EE 16B, SPRING 2018

NOTES FOR LECTURES 6B & 7A: CONTROLLER CANONICAL FORM & OBSERVABILITY

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CONTROLLER CANONICAL FORM (CCF)

→ Recall a previous controllability example : $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$

→ its characteristic polynomial $c(\lambda) \triangleq \det(A - \lambda I)$ had a nice form:

→ $-\lambda(a_2 - \lambda) - a_1 \equiv \boxed{\lambda^2 - a_2\lambda - a_1}$

→ A rather elegant observation someone made long ago is this:

→ suppose you have a size n matrix in the same form:

$$\underbrace{A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}; \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{TOGETHER CALLED CONTROLLER CANONICAL FORM (CCF)}}$$

→ then, $\boxed{c(\lambda) \triangleq \det(A - \lambda I) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_3\lambda^2 - a_2\lambda - a_1}$

→ proof: easily (though laboriously) obtained by applying the formula for determinants involving expansion using minors to $(A - \lambda I)$. Try it yourself if you have some time on your hands and you like getting to the bottom of things.

→ Now, suppose you have a scalar input to your system $(u(t))$ and the system is: $\underbrace{\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}\,u(t)}$

this (the original system without feedback) is called the OPEN LOOP system

→ Now, if you apply feedback $u(t) \mapsto u(t) - \vec{k}^T \vec{x}(t)$, then the "closed loop system" becomes:

$$\dot{\vec{x}} = (A - \vec{b}\vec{k}^T)\vec{x} + \vec{b}\,u(t),$$

with

$$(A - \vec{b}\vec{k}^T) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ a_1-k_1 & a_2-k_2 & a_3-k_3 & \cdots & a_n-k_n \end{bmatrix}$$

→ with characteristic polynomial: $C_f(\lambda) \triangleq \det(A - \vec{b}\vec{k}^T - \lambda I)$

$$= \lambda^n - (a_n-k_n)\lambda^{n-1} - (a_{n-1}-k_{n-1})\lambda^{n-2} - \cdots - (a_3-k_3)\lambda^2 - (a_2-k_2)\lambda - (a_1-k_1)$$

→ Now, our goal is to choose $k_1, \dots, k_n$ (the feedback) to place the eigenvalues of $(A - \vec{b}\vec{k}^T)$, i.e, the roots of $C_f(\lambda)$, wherever we want.

→ Suppose we want the roots to be $\underbrace{\lambda_1, \lambda_2, \dots, \lambda_n}$

↳ remember that if complex, the conjugate must also be present.

→ i.e, the desired char. poly. is:

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\cdots(\lambda-\lambda_n)$$

$$= \lambda^n \underbrace{- (\lambda_1+\lambda_2+\cdots+\lambda_n)}_{b_n}\lambda^{n-1} + \underbrace{\left(\begin{array}{l} \lambda_1\lambda_2+\lambda_1\lambda_3+\cdots\lambda_1\lambda_n \\ +\lambda_2\lambda_3+\lambda_2\lambda_4+\cdots\lambda_2\lambda_n \\ +\lambda_3\lambda_4+\cdots\quad\lambda_3\lambda_n \\ +\lambda_{n-1}\lambda_n \end{array}\right)}_{b_{n-1}}\lambda^{n-2} + \cdots + \underbrace{(-1)^n\lambda_1\lambda_2\lambda_3\cdots\lambda_n}_{b_1}$$

→ i.e., $b_1, \dots, b_n$ can be calculated from $\lambda_1, \dots, \lambda_n$, though the process may be extremely tedious.

→ now all you have to do to devise the feedback is equate the coefficients of the 2 char. poly. expressions:

$$a_n - k_n = -b_n$$

$$a_{n-1} - k_{n-1} = -b_{n-1}$$

$$\vdots$$

$$a_1 - k_1 = -b_1$$

→ I.e, if $(A, \vec{b})$ is in controllability canonical form, then it is always possible to devise feedback to place the eigenvalues wherever you like.

→ Size 3 example:

→ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$; $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

↓

→ $A - \vec{b}\vec{k}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-k_1 & 2-k_2 & 3-k_3 \end{bmatrix}$

→ char. poly.: $\lambda^3 - (3-k_3)\lambda^2 - (2-k_2)\lambda - (1-k_1)$

→ Suppose we want the 3 new eigenvalues to be:

$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$

$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3$

→ Therefore we should set: $k_3 = 3, k_2 = 2, k_1 = 1$

→ Suppose we want $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$

$$\Rightarrow (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \lambda(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) - \lambda_1\lambda_2\lambda_3$$

$$= \lambda^3 + 6\lambda^2 + \lambda(\underbrace{2+6+3}_{11}) + 6$$

$$\Rightarrow -(3-k_3) = -6 \Rightarrow k_3 = -3$$

$$\Rightarrow -(2-k_2) = -11 \Rightarrow k_2 = -9$$

$$\Rightarrow -(1-k_1) = -6 \Rightarrow k_1 = -5$$

→ But can any system be put in controller canonical form?

→ A: yes, if it is controllable!

→ Here is how you put any controllable system in c.c.f.:

0. Given a state-space system $\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u(t)$ (not necessarily in CCF)
   ↳ or $\frac{d\vec{x}}{dt} = A\vec{x}(t) + \vec{b}u(t)$
1. Form the controllability matrix:
   - $R_n \triangleq [\vec{b}, A\vec{b}, A^2\vec{b}, \dots, A^{n-1}\vec{b}]$ → a square matrix
   - if the system is controllable, then R_n is full rank, i.e., invertible.

2. Calculate R_n^{-1}
3. Take the last row of R_n^{-1} — call this last row $\vec{q}^T$ ($\vec{q}$ is a col. vector; $\vec{q}^T$ a row vector)
4. Form the following matrix, row by row:

$$T \triangleq \begin{bmatrix} \longleftarrow \vec{q}^T \longrightarrow \\ \longleftarrow \vec{q}^T A \longrightarrow \\ \longleftarrow \vec{q}^T A^2 \longrightarrow \\ \vdots \\ \longleftarrow \vec{q}^T A^{n-1} \longrightarrow \end{bmatrix} \begin{matrix} \longleftarrow \text{this is an } n \times n \text{ matrix} \\ \longleftarrow \text{it will be full rank and invertible} \end{matrix}$$

5. Define $\vec{z} = T\vec{x} \Longleftrightarrow \vec{x} = T^{-1}\vec{z}$ (this is called a basis transformation from $\vec{x}$ to $\vec{z}$, using T)

6. Using the definition of $\vec{z}$ in $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}u(t)$, you get, equivalently,

$$\frac{d\vec{z}}{dt} = \overbrace{TAT^{-1}}^{\hat{A}}\vec{z} + \overbrace{T\vec{b}}^{\hat{b}}\,u(t)$$

7. then:

→ $\hat{A} \triangleq TAT^{-1}$, with $\hat{b} \triangleq T\vec{b}$ will be in c.c.f !

→ Proof of the above procedure and claims:

→ from definition, $R_n^{-1} R_n = I$, or

$$\rightarrow \overset{R_n^{-1}}{\begin{bmatrix} \text{———} \\ \text{———} \\ \vdots \\ \text{———} \\ \leftarrow \vec{q}^T \rightarrow \end{bmatrix}} \overset{R_n}{\begin{bmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & | & & | \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & & 0 & 0 \\ 0 & 1 & & & 0 \\ \cdot & 0 & \ddots & 0 & 0 \\ & \vdots & & 0 & \cdot \\ & & & 1 & \cdot \\ 0 & 0 & & 0 & 1 \end{bmatrix}}$$

$$\longrightarrow \text{Hence: } \vec{q}^T \overset{R_n}{\begin{bmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & | & & | \end{bmatrix}} = [0, 0, \ldots\ 0,\ 1]$$

$$\Longrightarrow \vec{q}^T\vec{b} = 0,\ \vec{q}^T A\vec{b} = 0, \ldots, \vec{q}^T A^{n-2}\vec{b} = 0$$

and $\vec{q}^T A^{n-1}\vec{b} = 1$

→ also from definition, $T \triangleq \begin{bmatrix} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{bmatrix} \Longrightarrow \begin{bmatrix} \vec{q}^T\vec{b} \\ \vec{q}^T A\vec{b} \\ \vec{q}^T A^2\vec{b} \\ \vdots \\ \vec{q}^T A^{n-1}\vec{b} \end{bmatrix} = \overbrace{T\vec{b}}^{\hat{b}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

→ Consider TA:

$$\longrightarrow TA: \begin{bmatrix} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{bmatrix} A = \begin{bmatrix} \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \leftarrow \vec{q}^T A^3 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \\ \leftarrow \vec{q}^T A^n \rightarrow \end{bmatrix}$$

→ notice that the top $(n-1)$ rows of are just the last $(n-1)$ rows of T, i.e, the latter are shifted up by 1.

→ The last entry, $\vec{q}^T A^n$, can be expressed as a linear combination of the others, using the Cayley-Hamilton Theorem

→ C-H. Thm: A satisfies its own characteristic polynomial.

→ i.e., if $c(\lambda) = \lambda^n + a_n \lambda^{n-1} + \cdots + a_1$ is the char. poly. of A,

then $A^n + a_n A^{n-1} + \cdots + a_1 I = 0$, or

$$A^n = -a_n A^{n-1} - a_{n-1} A^{n-2} - \cdots - a_1 I$$

$$\rightarrow \vec{q}^T A^n = -a_n \vec{q}^T A^{n-1} - a_{n-1} \vec{q}^T A^{n-2} - \cdots - a_n \vec{q}^T A - a_1 \vec{q}^T$$

→ These two observations can be encapsulated in matrix form as:

$$\rightarrow TA = \overset{TA}{\begin{bmatrix} \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \leftarrow \vec{q}^T A^3 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \\ \leftarrow \vec{q}^T A^n \rightarrow \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & a_{n-1} & a_n \end{bmatrix}}_{\text{call this } \hat{A}} \overset{T}{\begin{bmatrix} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{bmatrix}}$$

→ thus we have $TA = \hat{A} T$ or $\boxed{\hat{A} = TAT^{-1}}$

we can write this form ONLY if T is invertible

↳ this is proved on the next page.

→ and $(\hat{A}, \hat{b})$ are in c.c.f.

→ A note about a fallacious proof:

→ in some places, a procedure is proposed that:

1. starts from some $(\hat{A}, \hat{b})$ in c.c.f. (with $a_1, \ldots, a_n$ arbitrary)
2. builds $\hat{R}_n \triangleq [\hat{A}^{n-1}\hat{b}, \hat{A}^{n-2}\hat{b}, \ldots, \hat{b}]$ and shows that it is lower triangular, with 1s on the diagonal,
3. defines $R_n \triangleq [A^{n-1}\vec{b}, A^{n-2}\vec{b}, \ldots, \vec{b}]$
4. defines $T = \hat{R}_n R_n^{-1}$

and 5. claims that $\hat{A} = TAT^{-1}$

for example, choosing $a_1 = a_2 \cdots = a_n = 0$ would imply that $\hat{A}$, which is singular, is similar to any matrix A - which is obviously wrong.

because step 5 is NOT TRUE IN GENERAL. for ARBITRARY CHOICES of $a_1, \ldots, a_n$

→ This procedure DOES NOT PROVE that controllability of $(A, \vec{b}) \Rightarrow (TAT^{-1}, T\vec{b})$ is in c.c.f.

→ PROOF THAT T ON THE PREVIOUS PAGE IS INVERTIBLE

- To prove that T is full-rank (or non-singular), we will look at TR_n, and show that this is always non-singular.
  - → Since R_n is non-singular (it is the controllability matrix), this implies that T must be non-singular, i.e., invertible.

- From the above, we know that:
  - $T\vec{b} = \hat{b}$, and
  - $TA = \hat{A}T$,
    - where $(\hat{A}, \hat{b})$ are in CCF.

→ therefore:

$$\rightarrow TA\vec{b} = \hat{A}T\vec{b} = \hat{A}\hat{b} = (\text{last col. of } \hat{A}) = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow (n-1)^{th} \text{ entry} \\ \\ \end{array}$$

$$\rightarrow TA^2\vec{b} = TAA\vec{b} = \hat{A}TA\vec{b} = \hat{A}\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \\ a_n^2 + a_{n-1} \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow (n-2)^{th} \text{ entry} \\ \\ \\ \end{array}$$

$$\rightarrow TA^3\vec{b} = TA(A^2\vec{b}) = \hat{A}TA^2\vec{b} = \hat{A}\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \\ a_n^2 + a_{n-1} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ a_n \\ a_n^2 + a_{n-1} \\ a_{n-2} + a_n a_{n-1} + a_n(a_n^2 + a_{n-1}) \end{array}\right] \begin{array}{l} \\ \\ \leftarrow (n-3)^{th} \text{ entry} \\ \\ \\ \\ \end{array}$$

⋮ all the way to

$$\rightarrow TA^{n-1}\vec{b} = \left[\begin{array}{c} 1 \\ * \\ \vdots \\ * \end{array}\right] \begin{array}{l} \leftarrow 1^{st} \text{ entry} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \end{array}$$

$*$s represent potential non-zero numbers.

— Using the above, we have:

$$TR_n = T\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{b}, & A\vec{b}, & \cdots, & A^{n-1}\vec{b} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ T\vec{b}, & TA\vec{b}, & TA^2\vec{b}, & \cdots, & TA^{n-1}\vec{b} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & - & & & 0 & 1 \\ 0 & 0 & 0 & 0 & - & & & 1 & * \\ 0 & 0 & 0 & 0 & & & 1 & * & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & - & * & * & * \\ 0 & 0 & 0 & 1 & * & \cdots & * & * & * \\ 0 & 0 & 1 & * & * & \cdots & * & * & * \\ 0 & 1 & * & * & * & \cdots & * & * & * \\ 1 & * & * & * & * & \cdots & * & * & * \end{bmatrix}}_{} \longleftarrow \text{call this } L$$

L is a strictly lower triangular matrix, with 1s on a diagonal $\implies$ it is full rank.

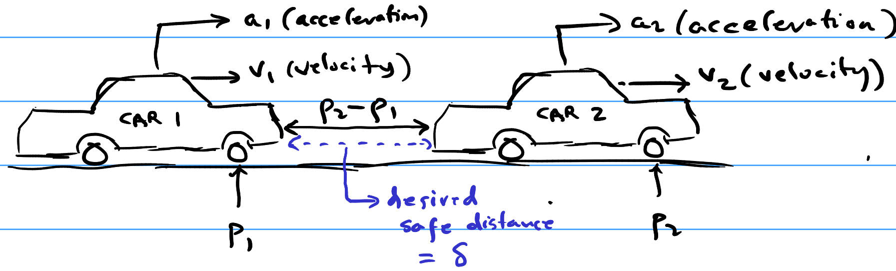
$\longrightarrow$ therefore, we have $TR_n = L$, with R_n and L both full rank / invertible

$$\implies \boxed{\begin{array}{l} T \text{ itself has to be full rank and invertible, with} \\ T = R_n^{-1}L, \quad \text{and } T^{-1} = L^{-1}R_n \end{array}}$$

- Why are the above results useful?
  - Because they tell us that if a system is controllable, we can always devise feedback to place its eigenvalues wherever we want.
  - but we don't necessarily have to put it in c.c.f first to do eigenvalue placement.
    - we can do it directly – ie, writing out $c(\lambda) \triangleq \det(A - \vec{b}\vec{k}^T - \lambda I)$ and matching its coefficients w/ those of $(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$

      ↳ leads to a linear system of eqns in the unknowns $k_1, \dots, k_n$, which can be solved (e.g.) numerically.

- **Example**: co-operative adaptive cruise control



CAR 1	CAR 2
$v_1 = \frac{dp_1}{dt}$	$v_2 = \frac{dp_2}{dt}$
$a_1 = \frac{dv_1}{dt}$	$a_2 = \frac{dv_2}{dt}$

$$\frac{d}{dt}(\underbrace{p_2 - p_1}_{\text{call this } x_1 + \delta, \text{ or } x_1 = \underbrace{p_2 - p_1 - \delta}}) = v_2 - v_1$$

↳ will be 0 if cars at desired safe distance

$$\frac{d}{dt}(\underbrace{v_2 - v_1}_{x_2}) = \underbrace{a_2 - a_1}_{u}$$

↓

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} u(t)$$

desired distance

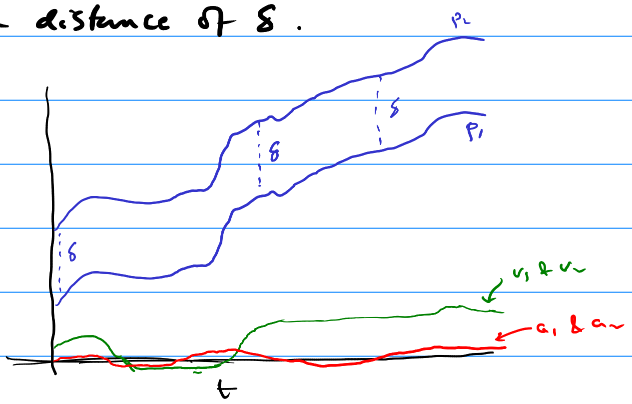
– Suppose, at $t=0$: $x_1 = 0 \Rightarrow p_2 - p_1 = \delta$

$x_2 = 0 \Rightarrow v_2 = v_1 \longleftarrow$ same velocities

→ and car 1's driver (somehow) perfectly matches car 2's acceleration : i.e, $a_1(t) \equiv a_2(t) \Rightarrow u(t) = 0$.

→ then $\frac{d\vec{x}}{dt} = 0$, and nothing changes.

→ they both go at the same velocities, keeping a distance of δ .



– Now, suppose there is a small acceleration error made by the driver of car 1: $a_1(t) = a_2(t) + \underline{\Delta a(t)} \Rightarrow u(t) = \Delta a(t)$

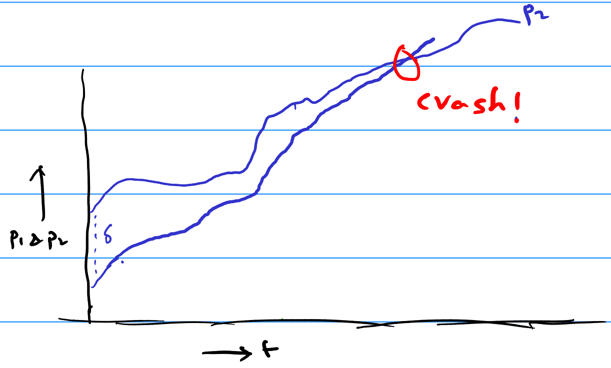
→ solving the system : we have

→ $\Delta v(t) = v_2(t) - v_1(t) = \int_0^t \Delta a(\tau)\,d\tau$

→ $\Delta p(t) = p_2(t) - p_1(t) - \delta = \int_0^t \Delta v(\tau)\,d\tau$

→ example : Δa = small constant ϵ

$\Rightarrow \Delta v(t) = \epsilon t$, and $\Delta p(t) = \boxed{\frac{\epsilon t^2}{2}}$ → keeps increasing without bounds, cars will hit each other if $\epsilon < 0$



— How can feedback help?

$$\rightarrow \quad A \mapsto \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_{} \; ; \quad c(\lambda) = \lambda(\lambda+k_2) + k_1 = 0 \Rightarrow \lambda^2 + k_2\lambda + k_1 = 0$$

$-k_1$ ↓ position feedback

$-k_2$ ↘ velocity feedback

Doppler radar provides both.

$$\lambda_{1,2} = -\frac{k_2}{2} \pm \frac{1}{2}\sqrt{k_2^2 - 4k_1}$$

→ make $k_2 > 0,\ k_1 > 0$

→ $\lambda_{1,2}$ both will have -ve real parts

$$\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = \lambda x_1$$

$$-(k_1 x_1 + k_2 x_2) = \lambda x_2$$

$$\Rightarrow -k_1 x_1 - k_2 \lambda x_1 = \lambda^2 x_1 \Rightarrow \underbrace{(\lambda^2 + k_2\lambda + k_1)}_{\to 0} x_1 = 0$$

say $x_1 = 1$

$\dot{\vec{x}} = Ax + \vec{b}u$

$P\dot{\vec{z}} = AP\vec{z} + \vec{b}u$

$\dot{\vec{z}} = \underbrace{P^{-1}AP}_{} z + P^{-1}\vec{b}u$

$\tilde{A} = P \Lambda P^{-1}$

$\Updownarrow$

$\vec{z} = P^{-1}\vec{x} \Leftrightarrow \vec{x} = P\vec{z}$ ✓

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} ; \quad P^{-1} = \frac{1}{\lambda_2 - 1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \int_0^t e^{\lambda_1(t-\tau)}\Delta a(\tau)\,d\tau \\ \lambda_2 \int_0^t e^{\lambda_2(t-\tau)}\Delta a(\tau)\,d\tau \end{bmatrix}$$

if $\Delta a(t) \equiv \epsilon$

$$\int_0^t e^{\lambda(t-\tau)}\,d\tau \overset{\lambda \neq 0}{=} \frac{1}{\lambda}\left[1 - e^{\lambda t}\right]$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \frac{\epsilon}{\lambda_1}\left[1 - e^{\lambda_1 t}\right] \\ \epsilon\left[1 - e^{+\lambda_2 t}\right] \end{bmatrix}$$

first entry → always in the range $[0, \epsilon/\lambda_1]$

second entry → always in the range $[0, \epsilon]$

$$\text{hence } x_1(t) = \frac{\epsilon}{\lambda_1}\left[1 - e^{\lambda_1 t}\right] + \epsilon\left[1 - e^{\lambda_2 t}\right] \in \Big[0,\ 1 + \underset{\uparrow}{1/\lambda_1}\Big]$$

make λ_1 large

this → 1

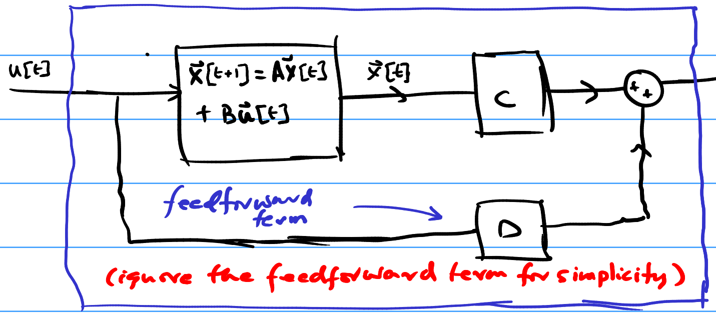
OBSERVABILITY AND OBSERVERS

→ So far, we have mostly been treating $\vec{x}(t)$ as the output $\vec{y}(t)$

→ but in most practical situations, the outputs are different from the state.

→ recall our output equation: $\vec{y}[t] = C\vec{x}[t] + D\vec{u}[t]$

(BACK TO DISCRETE SYSTEMS)



(ignore the feedforward term for simplicity)

$$\Rightarrow \underset{\text{size } m}{\vec{y}[t]} = \overset{m\times n \text{ matrix}}{C}\ \underset{\text{size } n}{\vec{x}[t]}$$

→ Q: from measurements of the output (and knowing the input) can you infer the state? (especially if $m < n$)

→ If yes, the system (A, B, C) is called **OBSERVABLE**.

→ Say, $\underline{m = 1}$ (just one output), while $\vec{x} \in \mathbb{R}^n$, $n > 1$

$$y[t] = \underset{[c_1\ c_2 \cdots c_n] \to 1\times n}{\vec{c}^T}\ \vec{x}[t]$$

$$y[0] = \vec{c}^T \vec{x}[0]$$

$$y[1] = \vec{c}^T \vec{x}[1] = \vec{c}^T(A\vec{x}[0] + B\vec{u}[0])$$

$$y[2] = \vec{c}^T \vec{x}[2] = \vec{c}^T(A^2\vec{x}[0] + AB\vec{u}[0] + B\vec{u}[1])$$

$$\vdots$$

$$y[i] = \vec{c}^T \vec{x}[i] = \vec{c}^T\left(A^i\vec{x}[0] + \sum_{j=1}^{i}(A^{i-j}B\vec{u}[j-1])\right)$$

$$y[n] = \vec{c}^T x[n] = \vec{c}^T\left(A^n\vec{x}[0] + \sum_{j=1}^{n}(A^{n-j}B\vec{u}[j-1])\right)$$

→ Now, say $u[t] \equiv 0$ — i.e., no input.

→ might you still be able to recover $\vec{x}[0], \vec{x}[1]$, etc. from $y[t]$?

→ boils down to recovery of just $\vec{x}[0]$, from which $\vec{x}[1], \vec{x}[2]$, etc. can be recovered

$y(0) = \vec{c}^T \vec{x}(0)$

$y(1) = \vec{c}^T \vec{x}(1) = \vec{c}^T A \vec{x}(0)$

$y(2) = \vec{c}^T \vec{x}(2) = \vec{c}^T A^2 \vec{x}(0)$

$\vdots$

$y(i) = \vec{c}^T \vec{x}(i) = \vec{c}^T A^i \vec{x}(0)$

$y(n) = \vec{c}^T x(n) = \vec{c}^T A^n \vec{x}(0)$

→
$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n] \end{bmatrix} = \begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \\ \leftarrow \vec{c}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{c}^T A^{n-1} \rightarrow \end{bmatrix} \vec{x}[0]$$

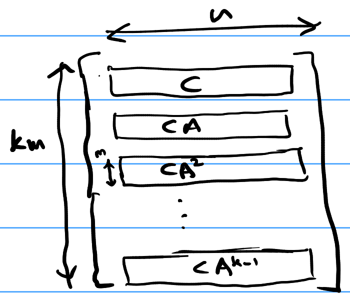
└→ observability matrix $\mathcal{O}$

→ if $\mathcal{O}$ is full rank (invertible), then yes, otherwise no.

$$\vec{x}[0] = \mathcal{O}^{-1} \begin{bmatrix} y(0) \\ \vdots \\ y(n) \end{bmatrix}$$

→ more generally, if $m > 1$ (several outputs), $\vec{y} = C\vec{x}$

$\mathcal{O} =$



← must be full rank, i.e., rank = n

→ Observability always boils down to finding just the IC, which we don't know

→ we know A, B, C, u(t) and y(t)

└→ CLARIFY THIS POINT IN THE BEGINNING.

this is a rotation matrix

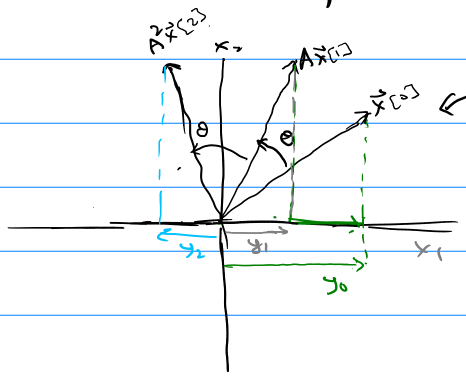
EXAMPLE :

$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^{A} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

$$y[t] = x_1[t] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\vec{c}^T} \vec{x}[t]$$

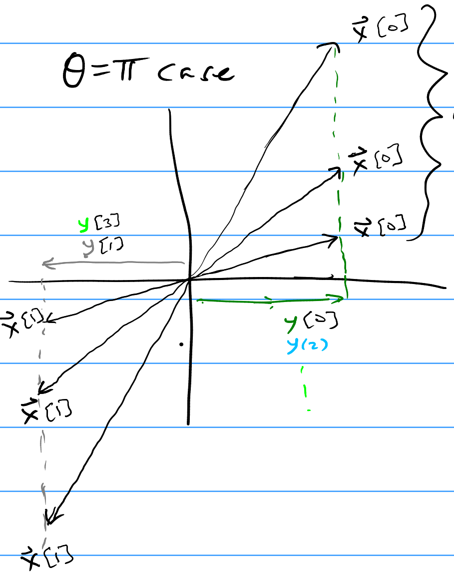
$$\begin{bmatrix} \leftarrow c^T \rightarrow \\ \leftarrow c^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & -\sin(\theta) \end{bmatrix} \quad : \quad \begin{array}{l} \text{rank} = 2 \text{ if } \sin(\theta) \neq 0 \\ \text{rank} = 1 \text{ if } \theta = 0, \pi, 2\pi, \ldots \end{array}$$

Graphical interpretation



$\theta \neq n\pi$ case

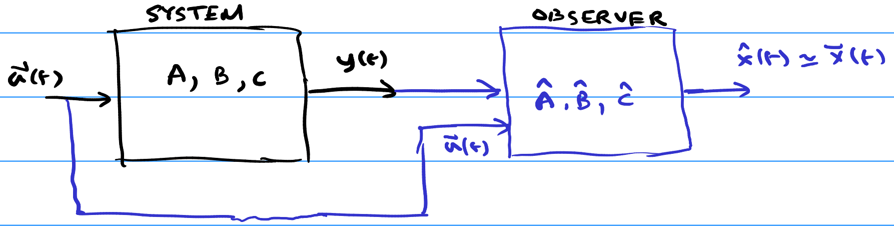
$\theta = \pi$ case



Any of these could be $\vec{x}(0)$, matching measured y_0 and the rotation matrix

OBSERVERS

→ A NEW SYSTEM (that you set up) TO ESTIMATE $\vec{x}(t)$ from the output

→ 

→ Form of observer: $\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + B\vec{u}[t] + \underbrace{\overset{n\times m}{L}\left(\overset{m\times n}{C}\hat{\vec{x}}[t] - \vec{y}[t]\right)}$

if $\hat{\vec{x}}[t] \simeq \vec{x}[t]$, then this term will be 0 and the observer will be just the original system.

— How good is the observer at doing its job (estimating $\hat{x}[t]$ well)?

- define the error $\vec{e}(t) \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$

— then $\vec{e}[t+1] = \hat{\vec{x}}[t+1] - \vec{x}[t+1]$

$$= A(\hat{\vec{x}}[t] - \vec{x}[t]) + LC(\hat{\vec{x}}[t] - \vec{x}[t])$$

$$\Longrightarrow \vec{e}[t+1] = A\vec{e}[t] + LC\vec{e}[t] = (A+LC)\vec{e}[t]$$

→ we would like $\vec{e}[t]$ to go to 0

$\Rightarrow$ choose L to make the eigenvalues of $(A+LC)$ stable!

→ exactly like feedback for controllability!

→ recall: $(A+BK)$ for controllabity.

→ you can re-use all that algebra and results:

— e.v.s of $(A+LC)$ = e.v.s of $(A+LC)^T = A^T + \overset{\text{treat like } -B}{C^T}\,\overset{\text{treat like } K}{L^T}$

→ i.e., can always place eigenvalues if $(A^T, -C^T)$ controllable

$\Rightarrow \left[(A^T)^{n-1}C^T, (A^T)^{n-2}C^T, \dots, A^TC^T, C^T\right]$ has full rank

$$\Rightarrow \left[(A^T)^{k-1}C^T, (A^T)^{k-2}C^T, \dots, A^TC^T, C^T\right]^T \text{ has full rank}$$

$$\hookrightarrow = \begin{bmatrix} — & CA^{k-1} & — \\ — & CA^{k-2} & — \\ & \vdots & \\ — & CA & — \\ — & C & — \end{bmatrix} \longleftarrow$$

simply the Observability criterion for the system

<u>Example</u>: same rotation matrix example

— $$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^{A} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

$$y[t] = x_1[t] = \underbrace{[1 \quad 0]}_{\vec{c}^T} \vec{x}[t]$$

— try $\theta = \pi/2$ as a special case: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\vec{c}^T = [1 \quad 0]$

→ $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $(\vec{c}^T)^T = \vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

→ e.v.s of A^T: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j \longleftarrow$ <u>magnitude = 1, (discrete-time) BIBO unstable</u>

→ feedback: $\vec{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \Rightarrow \vec{c}\vec{\ell}^T = \begin{bmatrix} \ell_1 & \ell_2 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow A^T + \vec{c}\vec{\ell}^T = \begin{bmatrix} \ell_1 & 1+\ell_2 \\ -1 & 0 \end{bmatrix}$$

eigenvalues: $(\ell_1 - \lambda)(-\lambda) + (1 + \ell_2) = 0$

$$\Rightarrow \lambda^2 - \ell_1\lambda + (1 + \ell_2) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{\ell_1}{2} \pm \frac{\sqrt{\ell_1^2 - 4(1+\ell_2)}}{2}$$

$$\Rightarrow \underline{\lambda_1 + \lambda_2 = \ell_1} \text{ and } \lambda_1 - \lambda_2 = +\sqrt{\ell_1^2 - 4(1+\ell_2)}$$

$$\Rightarrow (\lambda_1 - \lambda_2)^2 = \ell_1^2 - 4(1+\ell_2)$$

$$= (\lambda_1 - \lambda_2)^2 - (\lambda_1 + \lambda_2)^2 = -4(1+\ell_2)$$

$$\Rightarrow -4\lambda_1\lambda_2 = -4(1+\ell_2)$$

$$\Rightarrow \underline{l_2 = \lambda_1\lambda_2 - 1}$$

→ choose any 2 stable λs (respecting complex conjugacy) and set l_1 & l_2

→ what if $\theta = \pi$ (unobservable case)?

$$\rightarrow \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} ; \quad A^T = A$$

$$A^T + \vec{c}\,\vec{l}^T = \begin{bmatrix} -1+l_1 & l_2 \\ 0 & -1 \end{bmatrix}$$

$$\text{det: } -(\lambda+1)(-1+l_1-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1}, \quad \lambda_2 = l_1 - 1$$

↳ BIBO unstable and cannot be changed by choosing $\vec{l}$.

→ USING OBSERVERS FOR MORE ACCURATE POSITIONING & NAVIGATION

→ recall the accelerating car discrete-time example from the lecture on controllability:

$$\begin{bmatrix} x((t+1)T) \\ v((t+1)T) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}^{A} \begin{bmatrix} x(tT) \\ v(tT) \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}^{\vec{b}} a(t)$$

– Terminology: $\overset{\text{discrete}}{x[t]} \equiv \overset{\text{continuous}}{x(tT)}$; $v[t] \equiv v(tT)$; $a[t] \equiv a(tT)$

$$\longrightarrow \text{eigenvalues: } (1-\lambda^2) = 0 \Rightarrow \lambda_{1,2} = \pm 1 \longrightarrow$$ BIBO UNSTABLE

↳ TYPICAL FOR CARS, PLANES, ETC.

→ CONSIDER the scenario:

→ the car is NOT feedback stabilized

– there is (inevitable) error in the accel: $a[t] = \underset{\text{correct accel. for navigation}}{a_d[t]} + \underset{\text{accel. error (eg, pedal play)}}{\Delta a[t]}$ ← not precisely known, but small

→ we estimate its state using an observer:

$$\rightarrow \hat{x}[t+1] = A\hat{x}[t] + \vec{b}\, a_d[t]$$ → same as system, no $\vec{l}\vec{c}^T$ term

→ we don't know $\Delta a[t]$)

→ the estimation error follows:

$$\vec{e}[t+1] = A\vec{e}[t] + \vec{b}\,\Delta a[t]$$ ← unstable ⇒ $\vec{e}(t)$ grows larger and larger even if $\Delta a(t)$ remains small.

→ but, if we are able to measure the position of the actual car, say using GPS measurements:

$$y[t] = \underbrace{[1 \quad 0]}_{\vec{c}^T} \vec{x}[t] + \Delta p[t]$$

→ GPS measurement error, also not precisely known, but small.

→ <u>Our observer becomes:</u> $\vec{l}\,(\vec{c}^T\hat{\vec{x}} - y(t))$

$$\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + \vec{b}\,a_d[t] + \vec{l}\,(\vec{c}^T\hat{\vec{x}} - y(t))$$

$$= A\hat{\vec{x}}[t] + \vec{b}\,a_d[t] + \vec{l}\vec{c}^T\hat{\vec{x}}[t] - \vec{l}\vec{c}^T\vec{x}[t] - \vec{l}\,\Delta p[t]$$

$$= (A + \vec{l}\vec{c}^T)\hat{\vec{x}}[t] + \vec{l}\vec{c}^T(\hat{\vec{x}}[t] - \vec{x}[t]) + \vec{b}\,a_d[t] - \vec{l}\,\Delta p[t]$$

Hence the error $\vec{e}[t] \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$ obeys

$$\vec{e}[t+1] = A\vec{e}[t] + \vec{l}\vec{c}^T\vec{e}[t] + \underbrace{(\vec{b}\,a_d[t] - \vec{l}\,\Delta p[t])}_{\text{due to errors, but small}}$$

$$\Rightarrow \quad \vec{e}[t+1] = (A + \vec{l}\vec{c}^T)\vec{e}[t] + \underbrace{(\vec{b}\,a_d[t] - \vec{l}\,\Delta p[t])}_{\text{small}}$$

→ Now, if $(A, \vec{c})$ are observable, you can always choose $\vec{l}$ to stabilize the observer

→ which means that the estimation error can <u>always</u> be kept bounded and small, even taking into account small errors in GPS & acceleration

↳ this is what all navigational systems today do!

→ Actually, they use an additional improvement, whereby $\vec{l}$ changes w/ t (i.e, becomes $\vec{l}[t]$)

→ the updates are used to minimize the error due to GPS/pedal errors, other noise, etc, even further

↳ this is the famous KALMAN FILTER, the gold standard for accurate navigational estimation

— WHAT IT ACHIEVES:

- open-loop position/velocity estimation → blows up errors ∵ unstable
- estimation simply by GPS also has errors

- using a well-designed observer, you can make much better estimates than either of these 2 alone. ← quite remarkable

- and then, you can use this good estimate of $\hat{x}$ to feed back to the car (via controllability) to make its response stable.
  ↳ THIS IS WHAT IS GOING ON IN AUTONOMOUS VEHICLES (TESLA PICTURE)

— PICTURES AND HISTORY OF KALMAN.

→ ALSO: FEEDBACK STABILIZED LEVITRON DEMOS (earlier)