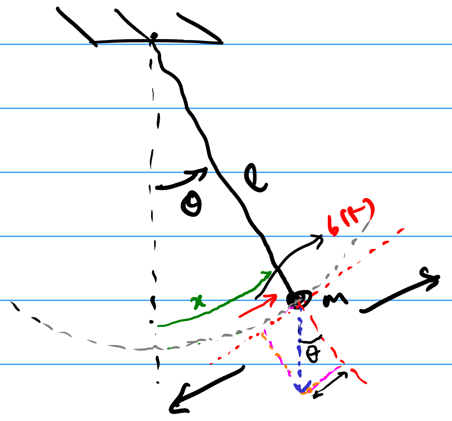
THE "SIMPLE" PENDULUM

$x(t) = \ell\theta(t)$ ← radians

$F = ma$



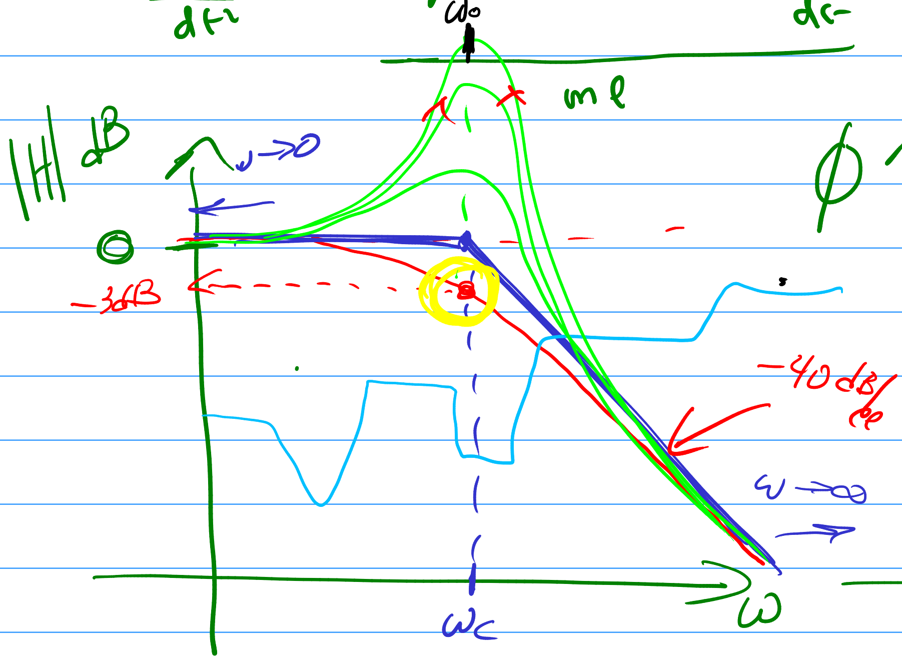
$-mg\sin(\theta) + b(t)$

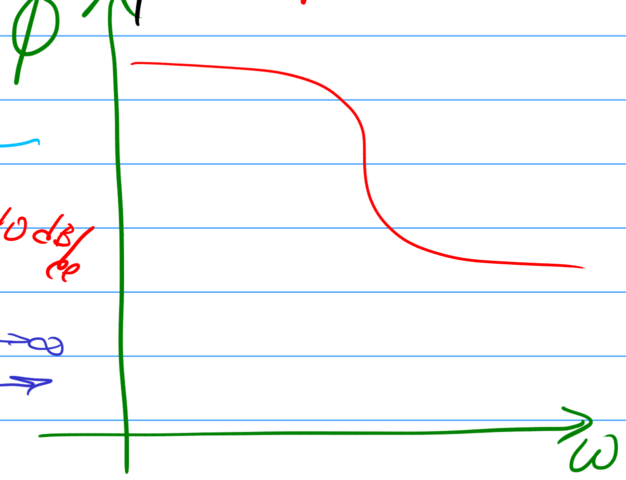
$$\text{accel.} = \frac{d^2x}{dt^2} = \ell\,\frac{d^2\theta(t)}{dt^2}$$

$$a = \frac{F}{m}$$

$$m\ell\,\frac{d^2\theta(t)}{dt^2} = \overset{F}{\checkmark}\; -mg\sin(\theta) + b(t) - k\ell\frac{d\theta}{dt}$$

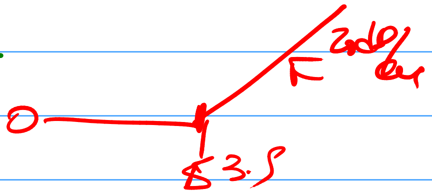
$$\frac{d^2\theta(t)}{dt^2} = \frac{-mg\sin(\theta) + b(t) - k\ell\frac{d\theta}{dt}}{m\ell}$$

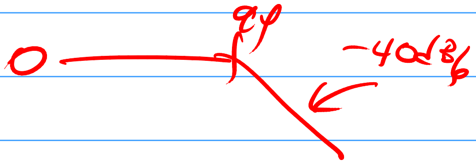


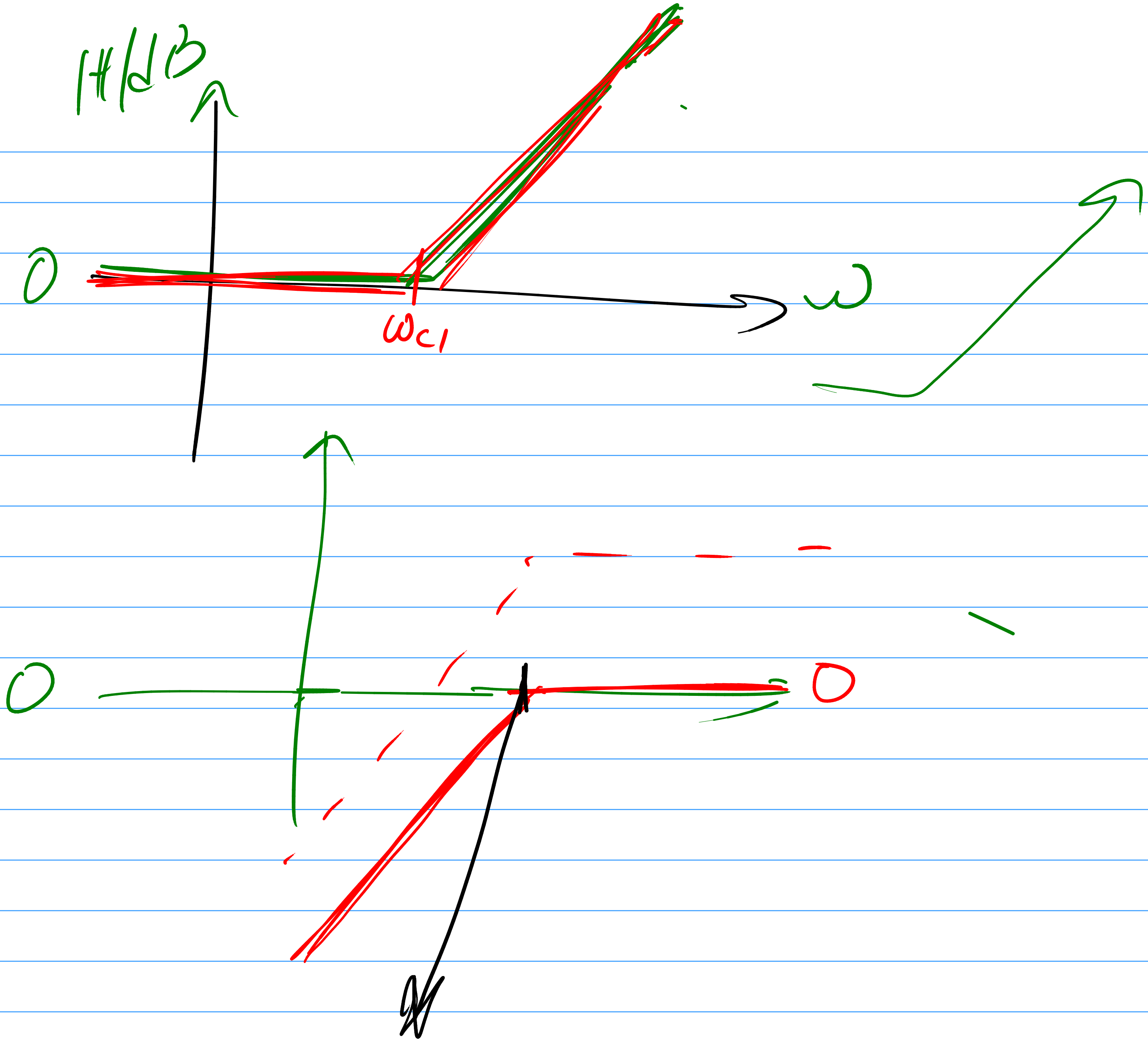


$$H(\omega) \Rightarrow \text{complex number} = |H|e^{j\phi} = |H|\angle\phi$$

$$H(\omega) = \frac{35.4\left(1 + j\,\omega/53.5\right)}{\left(1 + j\frac{\omega}{74}\right)^{2}}$$






|H|dB
0
ω
ω_{c1}
0
0

$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC} v_c = 0$$

$$x_1 \overset{\Delta}{=} v_c$$

$$x_2 \overset{\Delta}{=} \frac{dv_c}{dt}$$

$$\begin{cases} \dfrac{dx_2}{dt} + \dfrac{R}{L}x_2 + \dfrac{1}{LC}x_1 = 0 \\ \dfrac{dv_c}{dt} = x_2 \end{cases}$$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{1}{LC}x_1 - \dfrac{R}{L}x_2 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \overbrace{\begin{bmatrix} 0 & 1 \\ \underset{c}{\boxed{-1/LC}} & \underset{d}{\boxed{-R/L}} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{\xi} = \lambda\vec{\xi} \Rightarrow \overset{\text{def}}{\downarrow}(A - \lambda I)\vec{\xi} = 0$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ c & d-\lambda \end{bmatrix}; \quad \det(A-\lambda I) = -\lambda(d-\lambda) - c = 0$$

$$\Rightarrow \lambda^2 - \lambda d - c = 0$$

$$\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \qquad \lambda_{1,2} = \frac{+d \pm \sqrt{d^2 + 4c}}{2}$$

$$A = P\Omega P^{-1}$$

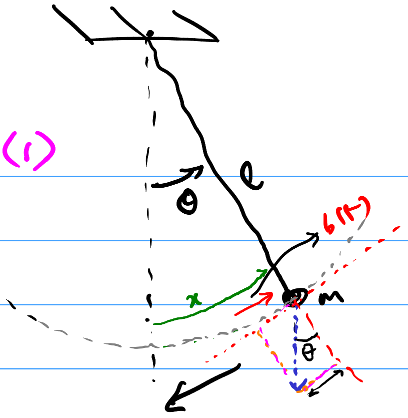
$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$$

$$A\vec{p} = \lambda\vec{p} \longrightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\Rightarrow (A - \lambda I)\vec{p} = 0$$

$-\lambda p_1 + p_2 = 0$ choose $p_1 = 1$, $p_2 = \lambda$ ↗ if it is an eigenvalue

THE "SIMPLE" PENDULUM



$$x(t) = l\overset{\swarrow \text{radians}}{\theta}(t) \qquad F = ma \qquad (1)$$

$$\text{accel.} = \frac{d^2x}{dt^2} = l\,\frac{d^2\theta(t)}{dt^2} \qquad (2)$$

$$a = \frac{F}{m} \qquad (3)$$

$$\overset{\swarrow F}{} \qquad -mg\sin(\theta) + b(t) \qquad (4)$$

$$m\,l\,\frac{d^2\theta(t)}{dt^2} = -mg\sin(\theta) + b(t) - kl\frac{d\theta}{dt} \qquad (5)$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\frac{d\theta}{dt} + \frac{b(t)}{ml} \qquad (6)$$

RECAP: Have got the "physics" eqns. for the "simple" pendulum.

NEXT: PUT (6) IN STATE-EQN FORM

$$\frac{d}{dt}\left[\frac{d\theta}{dt}\right] \nwarrow \quad \frac{d^2\theta(t)}{dt^2} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\overset{x_2}{\frac{d\theta}{dt}} + \overset{\text{input} = b(t)}{\frac{b(t)}{ml}} \qquad (6)$$

$$\vec{x}(t) = \begin{bmatrix} \theta(t) \\ \frac{d\theta}{dt} \end{bmatrix} \begin{matrix} \to x_1(t) \\ \to x_2(t) \end{matrix}$$

$$\to \underline{\text{define}}: \quad x_2(t) \triangleq \frac{d\overset{x_1}{\theta}(t)}{dt} \qquad (7)$$

$$\boxed{\begin{aligned} \frac{dx_1(t)}{dt} &= x_2 && (9) \\ \frac{d}{dt}x_2(t) &= -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) + \frac{b(t)}{ml} && (8) \end{aligned}}$$

$$\frac{d\vec{x}}{dt}$$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad u(t) \equiv b(t)$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} f_1(x_1, x_2; u) \\ f_2(x_1, x_2; u) \end{bmatrix} \triangleq \vec{f}(\vec{x}, u), \text{ with } \quad f_1(x_1, x_2; u) \triangleq x_2$$

$$f_2(x_1, x_2; u) \triangleq -\frac{g}{l}\sin(x_1(\overset{t}{t})) - \frac{k}{m}x_2(\overset{t}{t}) + \frac{\overset{u}{b(t)}}{ml}$$

ATTEMPT: can we write:

const matrix

const (vector)

$\vec{f}(\vec{x},u)$ as. $\underline{A}\vec{x} + \vec{b}u$?

VECTOR 2×1

$$\vec{f}(\vec{x},u) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) + \frac{\overset{u}{b(t)}}{ml} \end{bmatrix}$$

$$\xcancel{\overset{A}{\begin{bmatrix} 0 & 1 \\ - & -\frac{k}{m} \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} \cdot \\ x_2 \end{bmatrix}} + \overset{\vec{b}}{\begin{bmatrix} \\ \end{bmatrix}} u(t)}$$

STATE-SPACE FORM: (GENERAL - includes nonlinear)

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x},\vec{u}) \leftarrow \boxed{\text{MAY NOT BE EXPRESSIBLE AS } A\vec{x} + B\vec{u}}$$

- So our system is nonlinear, not in $A\vec{x} + B\vec{u}$ form?
  - WE'D LIKE VERY MUCH TO PUT IT IN ↗, EVEN IF WE HAVE TO APPROXIMATE

↓

LINEARIZATION

LINEARIZATION

- STEP 1: - select constant (wrt time) input $\vec{u}^*$: "DC"
  - ASSUME $\vec{x}(t)$ ALSO DC. $\equiv \vec{x}^*$

$$-\ \frac{d\overset{x^*}{x(t)}}{dt} = \vec{f}(\overset{x^*}{\vec{x}(t)}, \overset{u^*}{\vec{u}(t)})$$

$$\boxed{0 = \vec{f}(\vec{x}^*, \vec{u}^*)}$$

← DC op. pt.

← equilibrium pt

← quiescent point

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix} x_2^* \\ -\frac{g}{l}\sin(x_1^*) - \frac{k}{m}x_2^* + \frac{b^*}{ml}\end{bmatrix}$$

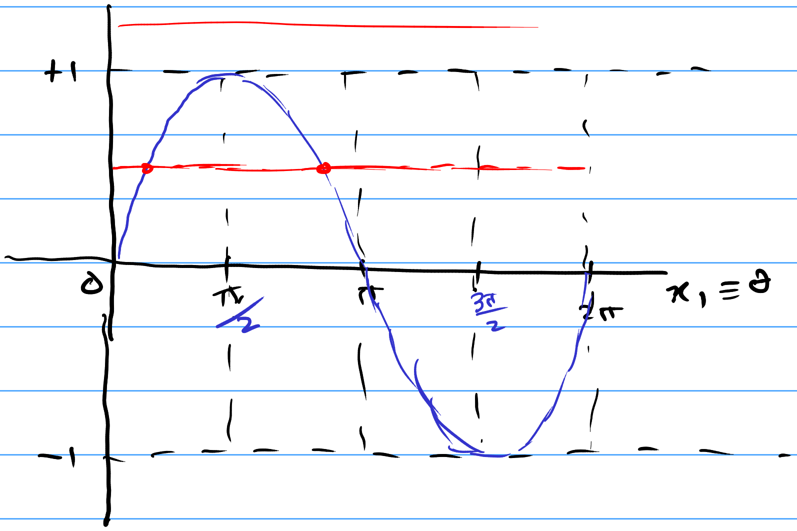
$$\frac{dx_1}{dt} = x_2^* = 0$$

$x_2^* = 0$

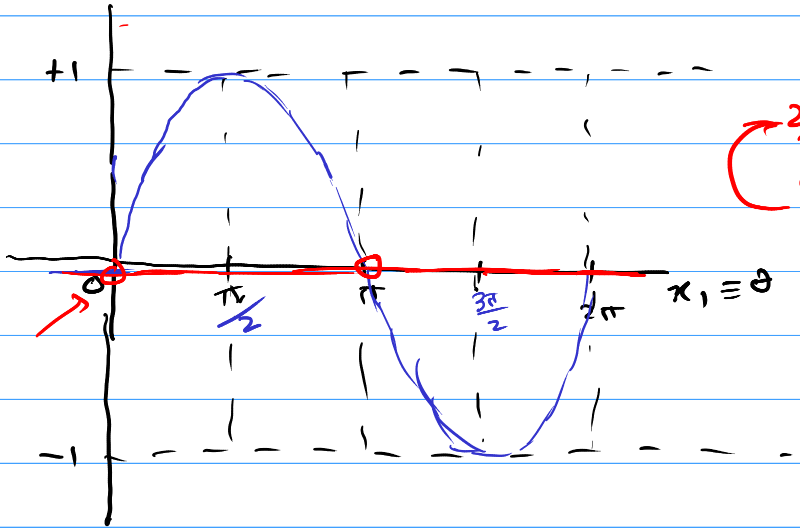
$$-\frac{g}{l}\sin(x_1^*) + \frac{b^*}{ml} = 0$$

$$\Rightarrow \boxed{\sin(x_1^*) = \frac{b^*}{mg}}$$

$\left(\frac{b^*}{mg} = 0.5\right)$ say



→ Now choose a "natural" $b^* = 0$ ← no tangential force applied by me



different 2 solutions

$\begin{cases} x_1^* = 0 \leftarrow \text{one solution} \\ x_1^* = \pi \end{cases}$

$$\frac{d}{dt}\left[\frac{d\theta}{dt}\right] \to x_2$$

$$=$$

$$\frac{d^3\theta}{dt^3} + a_1\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin(\theta) + \frac{k}{m}\frac{d\theta}{dt} - \frac{b(t)}{ml} = 0$$

x_1

$$\frac{d}{dt}\left[\frac{d^2\theta}{dt^2}\right] \to x_3$$

1) $\frac{dx_3}{dt} + a_1 x_3 + \frac{g}{l}\sin(x_1) + \frac{k}{m}x_2 - \frac{b(t)}{ml} = 0$

2) $\frac{dx_1}{dt} = x_2$

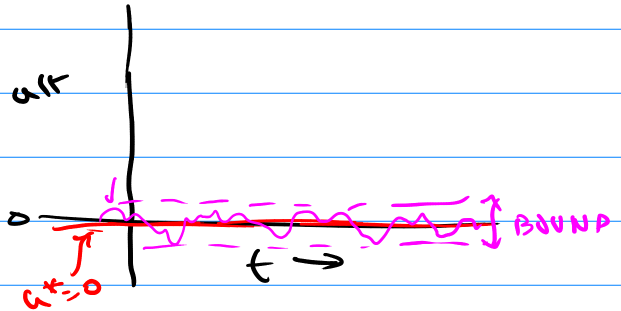
3) $\frac{dx_2}{dt} = x_3$

$x_1 = \theta$

$x_2 = \frac{d\theta}{dt}$

$x_3 = \frac{d^2\theta}{dt^2} = \frac{dx_2}{dt}$

— choose $\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → the other one is $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$ ← will deal with this one later



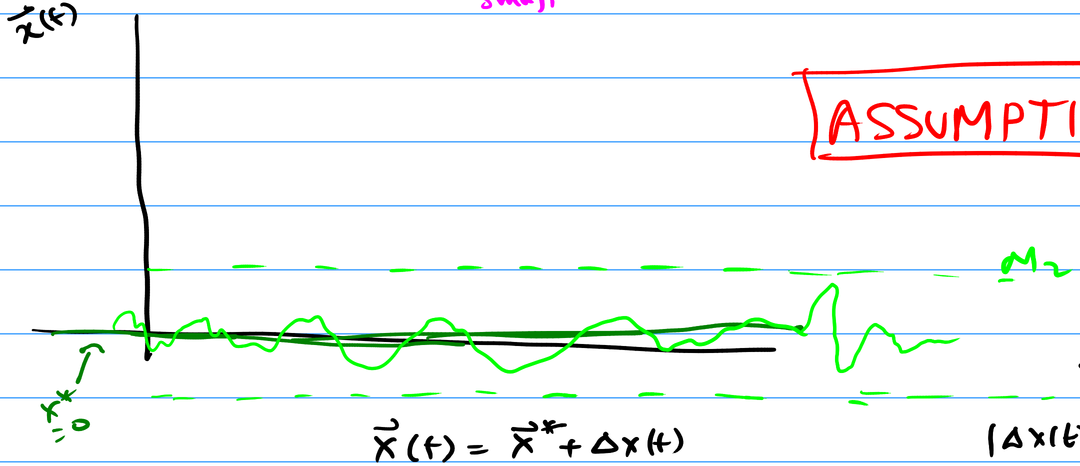
small const

$|\Delta u(t)| < M$ forall t

our current definition of "small"

$u(t) = u^* + \Delta u(t)$

"small"



ASSUMPTION

∀ → forall

$|\Delta x(t)| \le M_2 \;\forall t$

$\vec{x}(t) = \vec{x}^* + \Delta x(t)$

↳ small/bounded

RETURN TO FULL STATE EQUATIONS

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$$

→ $\boxed{\text{DC. o.p.}: \quad \vec{f}(\vec{x}^*, \vec{u}^*) = 0}$

→ $\vec{u}(t) = \vec{u}^* + \Delta\vec{u}(t)$
$\vec{x}(t) = \vec{x}^* + \Delta\vec{x}(t)$

$$\frac{d}{dt}\left[\vec{x}^* + \Delta\vec{x}(t)\right] = \vec{f}\left(\vec{x}^* + \Delta\vec{x}(t), \vec{u}^* + \Delta\vec{u}(t)\right)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cancel{\frac{d}{dt}\vec{x}^*} + \frac{d}{dt}\Delta\vec{x}(t) = \vec{f}\left(\vec{x}^* + \Delta\vec{x}(t), \vec{u}^* + \Delta\vec{u}(t)\right)$$

$$\rightarrow \frac{d}{dt}\Delta\vec{x}(t) = \vec{f}\left(\vec{x}^* + \Delta\vec{x}(t),\ \vec{u}^* + \Delta\vec{u}(t)\right) \qquad \Delta\vec{x} = \begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} \quad \Delta u$$

$$\vec{x}^* = \begin{bmatrix}x_1^*\\ x_2^*\end{bmatrix}$$

2×2 case as an example: $\overset{2}{\vec{f}}(\overset{2}{\vec{x}}, \overset{1}{u}) = \begin{bmatrix} f_1(x_1, x_2; u) \\ f_2(x_1, x_2; u) \end{bmatrix}$

$$\vec{f}(\vec{x}^* + \Delta\vec{x}, u^* + \Delta u) = \begin{bmatrix} f_1(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2;\ u^* + \Delta u) \\ f_2(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2;\ u^* + \Delta u) \end{bmatrix}$$

Taylor Series on : $f_1(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2;\ u^* + \Delta u)$

$$f_1(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2;\ u^* + \Delta u) =$$

APPROXIMATION

$$f_1(\underline{x_1^*} + \underline{\Delta x_1}) = \underline{f_1(x_1^*)} + \underline{\left.\frac{df_1}{dx}\right|_{x_1^*}}\Delta x_1 \cancel{+ \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x_1^*}\Delta x_1^2 + \frac{1}{6?}\left.\frac{d^3 f}{dx^3}\right|_{x_1^*}\Delta x^3}.$$

$$\frac{0.1}{0.01} \qquad \frac{0.01}{10^{-4}} \qquad \frac{0.001}{10^{-6}}$$

$$f_1(x_1^* + \Delta x_1,\ x_2^* + \Delta x_2;\ u^* + \Delta u) = \boxed{f_1(x_1^*, x^*, u^*)}$$

$$\boxed{+ \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1^*, x_2^*, u^*}\Delta x_1} \cancel{+ \frac{1}{2}\left.\frac{\partial^2 f_1}{\partial x_1^2}\right|_{x_1^*, x_2^*, u^*}\Delta x_1^2} + \cdots$$

$$\cancel{+ ?\left.\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\right|_{x_1^*, x_2^*, u^*}\underline{\Delta x_1 \Delta x_2}}$$

$$\boxed{+ \left.\frac{\partial f_1}{\partial x_2}\right|_{x_1^*, x_2^*, u^*}\Delta x_2} + \cdots \cancel{\ }.$$

$$\cancel{+ ?\left.\frac{\partial^2 f}{\partial x_1 \partial u}\right|_{\cdots}\underline{\Delta x_1}\,\underline{\Delta u}}$$

$$\boxed{+ \left.\frac{\partial f_1}{\partial u}\right|_{x_1^*, x_2^*, u^*}\Delta u} + \cancel{\cdots}\ .$$

FIRST ORDER APPROXIMATION

$$f_1(x_1^*+\Delta x_1, x_2^*+\Delta x_2; u^*+\Delta u) = f_1(x_1^*, x_2^*, u^*) + \underbrace{\frac{\partial f_1}{\partial x_1}^*\Delta x_1 + \frac{\partial f_1}{\partial x_2}^*\Delta x_2 + \frac{\partial f_1}{\partial u}^*\Delta u}$$

TOTAL DERIVATIVE

$$f_2(x_1^*+\Delta x_1, x_2^*+\Delta x_2; u^*+\Delta u) = f_2(x_1^*, x_2^*, u^*) + \frac{\partial f_2}{\partial x_1}^*\Delta x_1 + \frac{\partial f_2}{\partial x_2}\Delta x_2 + \frac{\partial f_2}{\partial u}\Delta u$$

$$\frac{d}{dt}\Delta\vec{x}(t) = \vec{f}\left(\vec{x}^* + \vec{\Delta x}(t), \vec{u}^* + \vec{\Delta u}(t)\right)$$

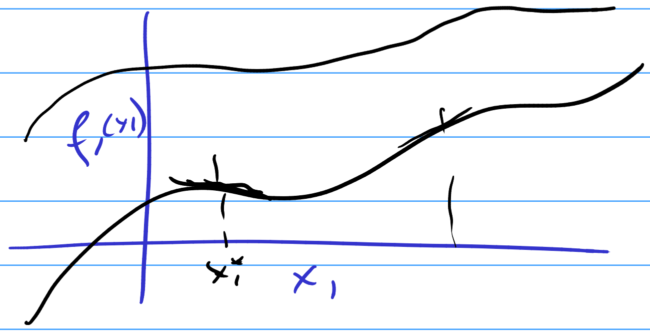
$$\frac{d}{dt}\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix} = \begin{bmatrix}\cancel{f_1(x_1^*, x_2^*, u^*)} + \frac{\partial f_1}{\partial x_1}^*\Delta x_1(t) + \frac{\partial f_1}{\partial x_2}^*\Delta x_2(t) + \frac{\partial f_1}{\partial u}^*\Delta u(t)\\ \cancel{f_2(x_1^*, x_2^*, u^*)} + \frac{\partial f_2}{\partial x_1}^*\Delta x_1(t) + \frac{\partial f_2}{\partial x_2}^*\Delta x_2(t) + \frac{\partial f_2}{\partial u}^*\Delta u(t)\end{bmatrix}$$

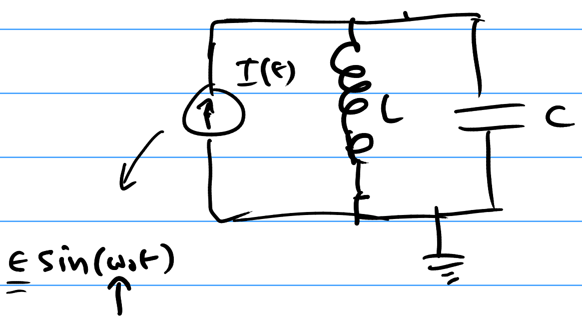
A

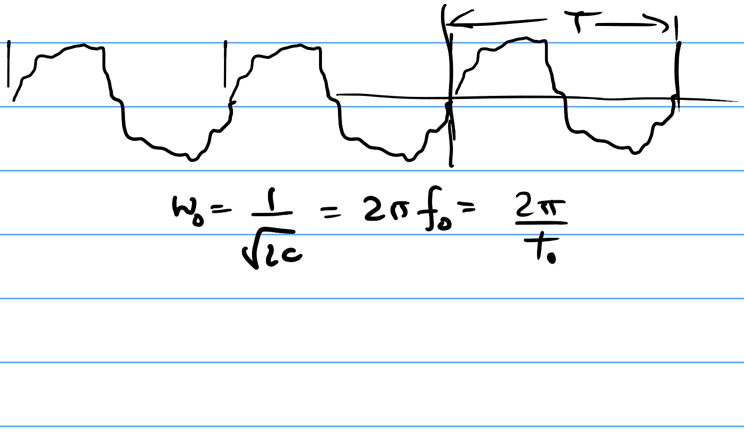
$$\vec{f}(\vec{x}^*, \vec{u}^*) = \vec{0} \leftarrow \text{DC o.p.}$$

$$\begin{bmatrix}f_1(x_1^*, x_2^*, u^*)\\ f_2(x_1^*, x_2^*, u^*)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\underset{\Delta\vec{x}}{\frac{d}{dt}\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix}} = \underset{2\times3}{\begin{bmatrix}\frac{\partial f_1}{\partial x_1}^* & \frac{\partial f_1}{\partial x_2}^* & \frac{\partial f_1}{\partial u}^*\\ \frac{\partial f_2}{\partial x_1}^* & \frac{\partial f_2}{\partial x_2}^* & \frac{\partial f_2}{\partial u}^*\end{bmatrix}} \underset{3\times1}{\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\\ \Delta u(t)\end{bmatrix}} = \underset{A}{\begin{bmatrix}\frac{\partial f_1}{\partial x_1}^* & \frac{\partial f_1}{\partial x_2}^*\\ \frac{\partial f_2}{\partial x_1}^* & \frac{\partial f_2}{\partial x_2}^*\end{bmatrix}} \underset{\Delta\vec{x}(t)}{\begin{bmatrix}\Delta x_1(t)\\ \Delta x_2(t)\end{bmatrix}} + \underset{\vec{b}}{\begin{bmatrix}\frac{\partial f_1}{\partial u}^*\\ \frac{\partial f_2}{\partial u}^*\end{bmatrix}}\Delta u(t)$$







$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f_0 = \frac{2\pi}{T_0}$$