $x_1 + \delta$, or $x_1 = p_2 - p_1 - \delta$
↳ will be 0 if cars at desired safe distance

$$\frac{d}{dt}\underbrace{(v_2 - v_1)}_{x_2} = \underbrace{a_2 - a_1}_{u}$$

↓

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} u(t)$$

desired distance

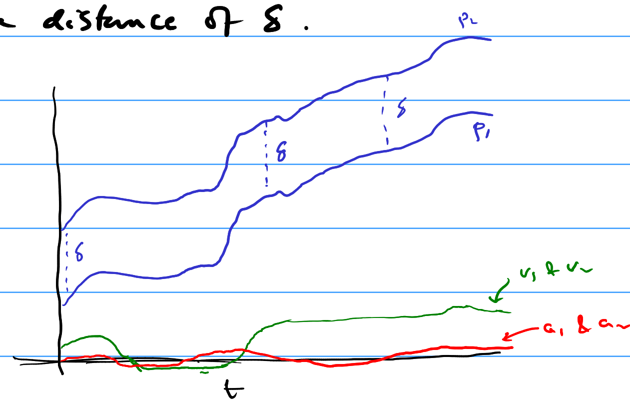
— Suppose, at $t=0$: $x_1 = 0 \Rightarrow p_2 - p_1 = \delta$

$x_2 = 0 \Rightarrow v_2 = v_1 \longleftarrow$ same velocities

$\rightarrow$ and car 1's driver (somehow) perfectly matches car 2's acceleration : i.e, $a_1(t) \equiv a_2(t) \Rightarrow u(t) = 0$.

$\rightarrow$ then $\frac{d\vec{x}}{dt} = 0$, and nothing changes.

$\rightarrow$ they both go at the same velocities, keeping a distance of δ.



— Now, suppose there is a small acceleration error made by the driver of car 1: $a_1(t) = a_2(t) + \underline{\Delta a(t)} \Rightarrow u(t) = \Delta a(t)$

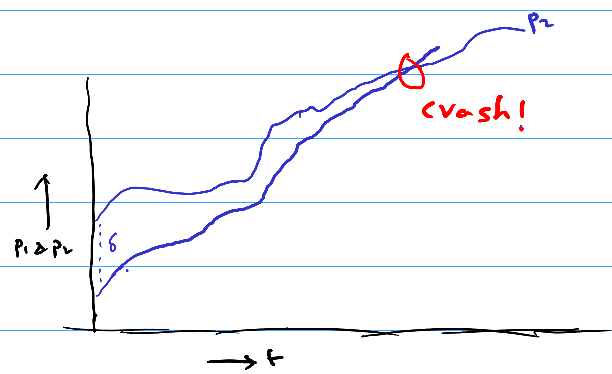
$\rightarrow$ solving the system : we have

$\rightarrow \Delta v(t) = v_2(t) - v_1(t) = \int_0^t \Delta a(\tau)\, d\tau$

$\rightarrow \Delta p(t) = p_2(t) - p_1(t) - \delta = \int_0^t \Delta v(\tau)\, d\tau$

$\rightarrow$ example : Δa = small constant ϵ

$\Rightarrow \Delta v(t) = \epsilon t$, and $\Delta p(t) = \frac{\epsilon t^2}{2}$ $\rightarrow$ keeps increasing without bounds, cars will hit each other if $\epsilon < 0$



— How can feedback help?

$$\rightarrow \quad A \mapsto \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \quad ; \quad c(\lambda) = \lambda(\lambda+k_2) + k_1 = 0 \Rightarrow \lambda^2 + k_2\lambda + k_1 = 0$$

position feedback ; velocity feedback

$\underbrace{\qquad}_{\text{Doppler radar provides both.}}$

$$\lambda_{1,2} = -\frac{k_2}{2} \pm \frac{1}{2}\sqrt{k_2^2 - 4k_1}$$

→ make $k_2 > 0$, $k_1 > 0$

→ $\lambda_{1,2}$ both will have -ve real parts

$$\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = \lambda x_1$$

$$-(k_1 x_1 + k_2 x_2) = \lambda x_2$$

$$\Rightarrow -k_1 x_1 - k_2 \lambda x_1 = \lambda^2 x_1 \Rightarrow \underbrace{(\lambda^2 + k_2\lambda + k_1)}_{\rightarrow 0} x_1 = 0$$

$\dot{\vec{x}} = Ax + \vec{b}u$

$P\dot{\vec{z}} = AP\vec{z} + \vec{b}u$

$\dot{\vec{z}} = P^{-1}AP z + P^{-1}\vec{b}u$

$A = P \Lambda P^{-1}$

$\Updownarrow$

$\vec{z} = P^{-1}\vec{x} \Leftrightarrow \vec{x} = P\vec{z}$ ✓

say $x_1 = 1$

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad ; \quad P^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)} \Delta a(\tau)\, d\tau \\ \lambda_2 \int_0^t e^{\lambda_2(t-\tau)} \Delta a(\tau)\, d\tau \end{bmatrix}$$

if $\Delta a(t) \equiv \epsilon$

$$\int_0^t e^{\lambda(t-\tau)} d\tau \overset{\lambda \neq 0}{=} \frac{1}{\lambda}\left[1 - e^{\lambda t}\right]$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \frac{\epsilon}{\lambda_1}\left[1 - e^{\lambda_1 t}\right] \\ \epsilon\left[1 - e^{+\lambda_2 t}\right] \end{bmatrix}$$

always in the range $[0, \epsilon/\lambda_1]$

always in the range $[0, \epsilon]$

hence $x_1(t) = \frac{\epsilon}{\lambda_1}\left[1 - e^{\lambda_1 t}\right] + \epsilon\left[1 - e^{\lambda_2 t}\right] \in \left[0, 1 + \frac{1}{\lambda_1}\right]$

make λ_1 large this $\rightarrow 1$

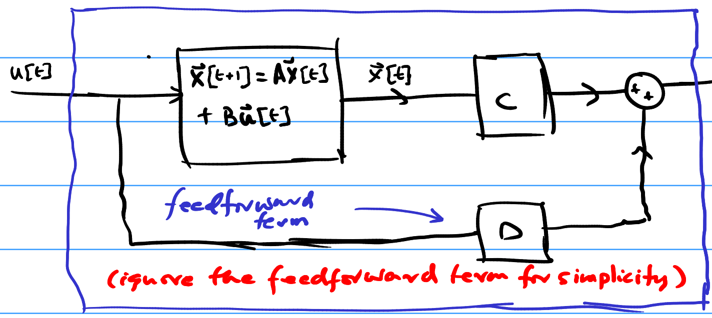
OBSERVABILITY AND OBSERVERS

→ So far, we have mostly been treating $\vec{x}(t)$ as the output $\vec{y}(t)$

→ but in most practical situations, the outputs are different from the state.

→ recall our output equation: $\vec{y}[t] = C\vec{x}[t] + D\vec{u}[t]$

(BACK TO DISCRETE SYSTEMS)



$\Rightarrow \vec{y}[t] = C\,\vec{x}[t]$

m×n matrix

size m · size n

→ Q: from measurements of the output (and knowing the input) can you infer the state? (especially if $m < n$)

→ If yes, the system (A, B, C) is called <u>OBSERVABLE</u>.

→ Say, <u>$m = 1$</u> (just one output), while $\vec{x} \in \mathbb{R}^n$, $n > 1$

$$y[t] = \vec{c}^{\,T}\vec{x}[t]$$

$[c_1\ c_2 \cdots c_n] \rightarrow 1\times n$

$$y[0] = \vec{c}^{\,T}\vec{x}[0]$$

$$y[1] = \vec{c}^{\,T}\vec{x}[1] = \vec{c}^{\,T}\left(A\vec{x}[0] + B\vec{u}[0]\right)$$

$$y[2] = \vec{c}^{\,T}\vec{x}[2] = \vec{c}^{\,T}\left(A^2\vec{x}[0] + AB\vec{u}[0] + B\vec{u}[1]\right)$$

$$\vdots$$

$$y[i] = \vec{c}^{\,T}\vec{x}[i] = \vec{c}^{\,T}\left(A^i\vec{x}[0] + \sum_{j=1}^{i}\left(A^{i-j}B\vec{u}[j-1]\right)\right)$$

$$y[n] = \vec{c}^{\,T}x[n] = \vec{c}^{\,T}\left(A^n\vec{x}[0] + \sum_{j=1}^{n}\left(A^{n-j}B\vec{u}[j-1]\right)\right)$$

→ Now, say $u[t] \equiv 0$ — i.e., no input.

→ might you <u>still</u> be able to recover $\vec{x}[0], \vec{x}[1]$, etc. from $y[t]$?

→ boils down to recovery of just $\vec{x}[0]$, from which $\vec{x}[1], \vec{x}[2]$, etc. can be recovered

$$y(0) = \vec{c}^T \vec{x}(0)$$
$$y(1) = \vec{c}^T \vec{x}(1) = \vec{c}^T A \vec{x}(0)$$
$$y(2) = \vec{c}^T \vec{x}(2) = \vec{c}^T A^2 \vec{x}(0)$$
$$\vdots$$
$$y(i) = \vec{c}^T \vec{x}(i) = \vec{c}^T A^i \vec{x}(0)$$
$$y(n) = \vec{c}^T x(n) = \vec{c}^T A^n \vec{x}(0)$$

→
$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \\ \leftarrow \vec{c}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{c}^T A^{n-1} \rightarrow \end{bmatrix}}_{\text{observability matrix } O} \vec{x}[0]$$

→ if O is full rank (invertible), then yes, otherwise no.

$$\vec{x}[0] = O^{-1} \begin{bmatrix} y(0) \\ \vdots \\ y(n) \end{bmatrix}$$

→ more generally, if $m > 1$ (several outputs), $\vec{y} = C\vec{x}$

$$O = km \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix} \leftarrow \text{must be full rank, i.e., rank} = n$$

→ Observability always boils down to finding just the IC, which we don't know

→ we know $A, B, C, u(t)$ and $y(t)$

└→ CLARIFY THIS POINT IN THE BEGINNING.

EXAMPLE:

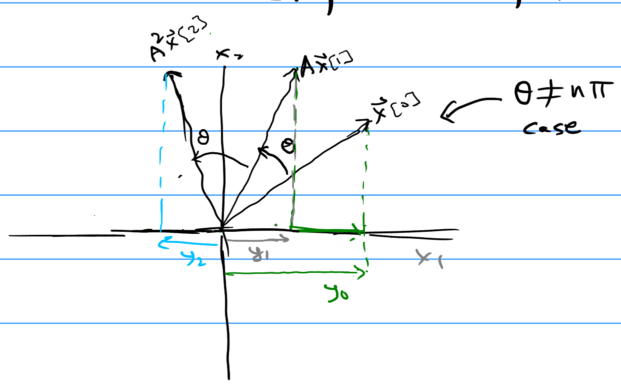
$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^{A} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

A → this is a rotation matrix

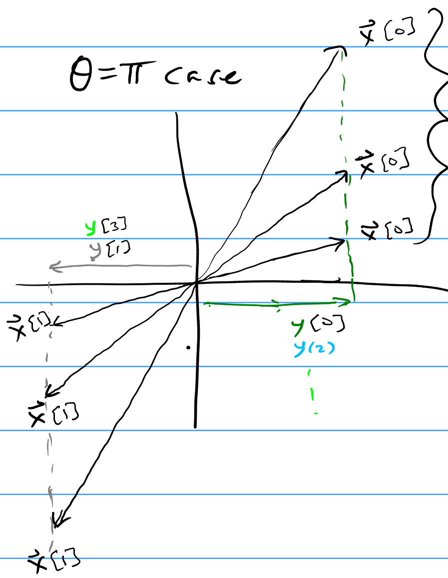
$$y[t] = x_1[t] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\vec{c}^T} \vec{x}[t]$$

$$\begin{bmatrix} \leftarrow c^T \rightarrow \\ \leftarrow c^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & -\sin(\theta) \end{bmatrix} : \quad \begin{array}{l} \text{rank} = 2 \text{ if } \sin(\theta) \neq 0 \\ \text{rank} = 1 \text{ if } \theta = 0, \pi, 2\pi, \ldots \end{array}$$

Graphical interpretation



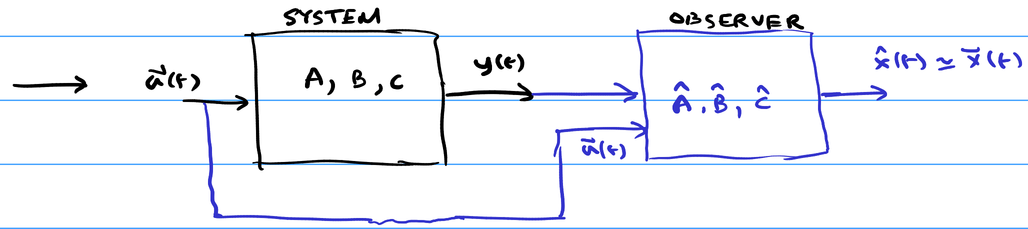
$\theta = \pi$ case



Any of these could be $\vec{x}(0)$, matching measured y_0 and the rotation matrix

OBSERVERS

→ A NEW SYSTEM (that you set up) TO ESTIMATE $\vec{x}(t)$ from the output



→ Form of observer:

$$\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + B\vec{u}[t] + \underbrace{\overset{n\times m}{L}\big(\overset{m\times n}{C}\hat{\vec{x}}[t] - \vec{y}[t]\big)}$$

if $\hat{\vec{x}}[t] = \vec{x}[t]$, then this term will be 0 and the observer will be just the original system.

— How good is the observer at doing its job (estimating $\hat{x}[t]$ well)?

- define the error $\vec{e}(t) \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$
  - then $\vec{e}[t+1] = \hat{\vec{x}}[t+1] - \vec{x}[t+1]$

$$= A(\hat{\vec{x}}[t] - \vec{x}[t]) + LC(\hat{\vec{x}}[t] - \vec{x}[t])$$

$$\Rightarrow \vec{e}[t+1] = A\vec{e}[t] + LC\vec{e}[t] = (A+LC)\vec{e}[t]$$

→ we would like $\vec{e}[t]$ to go to 0

⇒ choose L to make the eigenvalues of $(A+LC)$ stable!

→ exactly like feedback for controllability!

→ recall: $(A+BK)$ for controllability.

→ you can re-use all that algebra and results: treat like $-B$, treat like K

— e.v.s of $(A+LC)$ = e.v.s of $(A+LC)^T = A^T + C^T L^T$

→ i.e., can always place eigenvalues if $(A^T, -C^T)$ controllable

$$\Rightarrow \left[(A^T)^{n-1}C^T, (A^T)^{n-2}C^T, \dots, A^T C^T, C^T\right] \text{ has full rank}$$

$$\Rightarrow \left[(A^T)^{k-1}C^T, (A^T)^{k-2}C^T, \dots, A^TC^T, C^T\right]^T \text{ has full rank}$$

$$\hookrightarrow = \begin{bmatrix} — & CA^{k-1} & — \\ — & CA^{k-2} & — \\ & \vdots & \\ — & CA & — \\ — & C & — \end{bmatrix} \leftarrow \text{simply the observability criterion for the system}$$

Example: same rotation matrix example

—
$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^{A} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

$$y[t] = x_1[t] = \underbrace{[1 \quad 0]}_{\vec{c}^T} \vec{x}[t]$$

— try $\theta = \pi/2$ as a special case: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\vec{c}^T = [1 \quad 0]$

$\rightarrow$ $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $(\vec{c}^T)^T = \vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow$ e.v.s of A^T: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j \leftarrow$ magnitude $= 1$, (discrete-time) BIBO unstable

$\rightarrow$ feedback: $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \Rightarrow \vec{c}\vec{l}^T = \begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow A^T + \vec{c}\vec{l}^T = \begin{bmatrix} l_1 & 1 + l_2 \\ -1 & 0 \end{bmatrix}$$

eigenvalues: $(l_1 - \lambda)(-\lambda) + (1 + l_2) = 0$

$$\Rightarrow \lambda^2 - l_1\lambda + (1 + l_2) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{l_1}{2} \pm \frac{\sqrt{l_1^2 - 4(1+l_2)}}{2}$$

$$\Rightarrow \underline{\lambda_1 + \lambda_2 = l_1} \text{ and } \lambda_1 - \lambda_2 = +\sqrt{l_1^2 - 4(1+l_2)}$$

$$\Rightarrow (\lambda_1 - \lambda_2)^2 = l_1^2 - 4(1+l_2)$$

$$= (\lambda_1 - \lambda_2)^2 - (\lambda_1 + \lambda_2)^2 = -4(1+l_2)$$

$$\Rightarrow -4\lambda_1\lambda_2 = -4(1+l_2)$$

$$\Rightarrow \underline{l_2 = \lambda_1\lambda_2 - 1}$$

→ choose any 2 stable λs (respecting complex conjugacy)
and set l_1 & l_2

→ what if $\theta = \pi$ (unobservable case)?

$$\rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad A^T = A$$

$$A^T + \vec{c}\,\vec{l}^T = \begin{bmatrix} -1+l_1 & l_2 \\ 0 & -1 \end{bmatrix}$$

det: $-(\lambda+1)(-1+l_1-\lambda) = 0$

$$\Rightarrow \boxed{\lambda_1 = -1}, \quad \lambda_2 = l_1 - 1$$

↳ BIBO unstable and cannot be changed by choosing $\vec{l}$.

→ USING OBSERVERS FOR MORE ACCURATE POSITIONING & NAVIGATION

→ recall the accelerating car discrete-time example from the lecture on controllability:

$$\begin{bmatrix} x((t+1)T) \\ v((t+1)T) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}^{A} \begin{bmatrix} x(tT) \\ v(tT) \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}^{\vec{b}} a(t)$$

– Terminology: $\underset{\text{discrete}}{x[t]} \equiv \underset{\text{continuous}}{x(tT)}$; $v[t] \equiv v(tT)$; $a[t] \equiv a(tT)$

→ eigenvalues: $(1-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = \pm 1$ → BIBO UNSTABLE

↳ TYPICAL FOR CARS, PLANES, ETC.

→ CONSIDER the scenario:

→ the car is NOT feedback stabilized

– there is (inevitable) error in the accel: $a[t] = \underbrace{a_d[t]}_{\text{correct accel. for navigation}} + \underbrace{\Delta a[t]}_{\text{accel. error (eg, pedal play)}}$ ← not precisely known, but small

→ we estimate its state using an observer:

→ $\hat{x}[t+1] = A\hat{x}[t] + \vec{b}\, a_d[t]$ → same as system, no $\vec{l}\,\vec{c}^T$ term
→ we don't know $\Delta a(t)$

→ the estimation error follows:

$$\vec{e}[t+1] = A\vec{e}[t] + \vec{b}\,\Delta a[t]$$ ← unstable ⇒ $\vec{e}(t)$ grows larger and larger even if $\Delta a(t)$ remains small.

→ but, if we are able to measure the position of the actual car, say using GPS measurements:

$$y[t] = \underbrace{[1 \quad 0]}_{\vec{c}^T} \vec{x}[t] + \Delta p[t]$$

→ GPS measurement error, also not precisely known, but small.

→ <u>Our observer becomes:</u> $\vec{l}\,(\vec{c}^T\hat{\vec{x}} - y(t))$

$$\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + \vec{b}\,a_d[t] + \vec{l}\,(\vec{c}^T\hat{\vec{x}} - y(t))$$

$$= A\hat{\vec{x}}[t] + \vec{b}\,a_d[t] + \vec{l}\vec{c}^T\hat{\vec{x}}[t] - \vec{l}\vec{c}^T\vec{x}[t] - \vec{l}\,\Delta p[t]$$

$$= (A + \vec{l}\vec{c}^T)\hat{\vec{x}}[t] + \vec{l}\vec{c}^T(\hat{\vec{x}}[t] - \vec{x}[t]) + \vec{b}\,a_d[t] - \vec{l}\,\Delta p[t]$$

Hence the error $\vec{e}[t] \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$ obeys

$$\vec{e}[t+1] = A\vec{e}[t] + \vec{l}\vec{c}^T\vec{e}[t] + \underbrace{(\vec{b}\,a_d[t] - \vec{l}\,\Delta p[t])}_{\text{due to errors, but small}}$$

$$\Rightarrow \vec{e}[t+1] = (A + \vec{l}\vec{c}^T)\vec{e}[t] + \underbrace{(\vec{b}\,a_d[t] - \vec{l}\,\Delta p[t])}_{\text{small}}$$

→ Now, if $(A, \vec{c})$ are observable, you can always choose $\vec{l}$ to stabilize the observer

→ which means that the estimation error can <u>always</u> be kept bounded and small, even taking into account small errors in GPS & acceleration

↳ this is what all navigational systems today do!

→ Actually, they use an additional improvement, whereby $\vec{l}$ changes w/ t (i.e., becomes $\vec{l}[t]$)

→ the updates are used to minimize the error due to GPS/pedal errors, other noise, etc, even further

↳ this is the famous KALMAN FILTER, the gold standard for accurate navigational estimation

— WHAT IT ACHIEVES:

- open-loop position/velocity estimation → blows up errors ∵ unstable
- estimation simply by GPS also has errors

- using a well-designed observer, you can make much better estimates than either of these 2 alone. ← quite remarkable

- and then, you can use this good estimate of $\hat{x}$ to feed back to the car (via controllability) to make its response stable.
  → THIS IS WHAT IS GOING ON IN AUTONOMOUS VEHICLES (TESLA PICTURE)

— PICTURES AND HISTORY OF KALMAN.

→ ALSO: FEEDBACK STABILIZED LEVITRON DEMOS (earlier)