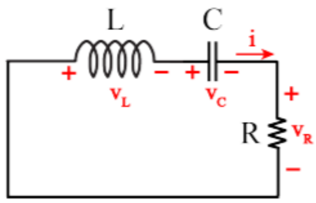
et's solve for $v_C(t)$ for $t > 0$ by writing the KVL and using i-v characteris



$$v_L + v_C + v_R = 0$$

$$L\frac{di}{dt} + v_C + iR = 0$$

$$L\frac{d}{dt}\left(C\frac{dv_C}{dt}\right) + v_C + C\frac{dv_C}{dt}R = 0$$

$$\boxed{LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0} \leftarrow \text{THE GOAL IS TO SOLVE THIS} \qquad (1)$$

1. Put it in state equation form:

Define: $- \; x_1(t) \triangleq v_C(t)$ $\qquad$ (2.1)

$- \; x_2(t) \triangleq \frac{dv_C(t)}{dt} = \frac{dx_1}{dt}$ $\qquad$ (2.2)

(2)

− using (2), (1) becomes

$$LC\frac{dx_2}{dt} + RC\,x_2 + x_1 = 0 \qquad (3)$$

− putting (2.2) & (3) together, we get

$$\frac{dx_1}{dt} = x_2 \qquad (4.1)$$

$$\frac{dx_2}{dt} = \frac{-1}{LC}\left(x_1 + RC\,x_2\right) \qquad (4.2)$$

(4)

− (4) can be expressed in matrix vector form:

− define $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ $\qquad$ (5)

$$\boxed{-\;\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{1}{LC} & -\frac{R}{L} \end{array}\right]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} \qquad (6)$$

STATE EQNS (NO INPUTS)

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

2. Solve (6)

2.1 An aside: eigendecomposition of an arbitrary 2x2 matrix

- Say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (7)

- eigen-equation: $A\vec{p} = \lambda\vec{p} \iff (A - \lambda I)\underbrace{\vec{p}}_{\neq \vec{0}} = \vec{0}$ (8)

- to allow (8) to have non-trivial solutions for $\vec{p}$, $\underbrace{(A - \lambda I)}_{\text{square}}$ must be rank-deficient

$$\Rightarrow \det \underbrace{(A - \lambda I)}_{\text{"singular" means (9)}} = 0 \quad (9)$$

- $\det(A - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = (a-\lambda)(d-\lambda) - bc$ (10)

$$= \lambda^2 - (a+d)\lambda + \underbrace{ad-bc}_{\text{det. of the original matrix}} = 0 \quad (11)$$

- the solution of (11) is:

$$\boxed{\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}} \quad (12)$$

- we have found the λs.

- define $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (12.5)

— next, we want to find $\vec{p}$, using (again) (8): $(A-\lambda I)\vec{p}=\vec{0}$

$$\begin{array}{ccc} (A-\lambda I) & \vec{p} & \vec{0} \\ \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} & \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \qquad (13)$$

— we know that if $\lambda \neq \lambda_1$, or λ_2, the only solution possible is $p_1, p_2 = 0$, ($\because$ $(A-\lambda I)$ is non-singular)

— so keep in mind that we are interested only in $\lambda = \lambda_1$ or λ_2

— let's try $\lambda = \lambda_1$ first:

$$\begin{array}{ccc} & \vec{p}_1 & \vec{0} \\ \begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} & \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \qquad (14)$$

$\rightarrow (a-\lambda_1)p_{11} + b\,p_{12} = 0$ (15.1)

$\rightarrow c\,p_{11} + (d-\lambda_1)p_{12} = 0$ (15.2)

(15)

— Use (15.1) to write p_{12} in terms of p_{11}

— $p_{12} = \dfrac{-(a-\lambda_1)p_{11}}{b}$ (16)

— Now, use (15.2) to express p_{12} in terms of p_{11}

— $p_{12} = \dfrac{-c\,p_{11}}{(d-\lambda_1)}$ (17)

— <u>Claim:</u> ((16) & (17)) are exactly the same eqn.

—Proof: need to show that $\dfrac{-(a-\lambda_1)}{b} = -\dfrac{c}{(d-\lambda_1)}$

or $\div(a-\lambda_1)(d-\lambda_1) = \div bc$

or $+\lambda_1^2 - \lambda_1(a+d) + ad - bc = 0$

↳ which is exactly (11), of which λ_1 is a soln.

— What this means is that we can use either (15.1) or (15.2) to solve for p_{11} & p_{12}; it makes no difference which one we use. So let's just use (15.1):

$$(a-\lambda_1)p_{11} + b\,p_{12} = 0 \qquad (18)$$

— the above reasoning tells us that we can choose one of p_{11} or p_{12} arbitrarily, and the other one gets determined by (18).

— a convenient choice avoiding division by anything:

$$\boxed{p_{11} = b \implies p_{12} = (\lambda_1 - a)} \qquad (19)$$

— Hence we have: $\vec{p}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}$ (20)

— now we try $\lambda = \lambda_2$

— completely identical to the above, just replace λ_1 by λ_2 and p_{11}, p_{12} by p_{21} and p_{22}, and $\vec{p}_1$ by $\vec{p}_2$.

— we will get: $\vec{p}_2 = \begin{bmatrix} b \\ \lambda_2 - a \end{bmatrix}$ (21)

— Therefore, we have $P = \begin{bmatrix} | & | \\ \vec{p}_1 & \vec{p}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{bmatrix}$ (22)

— so $A = P \Lambda P^{-1}$ (22.5)

— where P is given by (22)

— Λ is given by (12.5)

— and P^{-1} is given by (23) below

— let's also write out P^{-1}

$$P^{-1} = \begin{bmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_2 - a & -b \\ a - \lambda_1 & b \end{bmatrix} \times \frac{1}{b(\lambda_2 - a) - b(\lambda_1 - a)}$$

$$= \begin{bmatrix} \lambda_2 - a & -b \\ a - \lambda_1 & b \end{bmatrix} \times \frac{1}{b(\lambda_2 - \lambda_1)} \qquad (23)$$

TO SUMMARIZE:

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \overbrace{\begin{bmatrix} \lambda_2 - a & -b \\ a - \lambda_1 & b \end{bmatrix} \times \frac{1}{b(\lambda_2 - \lambda_1)}}^{P^{-1}}} \qquad (24)$$

- this is valid only if $b \neq 0$, but if it is, just compute $\begin{bmatrix} \frac{\lambda_2 - a}{b} & -1 \\ \frac{a - \lambda_1}{b} & +1 \end{bmatrix}$ as $b \to 0$ and it will be well-defined*

  * if $\lambda_1 \neq \lambda_2$

- if $\lambda_1 = \lambda_2$, then eigendecomposition not possible, have to use Jordan canonical form

- Let's return to the specific A for our RLC circuit: (6)

$$A = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{1}{LC} & -\frac{R}{L} \end{array}\right] \Rightarrow a = 0,\ b = 1,\ c = -\frac{1}{LC},\ d = -\frac{R}{L}$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

$$\boxed{\lambda_{1,2} = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}} \qquad (25)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_2 - a & -b \\ a - \lambda_1 & b \end{bmatrix} \times \frac{1}{b(\lambda_2 - \lambda_1)}$$

— and (24) becomes $\qquad a = 0,\ b = 1$

$$\boxed{A = \left[\begin{array}{c:c} 0 & 1 \\ \hdashline -\frac{1}{LC} & -\frac{R}{L} \end{array}\right] = \overset{P}{\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}} \overset{\Omega}{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}} \overbrace{\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \times \frac{1}{\lambda_2 - \lambda_1}}^{P^{-1}}} \qquad (26)$$

2.2 Now we can return to solving (6)

$$\frac{d}{dt} \overset{\vec{x}}{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} = \overbrace{\left[\begin{array}{c:c} 0 & 1 \\ \hdashline -\frac{1}{LC} & -\frac{R}{L} \end{array}\right]}^{A} \overset{\vec{x}}{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} \qquad (6)$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$\Rightarrow \frac{d\vec{x}}{dt} = P \Omega P^{-1} \vec{x}$$

$\Rightarrow$ premultiplying by P^{-1}: $\Rightarrow \frac{d}{dt}(P^{-1}\vec{x}) = \Omega (P^{-1}\vec{x})$

– define $\vec{y} \triangleq P^{-1}\vec{x} \Leftrightarrow \vec{x} = P\vec{y}$ $\qquad (27)$

$$\Rightarrow \frac{d\vec{y}}{dt} = \Omega \vec{y} \qquad (28)$$

$$\Rightarrow \boxed{\begin{aligned} \frac{dy_1(t)}{dt} &= \lambda_1 y_1(t) \\ \frac{dy_2(t)}{dt} &= \lambda_2 y_2(t) \end{aligned}} \qquad (29) \begin{cases} (29.1) \\ (29.2) \end{cases}$$

— Solution to (29)

$$\boxed{\begin{aligned} y_1(t) &= y_1(0)\, e^{\lambda_1 t} \\ y_2(t) &= y_2(0)\, e^{\lambda_2 t} \end{aligned}}$$

(30) $\begin{cases} (30.1) \\ (30.2) \end{cases}$

— the ICs $y_1(0)$ & $y_2(0)$ are given by (27):

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (31)$$

$$= \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \times \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 x_1(0) - x_2(0) \\ -\lambda_1 x_1(0) + x_2(0) \end{bmatrix} \quad (32)$$

— So (30) becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} (\lambda_2 x_1(0) - x_2(0))\, e^{\lambda_1 t} \\ (-\lambda_1 x_1(0) + x_2(0))\, e^{\lambda_2 t} \end{bmatrix} \quad (33)$$

— Finally, want $\vec{x}(t)$ from $\vec{y}(t)$, using (27)

$$\rightarrow \vec{x}(t) = P\vec{y}(t) \Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} (\lambda_2 x_1(0) - x_2(0))\, e^{\lambda_1 t} \\ (-\lambda_1 x_1(0) + x_2(0))\, e^{\lambda_2 t} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} (\lambda_2 x_1(0) - x_2(0))\, e^{\lambda_1 t} + (-\lambda_1 x_1(0) + x_2(0))\, e^{\lambda_2 t} \\ \lambda_1 (\lambda_2 x_1(0) - x_2(0))\, e^{\lambda_1 t} + \lambda_2 (-\lambda_1 x_1(0) + x_2(0))\, e^{\lambda_2 t} \end{bmatrix}} \quad (34)$$

— The underdamped case is when λ_1, λ_2 are complex (have nonzero imaginary parts)

— FIRST OBSERVATION: if λ_1 is complex, then $\lambda_2 = \overline{\lambda_1}$ (check (12))

— SECOND ": if λ_1 is complex, then $\vec{p}_1 = \overline{\vec{p}_2}$ (check (14))

— So define $\lambda_1 = \alpha + j\beta$, (35)

then $\lambda_2 = \alpha - j\beta$

and

$$P = \begin{bmatrix} 1 & 1 \\ \alpha + j\beta & \alpha - j\beta \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \alpha + j\beta & 0 \\ 0 & \alpha - j\beta \end{bmatrix}$$

$$\left.\begin{aligned} &- \lambda_2 - \lambda_1 = -j2\beta \\ &- e^{\lambda_1 t} = e^{\alpha t} e^{j\beta t} \\ &- e^{\lambda_2 t} = e^{\alpha t} e^{-j\beta t} \end{aligned}\right\} \longrightarrow \quad (36)$$

$$x_1(t) = \frac{1}{\lambda_2 - \lambda_1} (\lambda_2 x_1(0) - x_2(0)) e^{\lambda_1 t} + (-\lambda_1 x_1(0) + x_2(0)) e^{\lambda_2 t} \quad \text{(from (34))}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \left\{ \left[(\alpha - j\beta) x_1(0) - x_2(0)\right] e^{\alpha t} e^{j\beta t} + \left[(-\alpha - j\beta) x_1(0) + x_2(0)\right] e^{\alpha t} e^{-j\beta t} \right]$$

$$\Rightarrow x_1(t) = \frac{j e^{\alpha t}}{2\beta} \left\{ \Big[\underbrace{(\alpha - j\beta) e^{j\beta t}}_{z} \underbrace{-}_{-} \underbrace{(\alpha + j\beta) e^{-j\beta t}}_{\bar{z}} \Big] x_1(0) + \Big[\underbrace{- e^{j\beta t}}_{-w} \underbrace{+}_{+} \underbrace{e^{-j\beta t}}_{\bar{w}} \Big] x_2(0) \right] \quad (37)$$

$$= \frac{j e^{\alpha t}}{2\beta} \left[+2\,\mathrm{Im}(z)\, x_1(0) - 2\,\mathrm{Im}(w)\, x_2(0) \right] \quad (38)$$

$$\left.\begin{aligned} &\rightarrow \mathrm{Im}(z) = \mathrm{Im}\left[(\alpha - j\beta) e^{j\beta t}\right] = \mathrm{Im}\left[(\alpha - j\beta)(\cos(\beta t) + j\sin(\beta t))\right] \\ &\qquad = -j\beta\cos(\beta t) + j\alpha\sin(\beta t) \\ &\rightarrow \mathrm{Im}(w) = \mathrm{Im}\left[e^{j\beta t}\right] = j\sin\beta t \end{aligned}\right\} \quad (39)$$

- using (38) & (39):

$$\rightarrow x_1(t) = \frac{j e^{\alpha t}}{2\beta}\left[j2\left(-\beta\cos(\beta t) + \alpha\sin(\beta t)\right) x_1(0) + j\,2\sin(\beta t)\, x_2(0) \right]$$

$$\rightarrow x_1(t) = \frac{e^{\alpha t}}{\beta}\left[\left[\beta\cos(\beta t) - \alpha\sin(\beta t)\right] x_1(0) - \sin(\beta t)\, x_2(0) \right] \quad (40)$$

→ Now we solve for $x_2(t)$

→ you could use (36), but it may be easier to use the basic definition: $x_2 = \frac{dx_1}{dt}$.

$$x_1(t) = \frac{e^{\alpha t}}{\beta}\left[\left[\beta\cos(\beta t) - \alpha\sin(\beta t)\right] x_1(0) - \sin(\beta t)\, x_2(0) \right]$$

$$\frac{dx_1}{dt} = \frac{\alpha}{\beta} e^{\alpha t}\left[\left[\beta\cos(\beta t) - \alpha\sin(\beta t)\right] x_1(0) - \sin(\beta t)\, x_2(0) \right]$$
$$+ \frac{e^{\alpha t}}{\beta}\left[\left[-\beta^2\sin(\beta t) + \alpha\beta\cos(\beta t)\right] x_1(0) - \beta\cos(\beta t)\, X_2(0) \right]$$

$$= e^{\alpha t}\left[\begin{array}{l} \left(\alpha\cos(\beta t) - \frac{\alpha^2}{\beta}\sin(\beta t) - \beta\sin(\beta t) + \alpha\cos(\beta t)\right) x_1(0) \\ - \left(\frac{\alpha}{\beta}\sin(\beta t) + \cos(\beta t)\right) x_2(0) \end{array} \right]$$

$$\boxed{X_2(t) = e^{\alpha t}\left[\begin{array}{l} \left[2\alpha\, x_1(0) - x_2(0)\right]\cos(\beta t) \\ - \frac{1}{\beta}\left[\frac{\alpha^2 + \beta^2}{} x_1(0) + \alpha\, x_2(0)\right]\sin(\beta t) \end{array} \right]} \quad (41)$$