# HOW LINEAR(IZED) STATE-SPACE FORMS LEAD TO PHASORS & TRANSFER FUNCTIONS

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— Linear(ized) SSR: $\underbrace{\frac{d\vec{x}}{dt}}_{\text{size } n} = \underbrace{A}_{n\times n \text{ matrix}}\vec{x} + \underbrace{\vec{b}}_{n\times 1 \text{ vector}}\underbrace{u(t)}_{\text{scalar input}}$ [all quantities real] (1)

— with scalar output: $y(t) = \underbrace{\vec{c}^{\,T}}_{1\times n \text{ row vector}}\vec{x}(t)$ [all quantities real] (2)

— $\boxed{\text{ASSUME}}$ $\vec{x}(t)$ and $u(t)$ are sinusoidal at angular freq. $\omega$ (3)

$\rightarrow$ $u(t) = \underbrace{U}_{\text{capital } U;\ \text{phasor repr. of } u(t)} e^{j\omega t} + \underbrace{\bar{U}}_{\text{complex conjugate of } U} e^{-j\omega t}$ [U can now be complex] [u(t) is real] (4)

$\rightarrow$ $\vec{x}(t) = \underbrace{\vec{X}}_{\text{capital } X;\ n\times 1 \text{ vector of scalar phasors}} e^{j\omega t} + \bar{\vec{X}} e^{-j\omega t}$ [$\vec{X}$ can be complex] [$\vec{x}(t)$ is real] (5)

— Put (4) & (5) in (1):

$$\rightarrow \frac{d}{dt}\left[\vec{X}e^{j\omega t} + \bar{\vec{X}}e^{-j\omega t}\right] = A\left[\vec{X}e^{j\omega t} + \bar{\vec{X}}e^{-j\omega t}\right] + \vec{b}\left(Ue^{j\omega t} + \bar{U}e^{-j\omega t}\right) \quad (6)$$

$$\Rightarrow j\omega\vec{X}e^{j\omega t} + (-j\omega)\bar{\vec{X}}e^{-j\omega t} = (A\vec{X} + \vec{b}U)e^{j\omega t} + (A\bar{\vec{X}} + \vec{b}\bar{U})e^{-j\omega t} \quad (7)$$

$$\Rightarrow \left[(A - j\omega I)\vec{X} + \vec{b}U\right]e^{j\omega t} + \left[(A + j\omega I)\bar{\vec{X}} + \vec{b}\bar{U}\right]e^{-j\omega t} = \vec{0} \quad (8)$$

— Now, we use the fact that for any $\omega_1 \neq \omega_2$, $e^{j\omega_1 t}$ & $e^{j\omega_2 t}$ are LINEARLY INDEPENDENT FUNCTIONS (of $t$). (9)

— This means that if $a e^{j\omega_1 t} + b e^{j\omega_2 t} \equiv 0 \ \forall t$,
then $a = b = 0$. } (10)

— The concept of linear independence of functions is very similar to that for vectors. (Google it, if you like).

— (10) implies that the coeffs. of $e^{j\omega t}$ & $e^{-j\omega t}$ in (8) must each be $\vec{0}$:

$\omega_1 = \omega$, $\omega_2 = -\omega$

— $\left[(A - j\omega I)\vec{X} + \vec{b}\, U\right] = \vec{0}$ (11)

and

— $\left[(A + j\omega I)\overline{\vec{X}} + \vec{b}\,\overline{U}\right] = \vec{0}$ (12)

— note: the LHS (left hand side) of (12) is simply the complex conjugate of the LHS of (11) — here, we RELY on the fact that $A$ & $\vec{b}$ are real (from (1))

— therefore (11) and (12) are the same equation, so we need consider only one of them — say (11). (13)

— So, from (11), our phasor equations are:

— $(A - j\omega I)\vec{X} = -\vec{b}\, U \Rightarrow \boxed{\vec{X} = -(A - j\omega I)^{-1}\vec{b}\, U}$ (14)

— If we follow the same process as above for the output equation (2), we will get

$$Y = \vec{c}^{\,T}\vec{X},$$ (15)

where $Y$ is the phasor corresponding to the output $y(t)$.

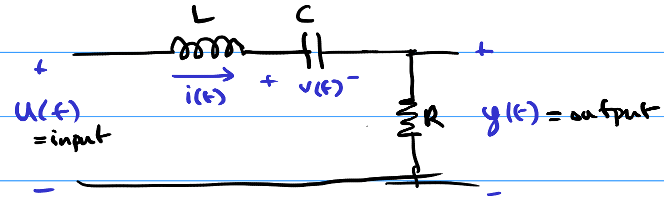
— putting (14) & (15) together, we get

$$Y = -\vec{c}^{\,T}(A - j\omega I)^{-1}\vec{b}\, U, \quad \text{or}$$

$$\boxed{H(\omega) \triangleq \frac{Y}{U} = -\vec{c}^{\,T}(A - j\omega I)^{-1}\vec{b}}$$ (16)

— $H(\omega)$ is the scalar transfer function from $u(t)$ to $y(t)$, expressed using $A$, $\vec{b}$ & $\vec{c}^{\,T}$

— **Example**: series RLC circuit with output = voltage across the resistor




(17)

KCL: $C\frac{dv}{dt} = i(t)$

SCRATCH WORK

KVL: $L\frac{di}{dt} + v(t) + R\,i(t) = u(t)$

$\frac{dv}{dt} = \frac{i(t)}{C}$

$\frac{di}{dt} = \frac{-v(t) - R\,i(t) + u(t)}{L}$

$y(t) = R\,i(t)$

— state-space equations:

$$\rightarrow \frac{d}{dt}\underbrace{\begin{bmatrix} v(t) \\ i(t) \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} v(t) \\ i(t) \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}} u(t) \qquad (18)$$

$$\rightarrow \vec{y}(t) = \underbrace{\begin{bmatrix} 0 & R \end{bmatrix}}_{\vec{c}^T}\underbrace{\begin{bmatrix} v(t) \\ i(t) \end{bmatrix}}_{\vec{x}} \qquad (19)$$

— evaluate (16):

$$\rightarrow (A - j\omega I) = \begin{bmatrix} -j\omega & \frac{1}{C} \\ -1/L & -\frac{R}{L} - j\omega \end{bmatrix} \Rightarrow (A - j\omega I)^{-1} = \begin{bmatrix} -\frac{R}{L} - j\omega & -\frac{1}{C} \\ \frac{1}{L} & -j\omega \end{bmatrix} \times \frac{1}{j\omega\left(\frac{R}{L} + j\omega\right) + \frac{1}{LC}} \qquad (20)$$

$$\rightarrow (A - j\omega I)^{-1}\vec{b} = \frac{1}{(j\omega)^2 + \frac{R}{L}j\omega + \frac{1}{LC}}\begin{bmatrix} -\frac{1}{LC} \\ -\frac{j\omega}{L} \end{bmatrix} \qquad (21)$$

$$\rightarrow H(\omega) = -\vec{c}^T(A - j\omega I)^{-1}\vec{b} = \frac{j\omega R/L}{(j\omega)^2 + \frac{R}{L}j\omega + \frac{1}{LC}} \qquad (22)$$

$$\Rightarrow \boxed{H(\omega) = \frac{j\omega RC}{LC(j\omega)^2 + RC(j\omega) + 1}} \qquad (23)$$

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

SCRATCH

$$= \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}$$

$$= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

— Finding the poles and zeroes:

— Factor the denominator of (23):

$$\underbrace{LC}_{a}(j\omega)^2 + \underbrace{RC}_{b}(j\omega) + \underbrace{1}_{c} = ((j\omega) - p_1)((j\omega) - p_2), \text{ where}$$

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2} \quad (24)$$

— define: $\alpha \triangleq \frac{R}{2L}$, $\omega_0 \triangleq \frac{1}{\sqrt{LC}}$, then we have (25)

$$p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad (26)$$

THE POLES

— zeroes: the numerator of (23) is merely $RC\, j\omega = RC(j\omega - 0)$

→ hence there is one zero, i.e., $z_1 = 0$ (27)

(End RLC circuit example)

## THE CONNECTION BETWEEN THE EIGENVALUES OF A and THE POLES OF H(ω)

— Returning to (11): $\left[(A - j\omega I)\vec{x} + \vec{b}\,U\right] = \vec{0}$ (11)

— suppose we can eigendecompose A (we will always assume this):

— then $A = P\Lambda P^{-1}$ (28)

— put (28) in (11): $(P\Lambda P^{-1} - j\omega I)\vec{x} = -\vec{b}\,U$ (29)

— now: note that $I = PP^{-1}$, so (29) becomes

$$(P\Lambda P^{-1} - j\omega PP^{-1})\vec{x} = -\vec{b}\,U \quad (30)$$

$$\Rightarrow P(\Lambda - j\omega I)P^{-1}\vec{x} = -\vec{b}\,U \quad (31)$$

$$\Rightarrow \vec{x} = -P(\Lambda - j\omega I)^{-1}P^{-1}\vec{b}\,U \quad (32)$$

→ using (15) — the output equation $Y = \vec{c}^T\vec{x}$ — with (32), we get

$$\rightarrow H(\omega) \triangleq \frac{Y}{U} = \underbrace{-\vec{c}^T P}_{\text{call this } \vec{s}^T}(\Lambda - j\omega I)^{-1}\underbrace{P^{-1}\vec{b}}_{\text{call this } \vec{r}} \quad (33)$$

→ note that $\vec{s}^T$ is a row vector: say $\vec{s}^T = [s_1, s_2, \dots, s_n]$
→ " " $\vec{r}$ " " col. " : say $\vec{r}^T = [r_1, r_2, \dots, r_n]$ (34)

→ and observe that $(\Lambda - j\omega I)$ is a <u>diagonal matrix</u>:

$$(\Lambda - j\omega I) = \begin{bmatrix} \lambda_1 - j\omega & & & \\ & \lambda_2 - j\omega & & \\ & & \ddots & \\ & & & \lambda_n - j\omega \end{bmatrix} \quad (35)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A.

→ using (35), the inverse $(\Lambda - j\omega I)^{-1}$ is easily written:

$$(\Lambda - j\omega I)^{-1} = \begin{bmatrix} \frac{1}{\lambda_1 - j\omega} & & & \\ & \frac{1}{\lambda_2 - j\omega} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n - j\omega} \end{bmatrix} \quad (36)$$

— using (36) & (34), we can re-express (33) as:

$$H(\omega) = \underbrace{[s_1, s_2, \cdots, s_n]}_{\vec{s}^T} \overbrace{\begin{bmatrix} \frac{1}{\lambda_1 - j\omega} & & & \\ & \frac{1}{\lambda_2 - j\omega} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n - j\omega} \end{bmatrix}}^{(A - j\omega I)^{-1}} \overbrace{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}}^{\vec{r}} \quad (37)$$

$$\Rightarrow H(\omega) = \frac{s_1 r_1}{\lambda_1 - j\omega} + \frac{s_2 r_2}{\lambda_2 - j\omega} + \cdots + \frac{s_n r_n}{\lambda_n - j\omega} \quad (38)$$

$$\Rightarrow H(\omega) = \frac{\left[ s_1 r_1 (\lambda_2 - j\omega)(\lambda_3 - j\omega) \cdots (\lambda_n - j\omega) + s_2 r_2 (\lambda_1 - j\omega)(\lambda_3 - j\omega) \cdots (\lambda_n - j\omega) + \cdots + \cdots + s_n r_n (\lambda_1 - j\omega)(\lambda_2 - j\omega) \cdots (\lambda_{n-1} - j\omega) \right]}{(\lambda_1 - j\omega)(\lambda_2 - j\omega)(\lambda_3 - j\omega) \cdots (\lambda_{n-1} - j\omega)(\lambda_n - j\omega)} \quad (39)$$

— From the form of the denominator of (39), we can readily see that $H(\omega)$ has $n$ poles, and that they are simply the eigenvalues of $A$. (40)

— Moreover, from the numerator expression (which is an $(n-1)^{th}$ degree polynomial), we can infer that there will be $(n-1)$ zeroes — which can be found by factorizing the numerator polynomial (41)

— Returning to the series RLC example:

— from (18), $A = \left[\begin{array}{c|c} 0 & \frac{1}{C} \\ \hline -\frac{1}{L} & -\frac{R}{L} \end{array}\right]$

— to find its eigenvalues: $\det(A - \lambda I) = 0$

char. poly. of A

$$\Rightarrow \det\left[\begin{array}{c|c} -\lambda & \frac{1}{C} \\ \hline -\frac{1}{L} & -\frac{R}{L} - \lambda \end{array}\right] = 0 \Rightarrow \lambda\left(\lambda + \frac{R}{L}\right) + \frac{1}{LC} = 0 \tag{42}$$

$$\Rightarrow \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \tag{43}$$

$$\Rightarrow \lambda_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{43}$$

— compare against the poles obtained earlier in (26) — THEY ARE THE SAME

— not surprising in hindsight, since (42) and the denominator in (20) are identical if you set $\boxed{\lambda = j\omega}$ in (42) (44)