EE16B, Spring 2018
UC Berkeley EECS

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Lectures 6B & 7A: Overview Slides

Controller Canonical Form
Observability

Controller Canonical Form (CCF)

- Recall prior example: $\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$

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- Generalization: **Controller Canonical Form (CCF)**

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- char poly: $\lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_2\lambda - a_1$
  - ➔ not difficult to show this (though a bit tedious)
    - apply determinant formula using minors to the last row

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  ➔ its roots are the eigenvalues that determine stability

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We just showed: if a system is in CCF, feedback can move its eigenvalues to any desired locations

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      - $\begin{matrix} -(3-k_3) = 6 \\ -(2-k_2) = 11 \\ -(1-k_1) = 6 \end{matrix} \Rightarrow \begin{cases} k_3 = 9 \\ k_2 = 13 \\ k_1 = 7 \end{cases}$

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- $R_n^{-1} = \begin{bmatrix} \rule{4cm}{0.4pt} \\ \rule{4cm}{0.4pt} \\ \vdots \\ \rule{4cm}{0.4pt} \\ \longleftarrow \quad \vec{q}^T \quad \longrightarrow \end{bmatrix}$; ($\vec{q}$ is a col. vector; $\vec{q}^T$ is a row vector)

Converting Systems to CCF (contd.)

5. Form the basis transformation matrix $T \triangleq \begin{bmatrix} \longleftarrow & \vec{q}^T & \longrightarrow \\ \longleftarrow & \vec{q}^T A & \longrightarrow \\ \longleftarrow & \vec{q}^T A^2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{q}^T A^{n-1} & \longrightarrow \end{bmatrix}$

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equivalent to the original system: $u(t) \mapsto \vec{x}(t)$ is the same

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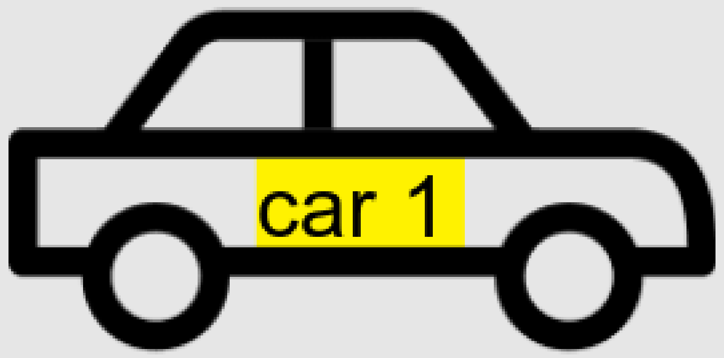
- Proof: see the handwritten notes

Example: co-operative car control



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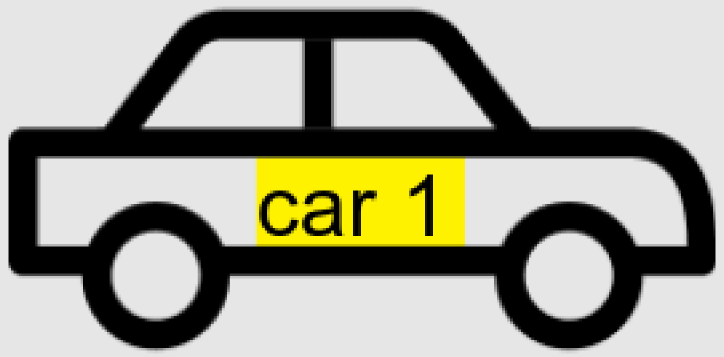
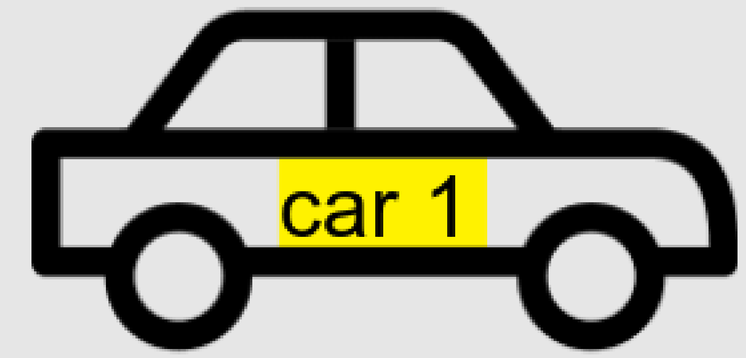


position: p_1 velocity: v_1 accel: a_1

Example: co-operative car control

position: p_2 velocity: v_2 accel: a_2



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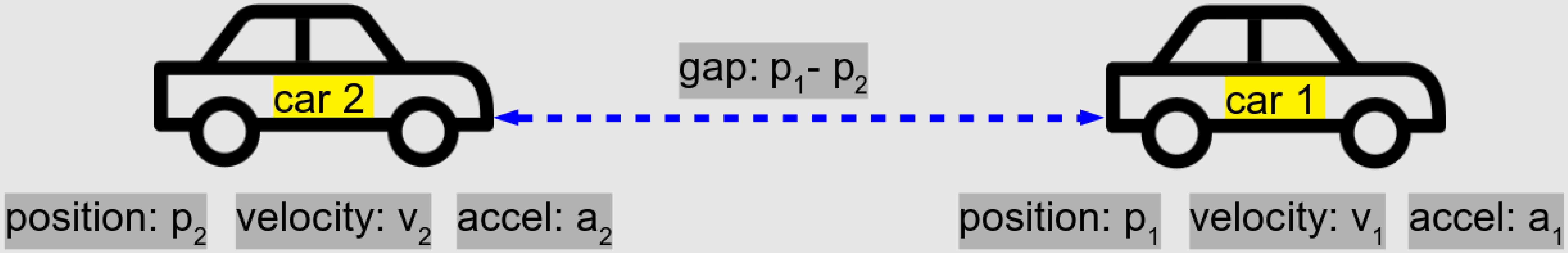

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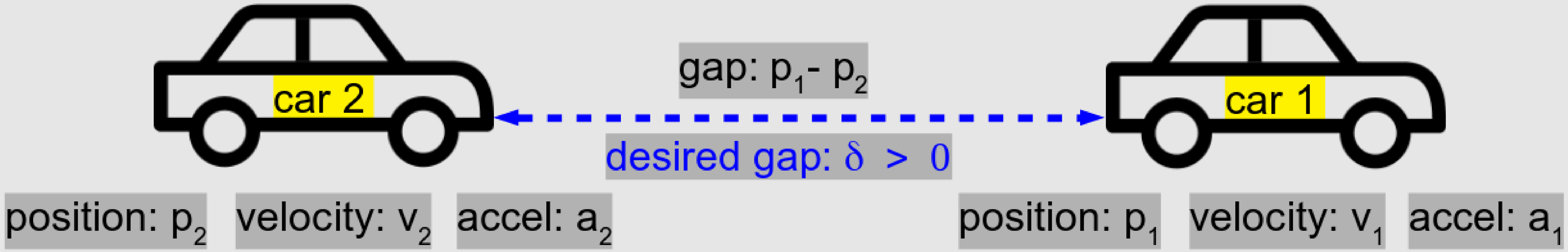

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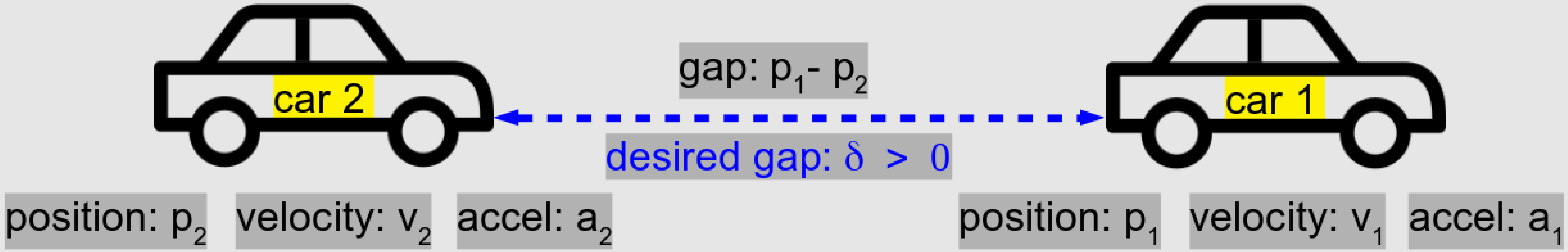

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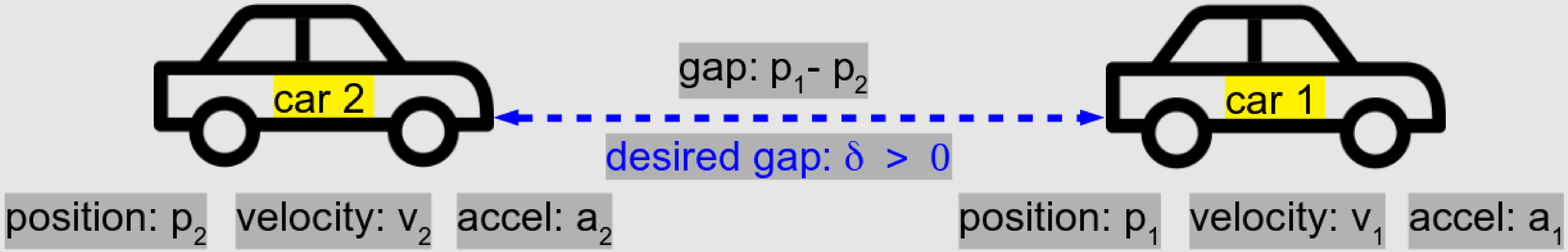

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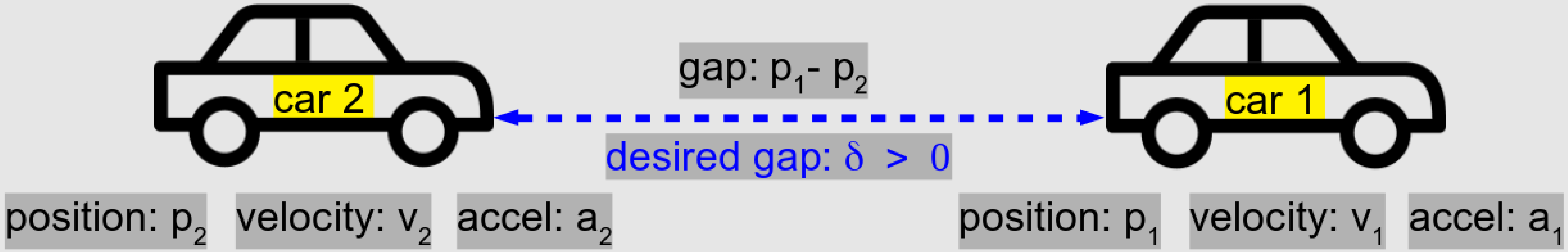

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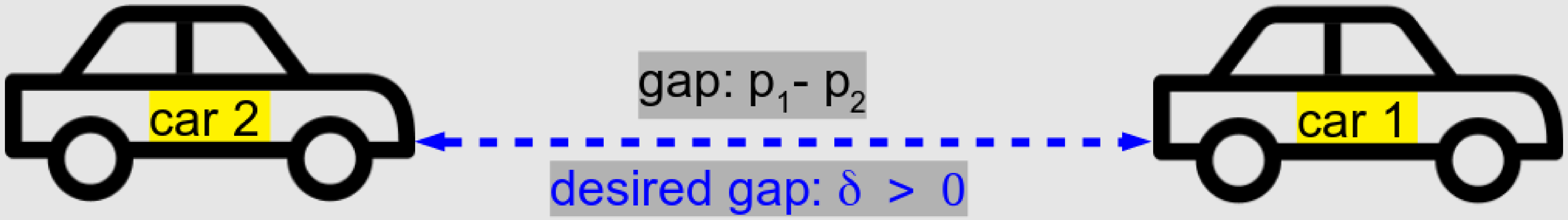

position: p_2 velocity: v_2 accel: a_2

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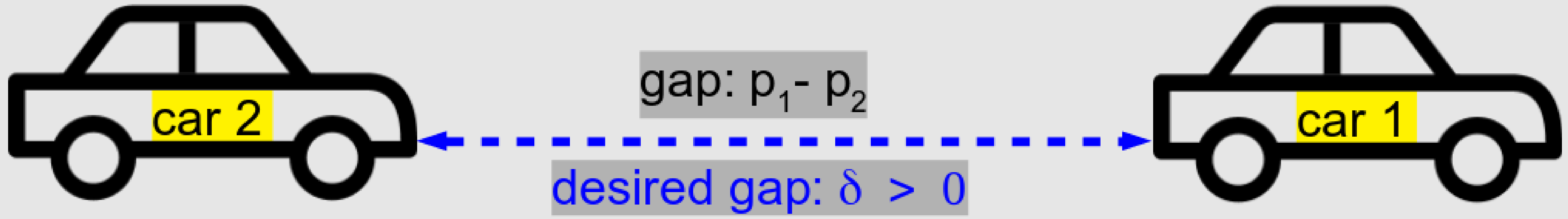

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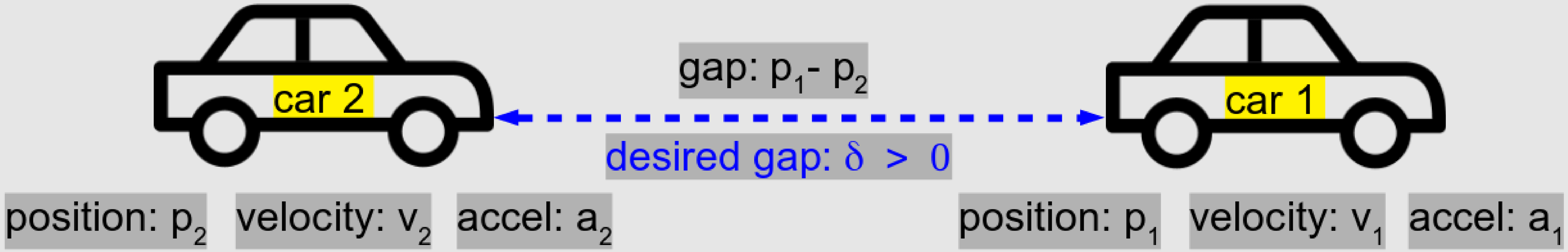

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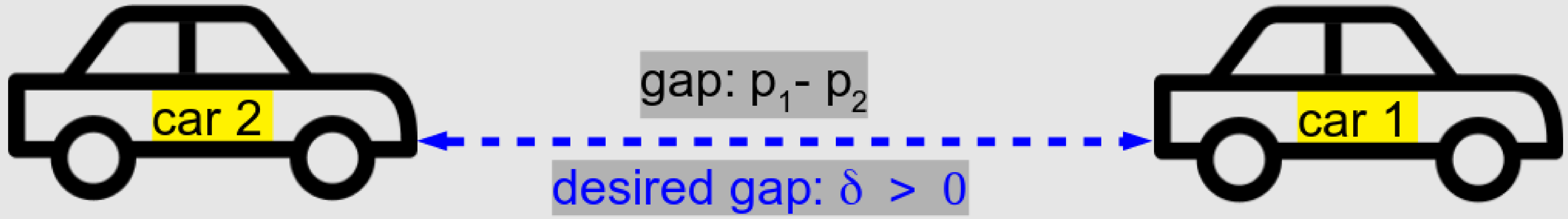

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$$\frac{d(\underbrace{p_1 - p_2}_{x_1(t)+\delta})}{dt} = v_1(t) - v_2(t) \longleftarrow \text{call this } x_2(t)$$

$$\frac{d(v_1 - v_2)}{dt} = a_1(t) - a_2(t) \longleftarrow \text{call this } u(t)\text{, the input}$$

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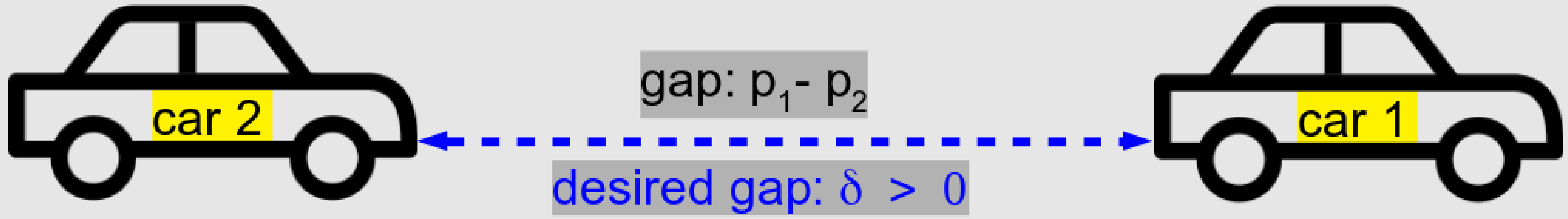

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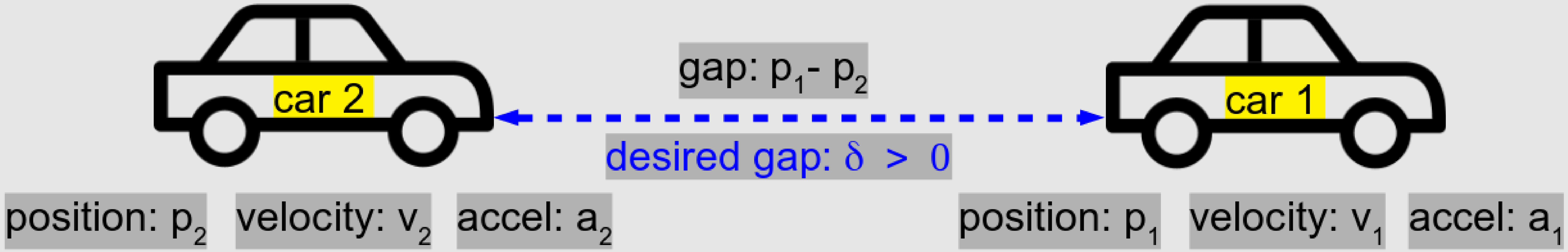

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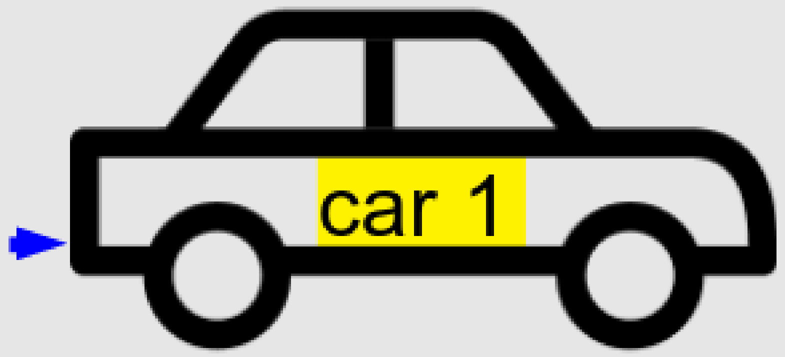
$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}^{\vec{x}(t)} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\vec{b}(t)} u(t)$$

Example: co-operative car control

gap: $p_1 - p_2$

desired gap: $\delta > 0$



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BIBO UNSTABLE

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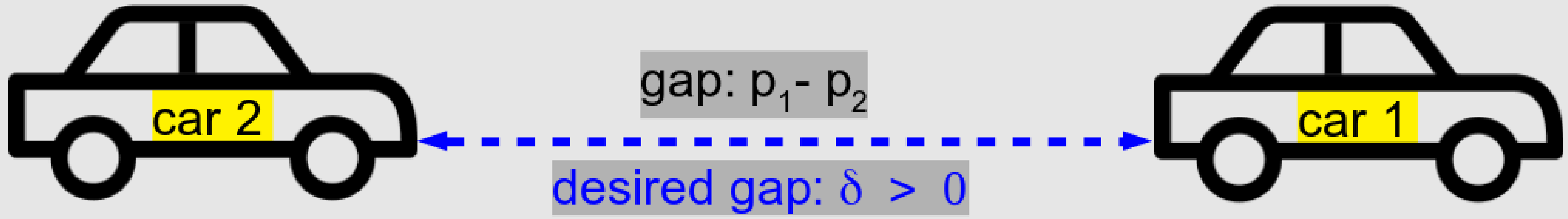

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with IC = $[0, 0]^T$ and u(t) = $-\varepsilon$, the cars will hit each other in

$$T = \sqrt{\frac{2\delta}{\epsilon}}$$

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BIBO UNSTABLE

Co-op. Car Control (contd.)

- introduce state feedback: $A \mapsto \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$

Co-op. Car Control (contd.)

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position feedback

Co-op. Car Control (contd.)

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position feedback

velocity feedback

Co-op. Car Control (contd.)

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position feedback

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both sensed by Doppler radar

Co-op. Car Control (contd.)

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position feedback — velocity feedback

both sensed by Doppler radar

- eigenvalues:
  - $\lambda_{1,2} = -\frac{k_2}{2} \pm \frac{1}{2}\sqrt{k_2^2 - 4k_1}$

Co-op. Car Control (contd.)

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position feedback

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  - $k_2 > 0, \quad k_1 > 0$ ensures eigenvalues have -ve real parts
- small errors in the acceleration u(t) → only small changes to the desired distance δ
  - see handwritten notes for details

Controllable Systems can be Stabilized

- So far, we have shown that:
  - CCF systems can be stabilized by feedback
  - Controllable systems can be put in CCF

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- suppose we have just a **SCALAR output** $y[t]$
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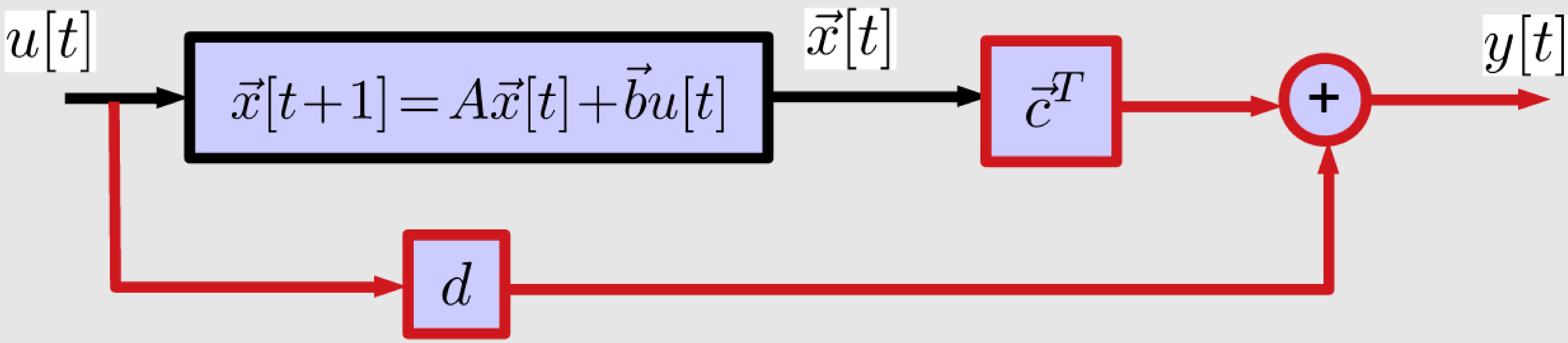

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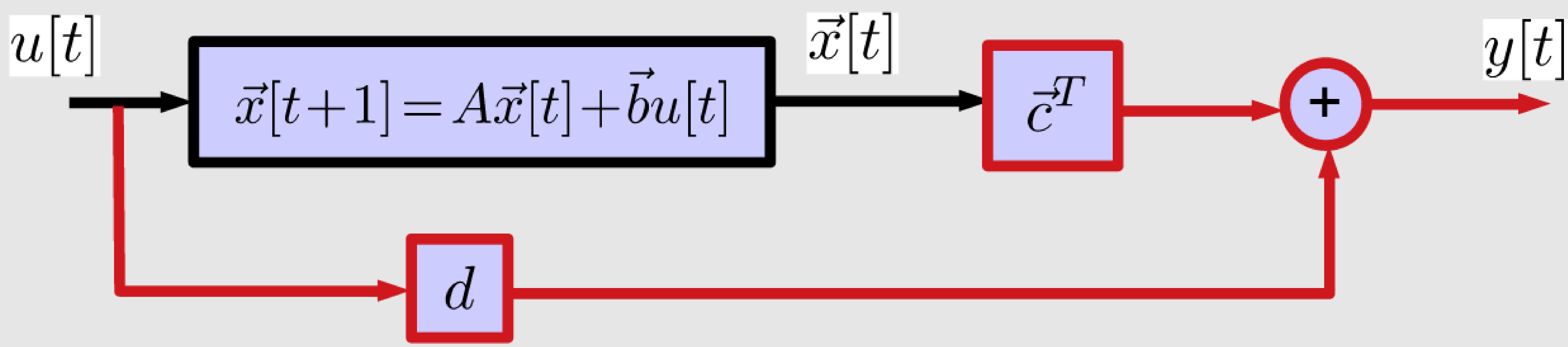

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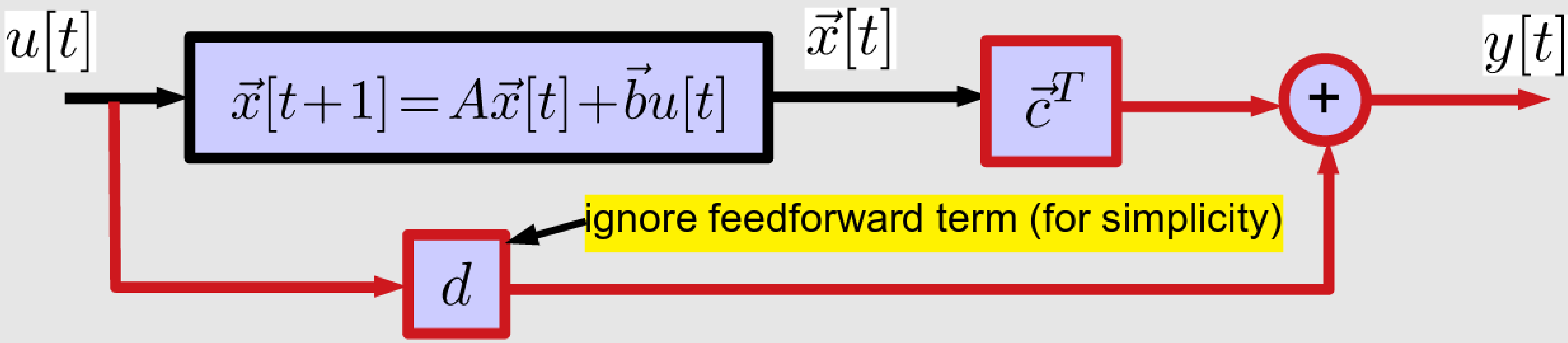

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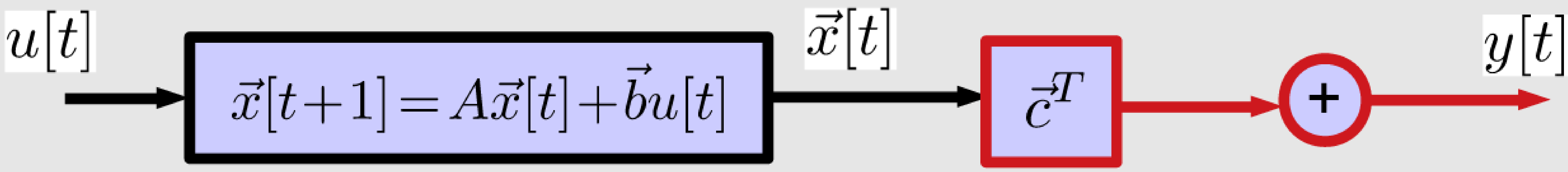

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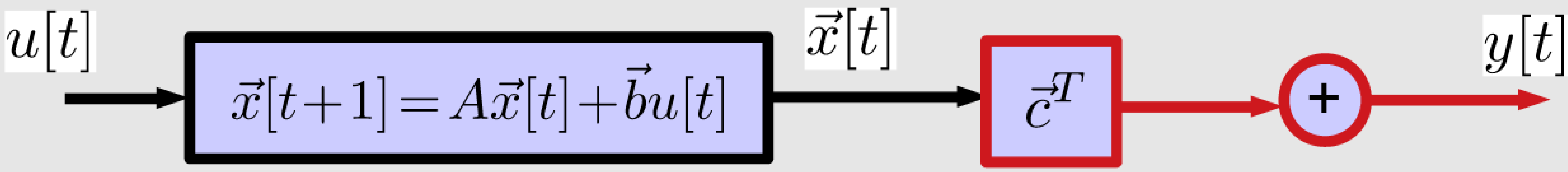

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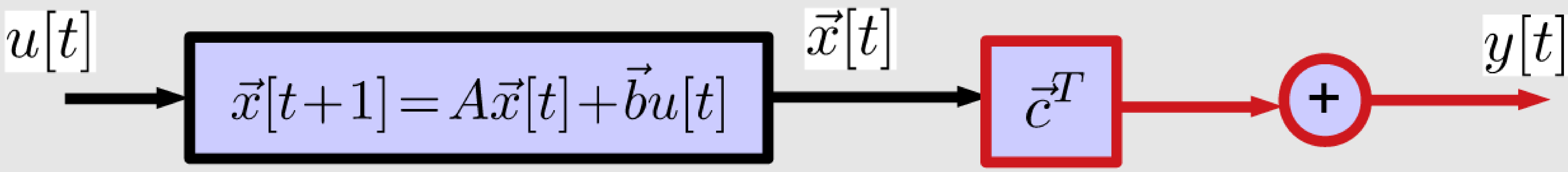

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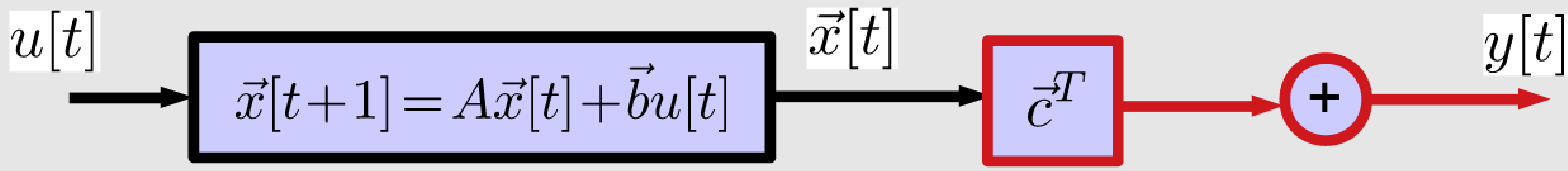

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If yes: the system is called **OBSERVABLE**

The Observability Matrix

- We know that $\vec{x}[t] = A^{t-1}\,\vec{x}[0] + \sum_{i=1}^{t} A^{t-i}\vec{b}u[i-1]$

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must be full-rank/non-singular/invertible to recover $\vec{x}(t)$ uniquely from measurements of y(t)

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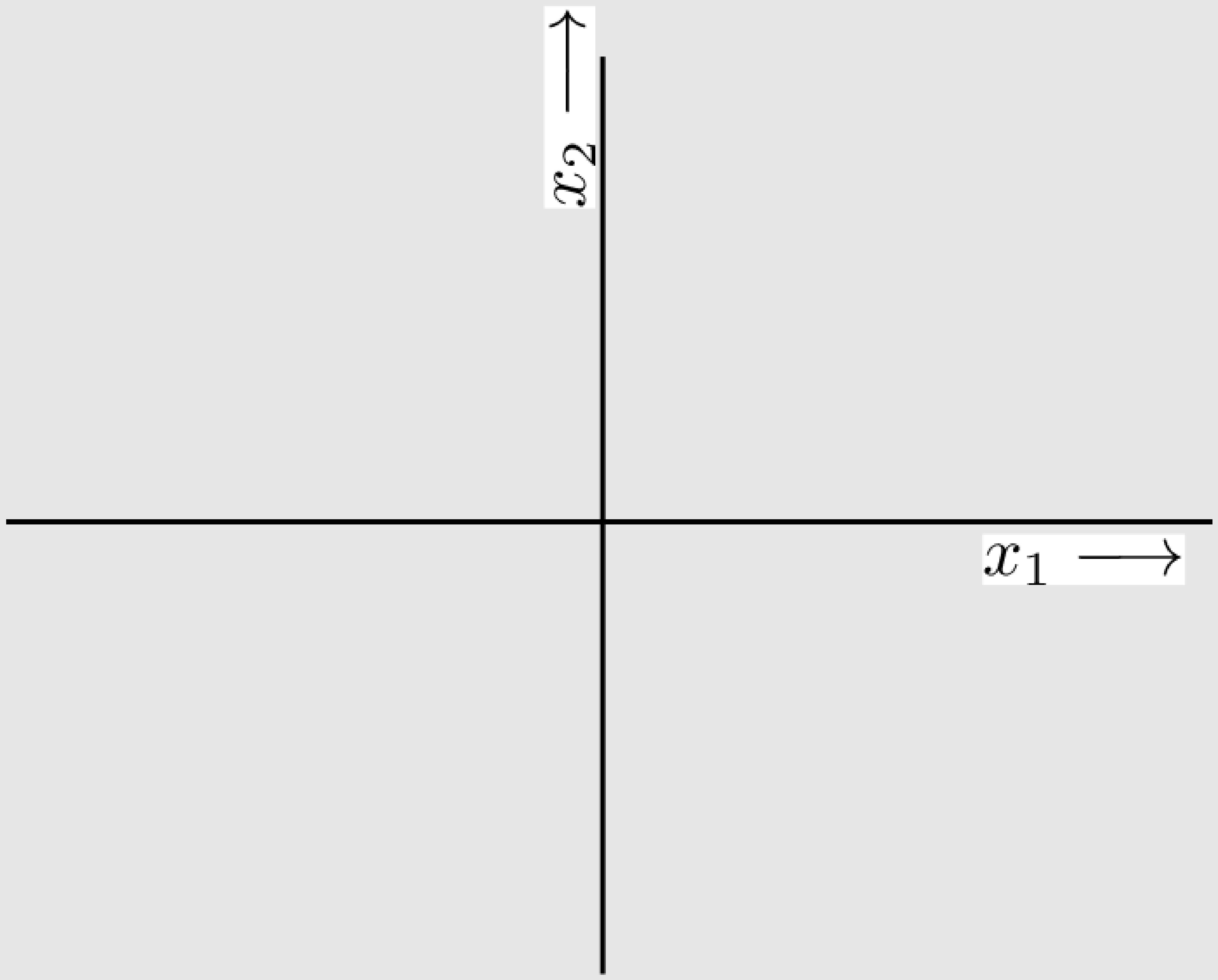

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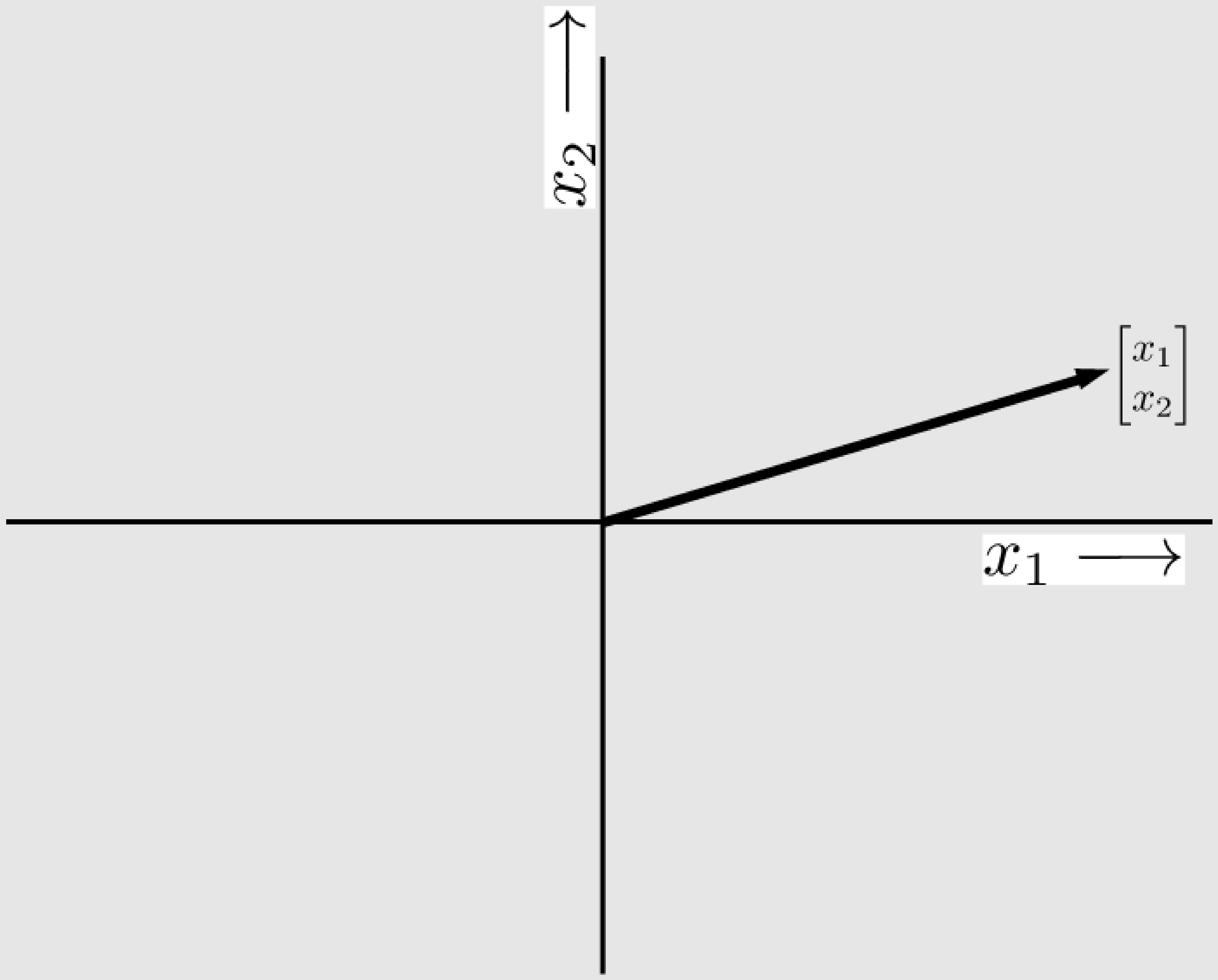

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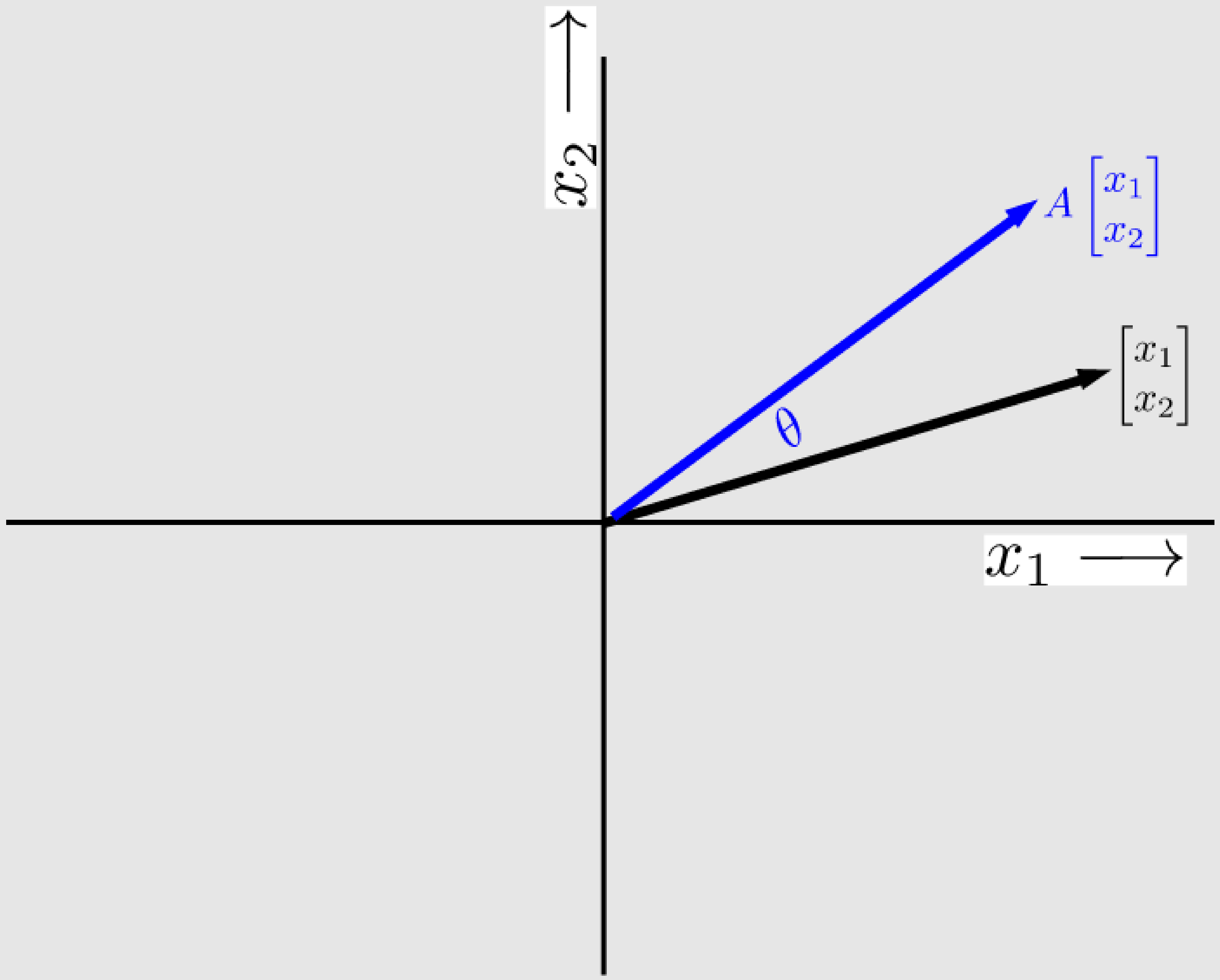

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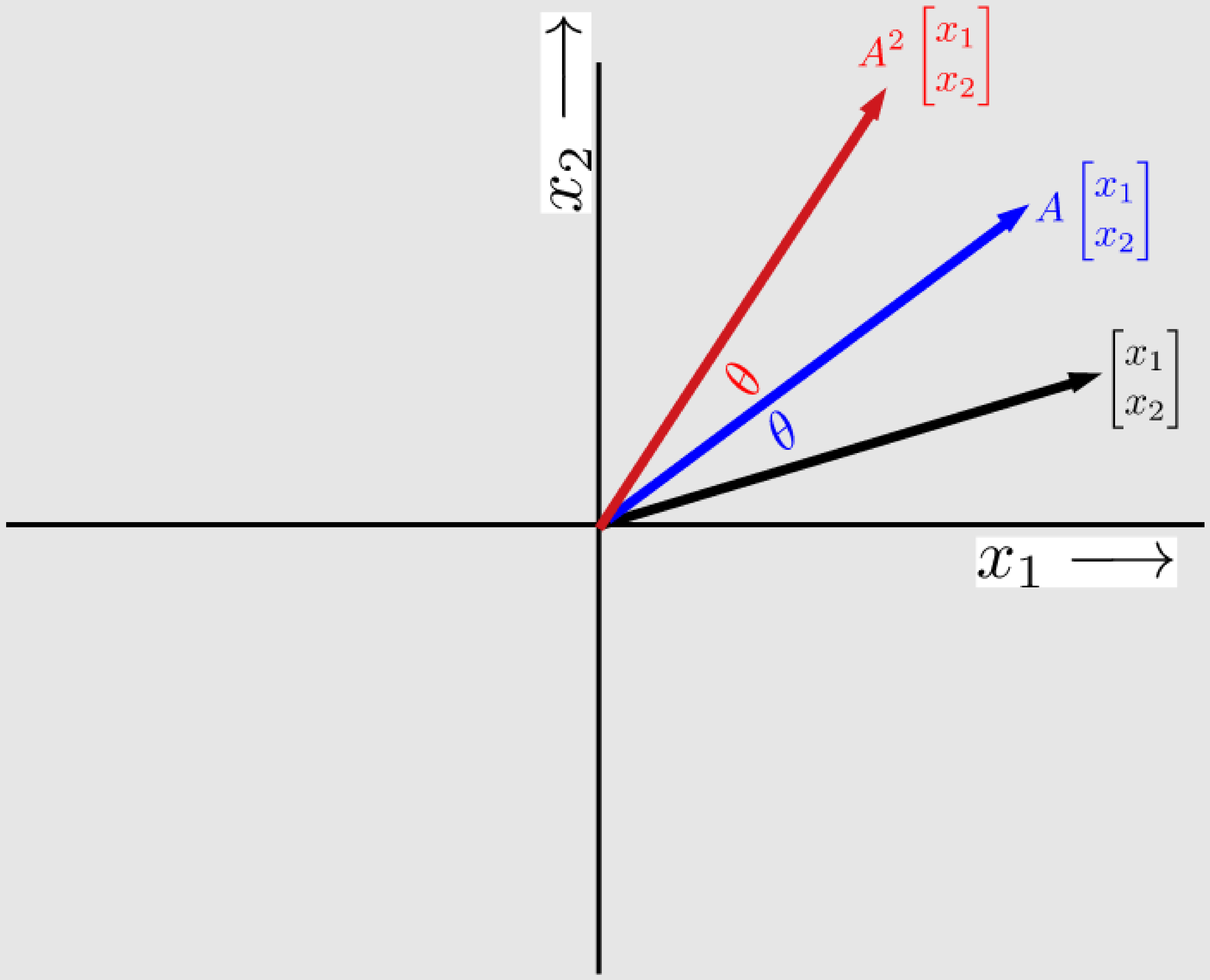

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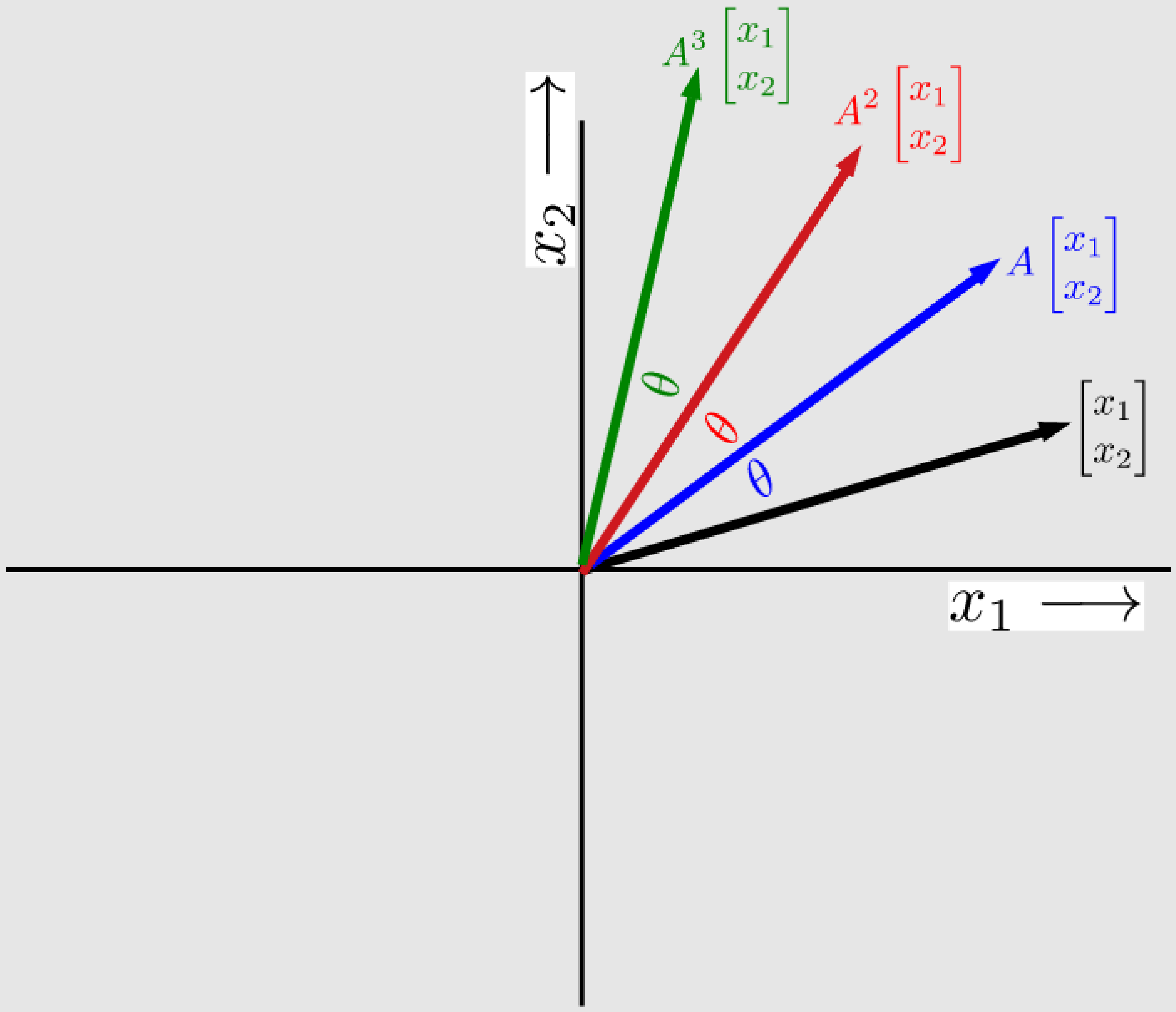

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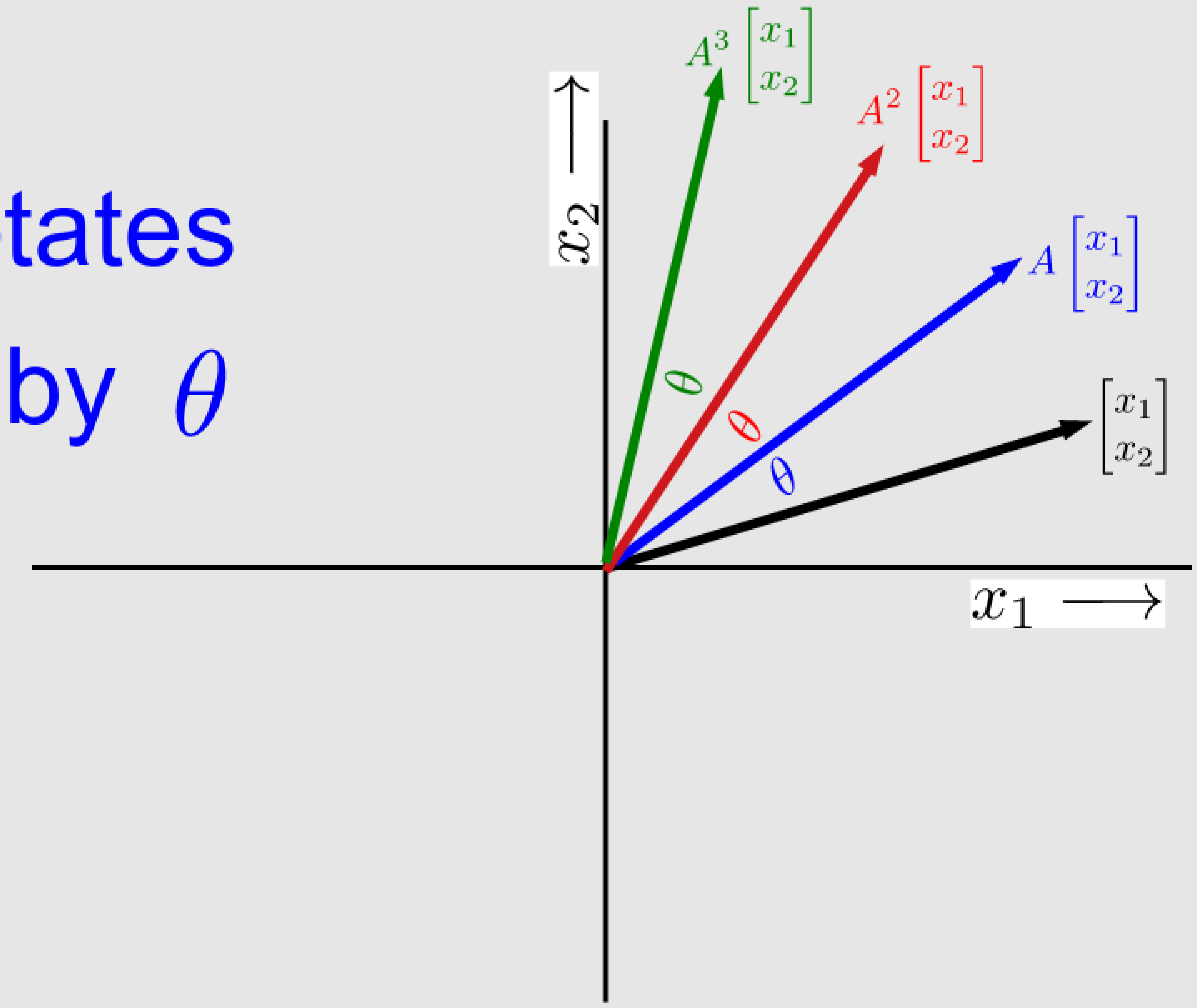
- Each application of A rotates by θ



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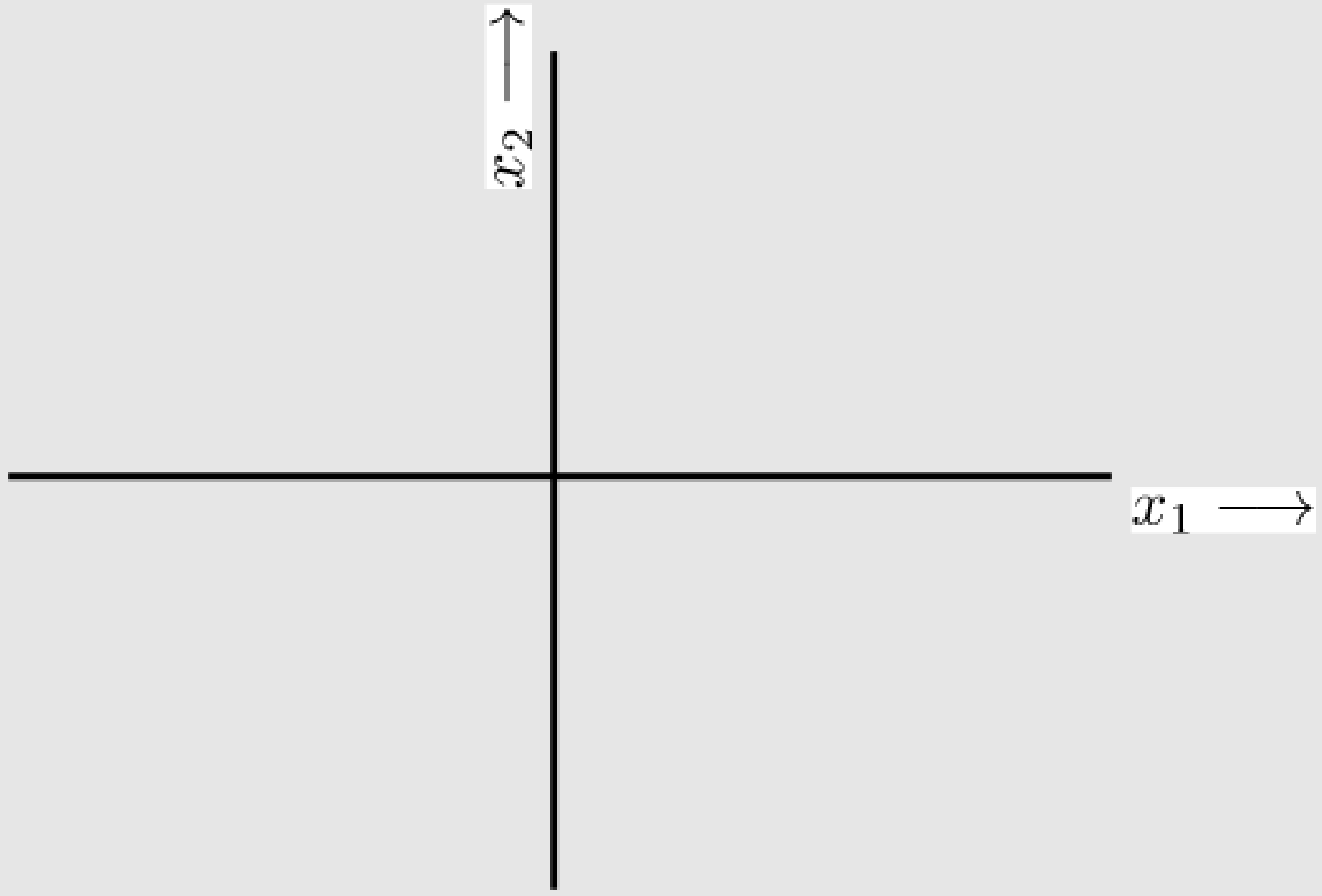

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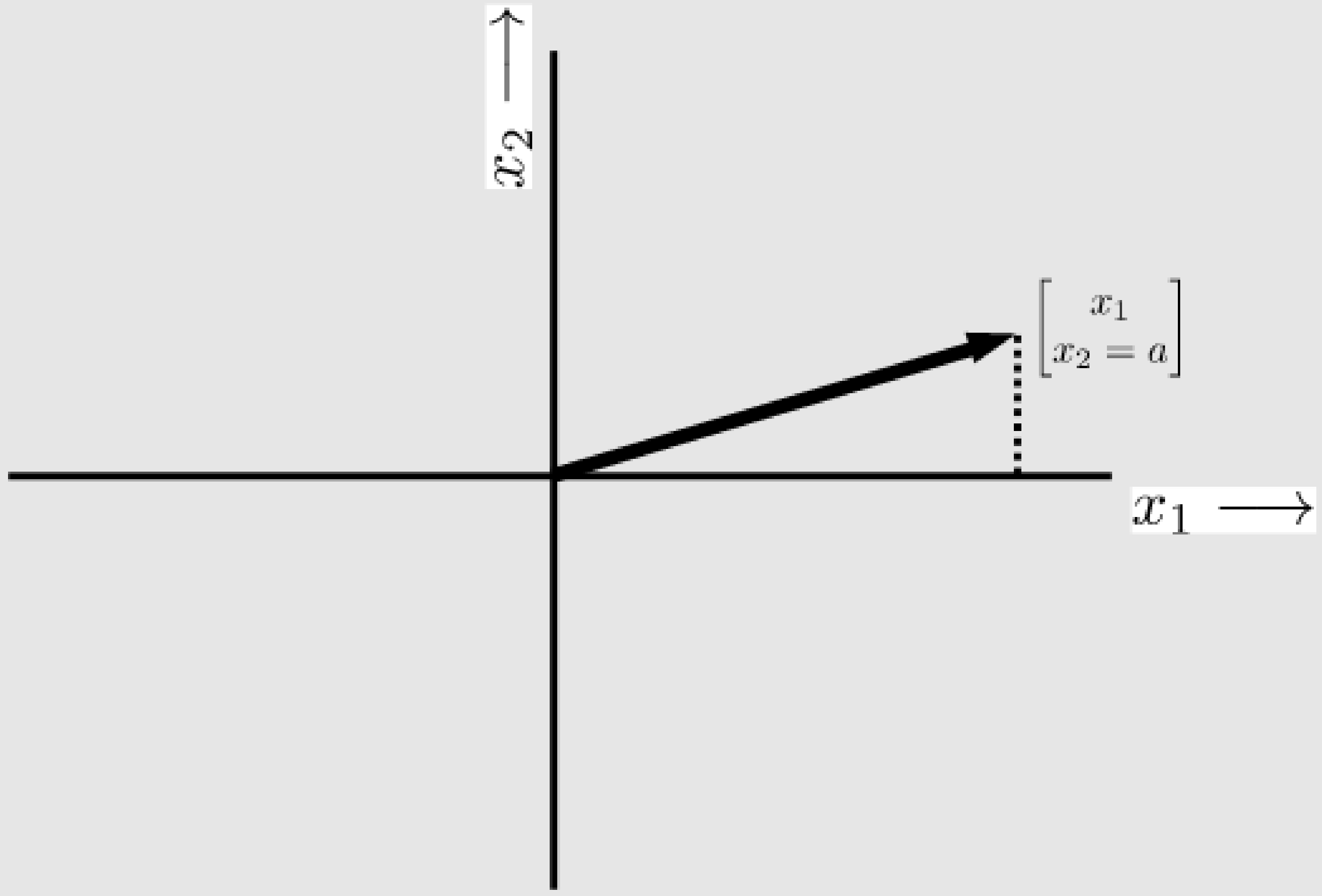

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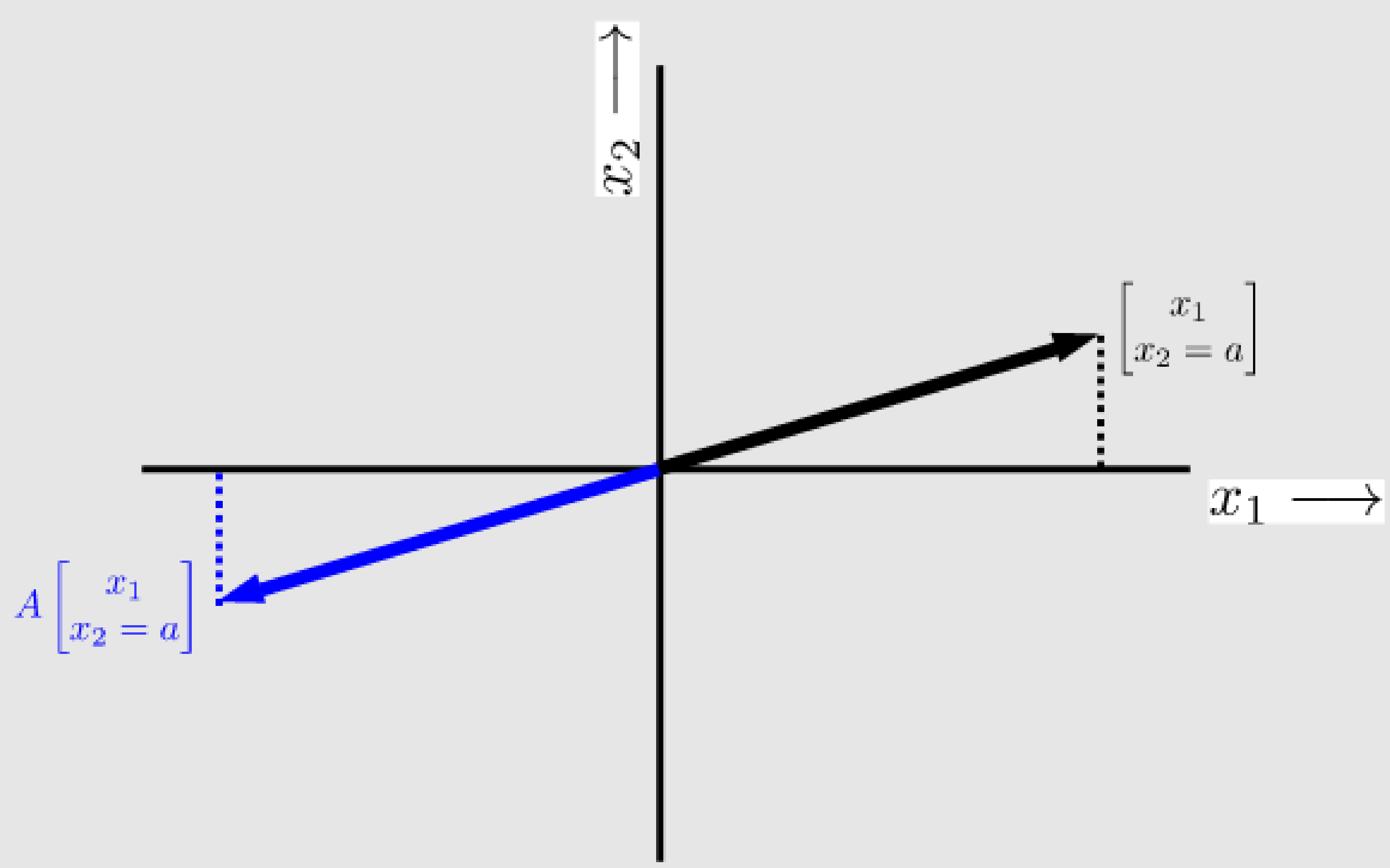

Observability: Example (contd.)

- $\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$, $y[t] = x_1[t] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\vec{c}^T} \vec{x}[t]$

- Observability matrix: $O \triangleq \begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & -\sin(\theta) \end{bmatrix}$

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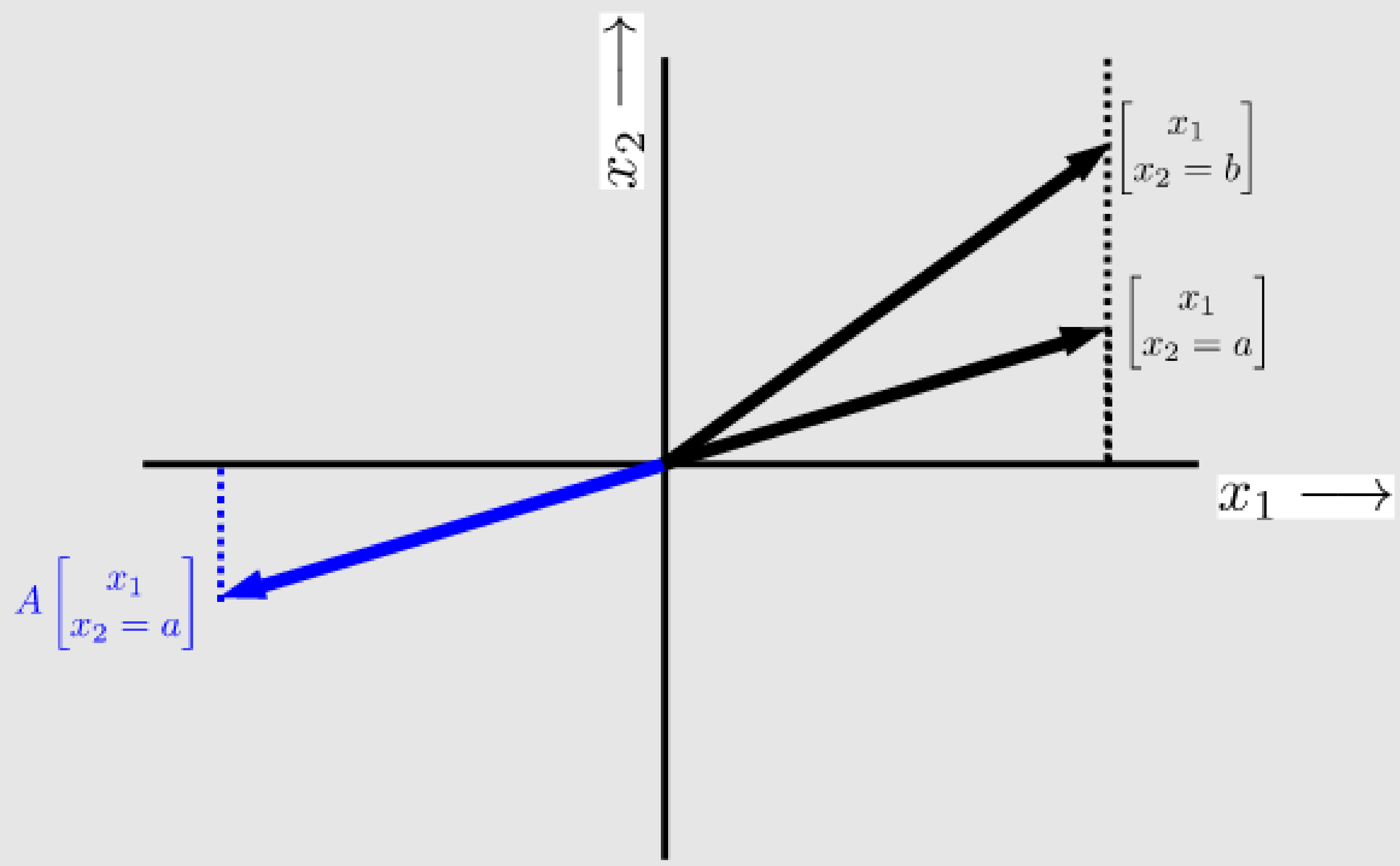

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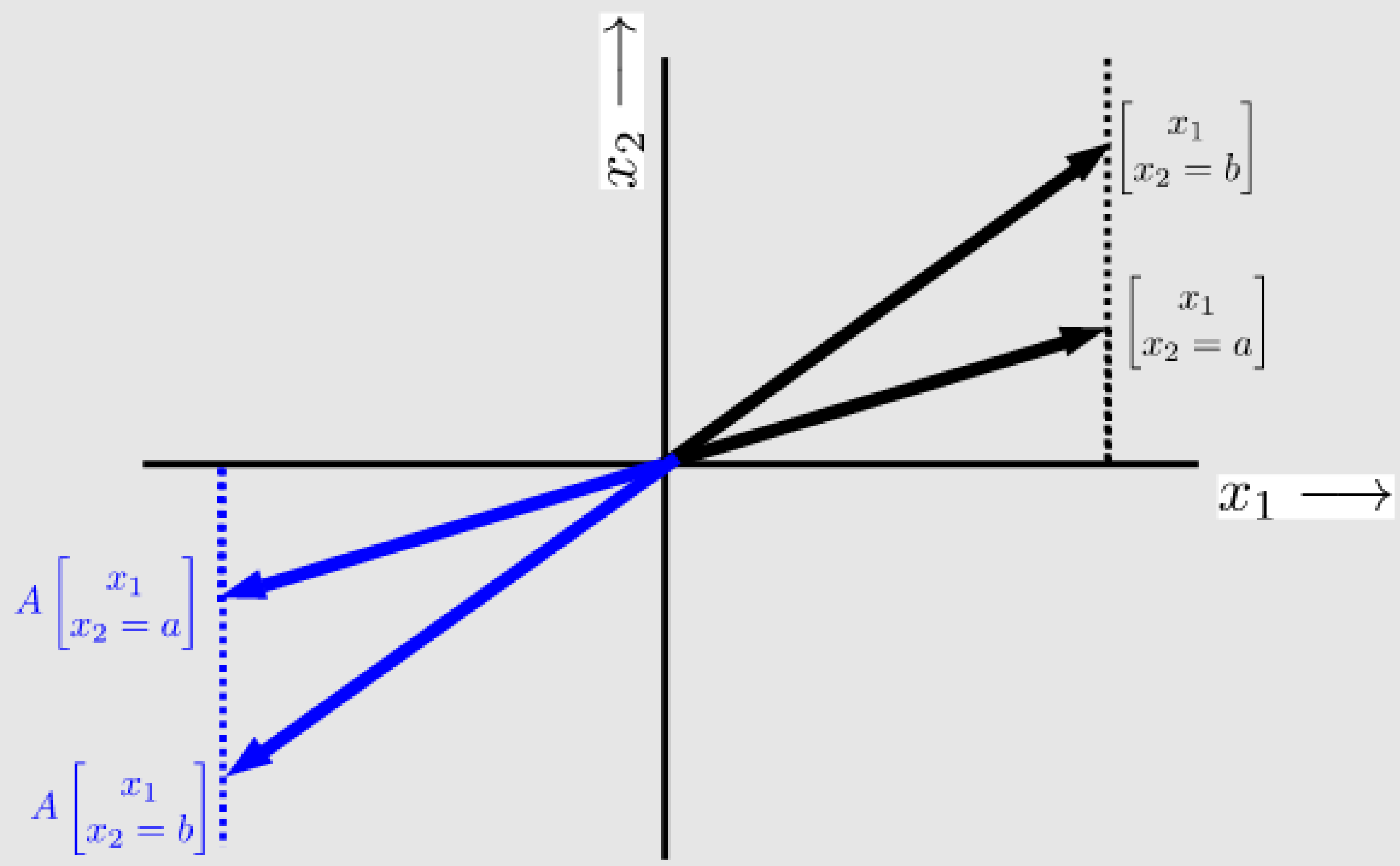

Observability: Example (contd.)

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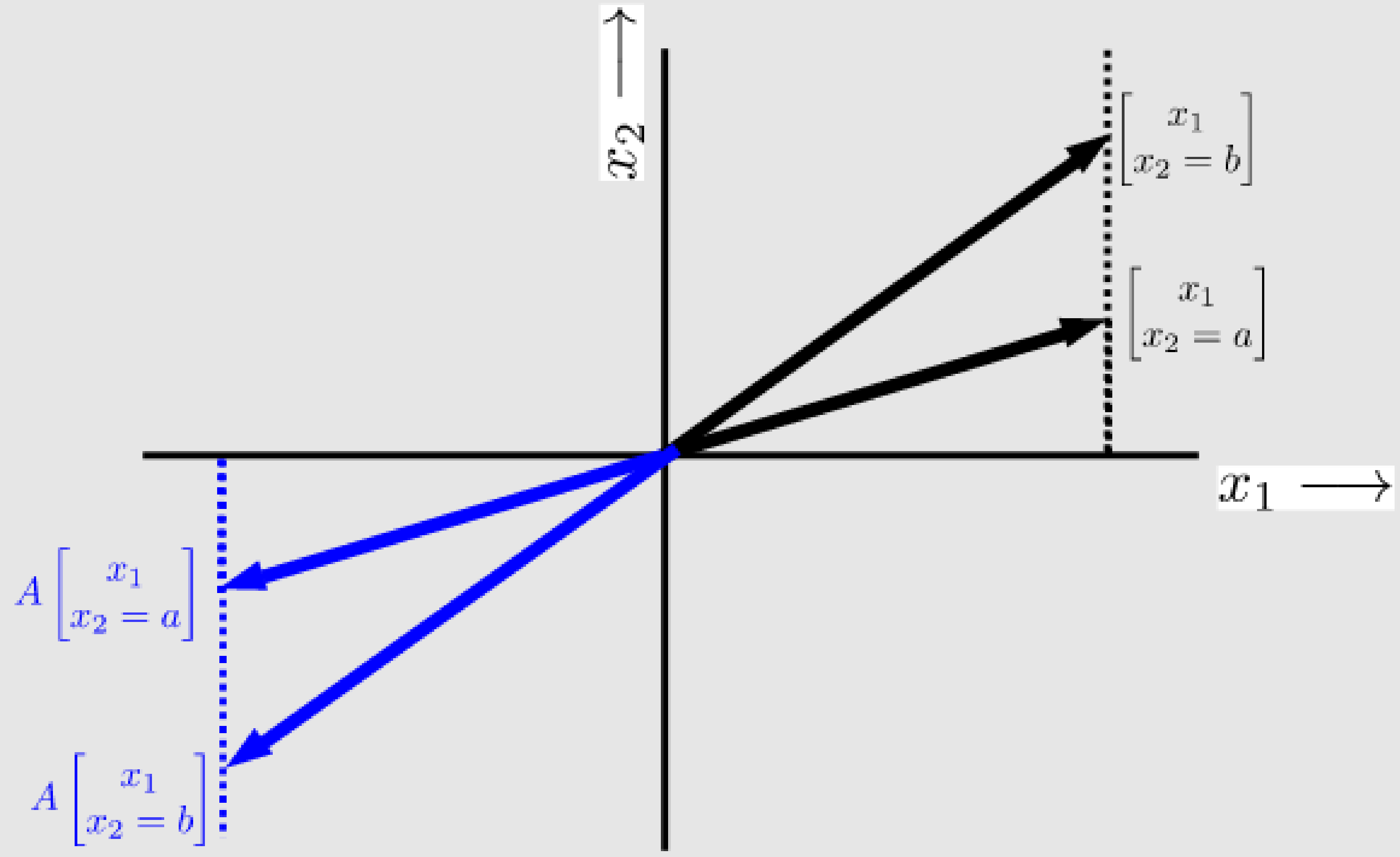
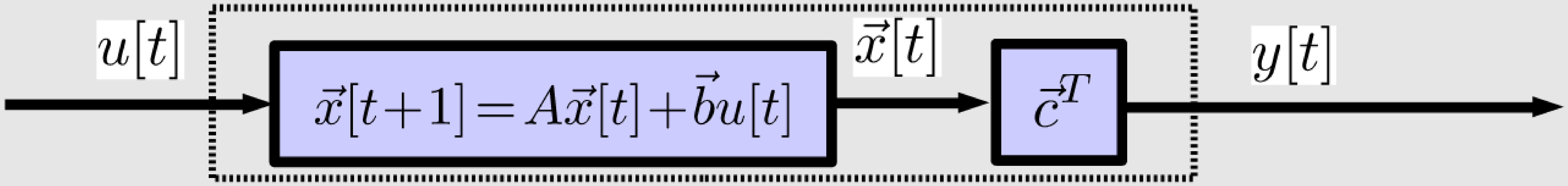
    - ➔ **cannot recover x_2 uniquely**



Observers

- Can we make a **system that recovers $\vec{x}[t]$ from y[t] in real time**?
  - (we can use our knowledge of A, $\vec{b}$, u[t] – and y[t])



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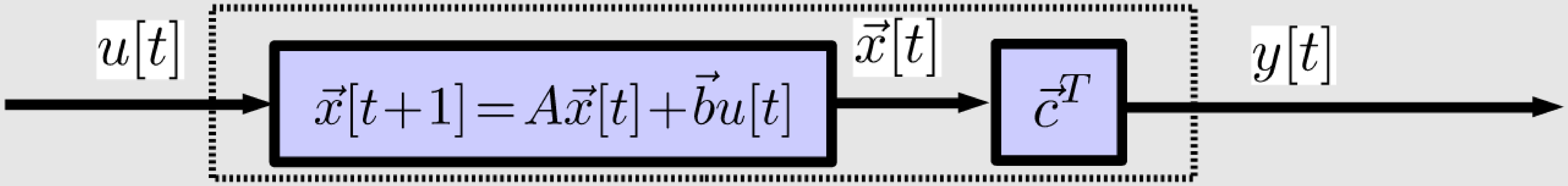

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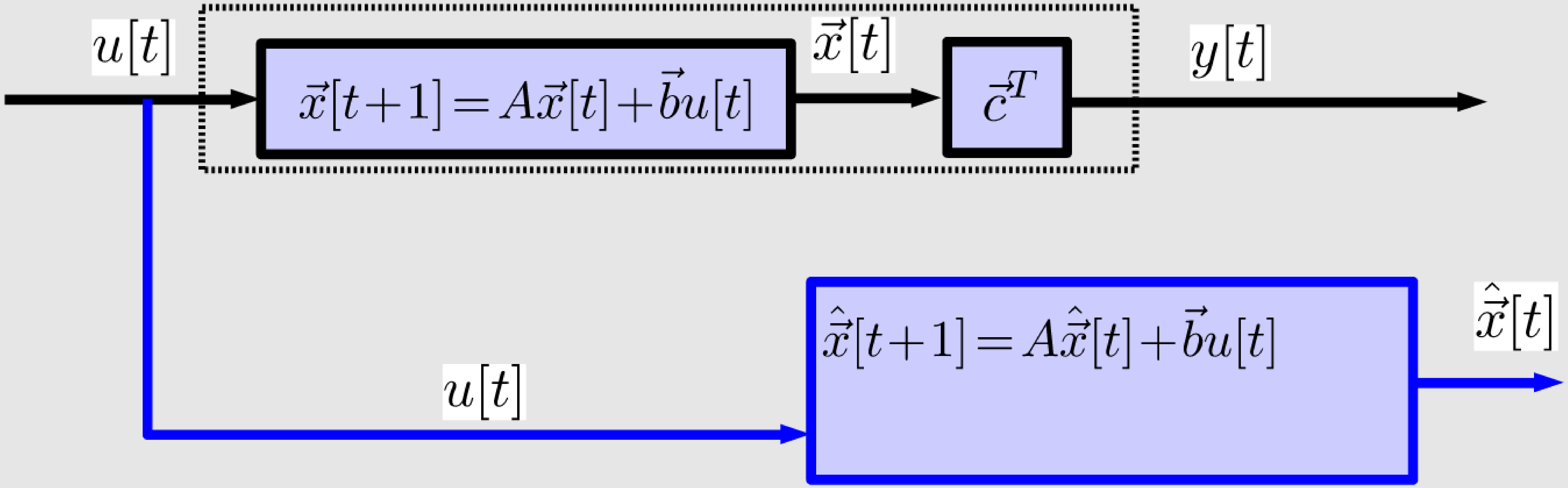

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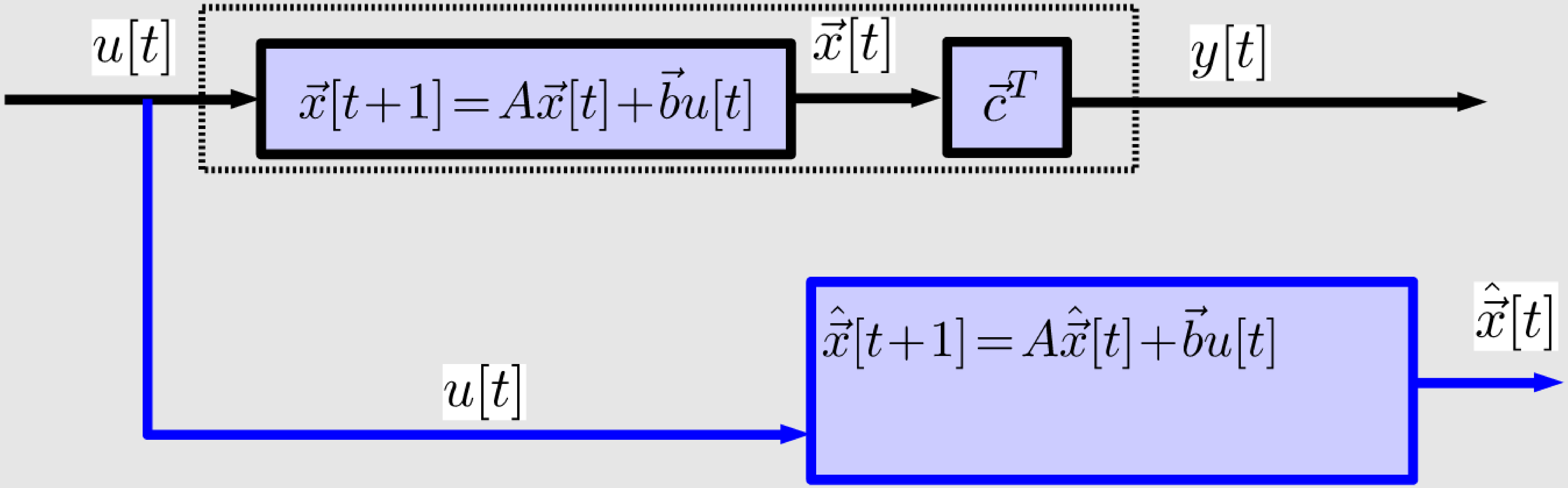

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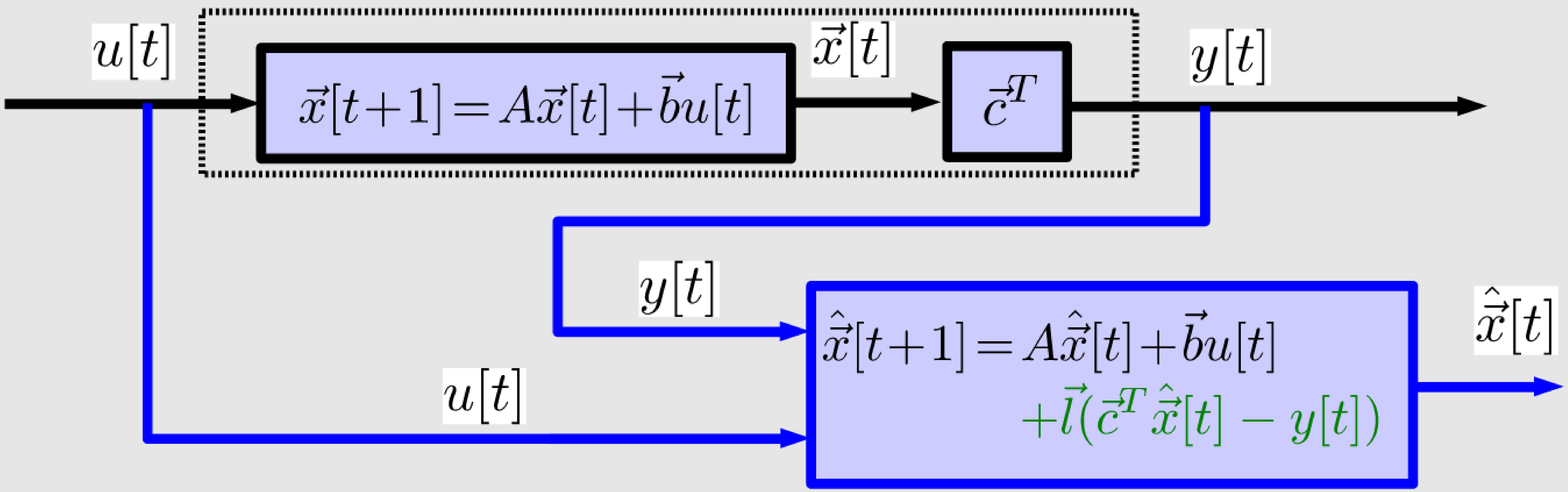

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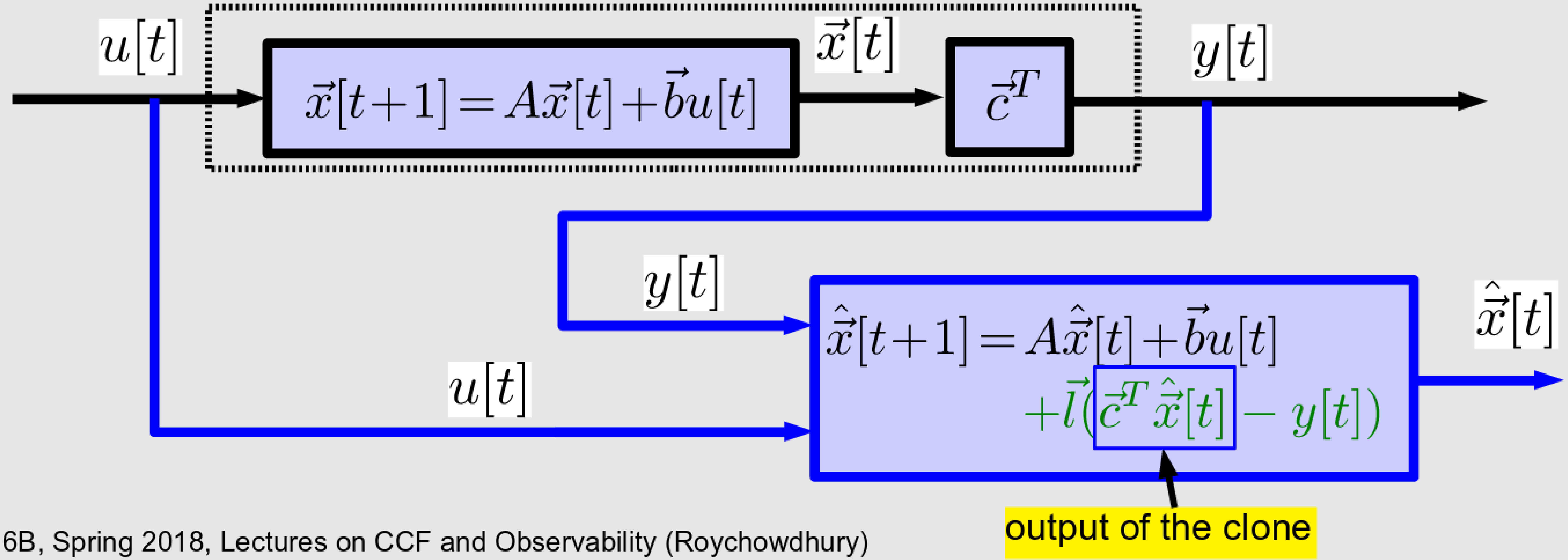

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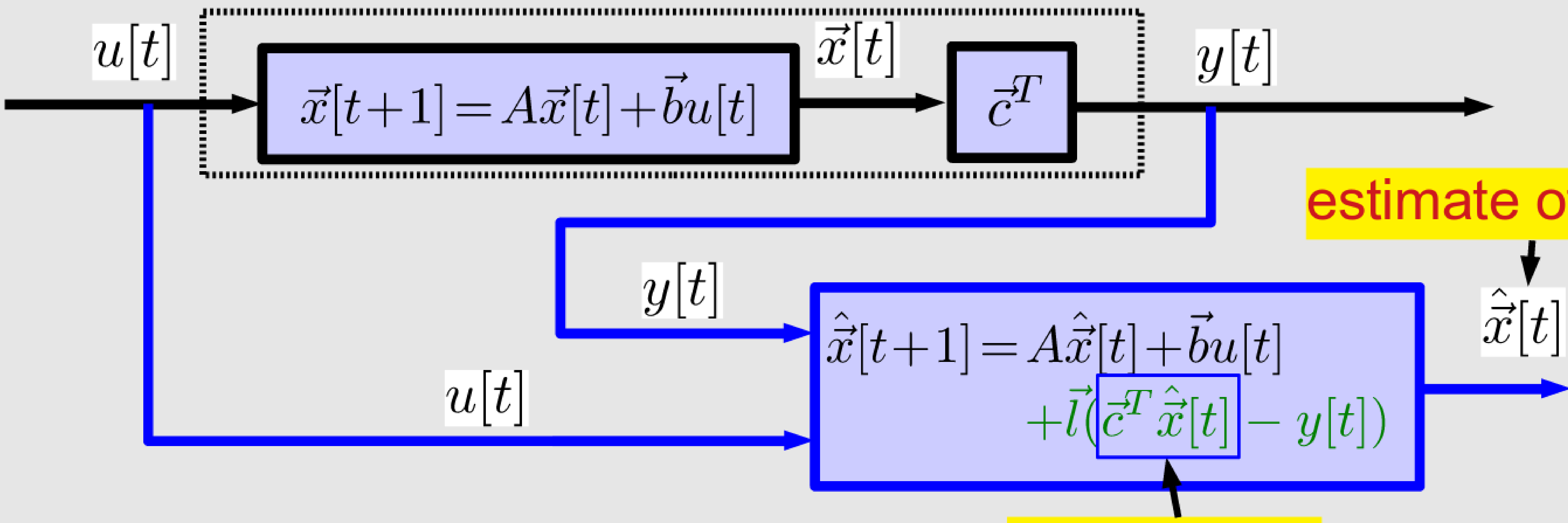

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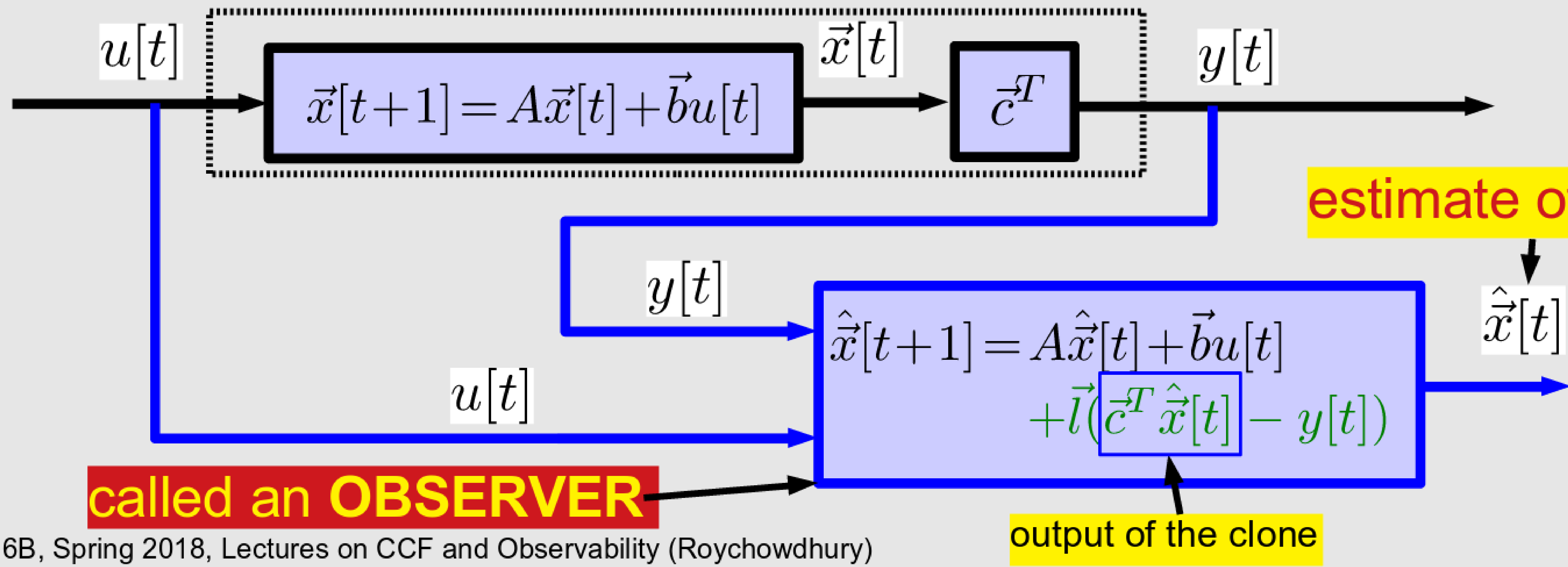

Observers – Why/How They Work

- Observer: $\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + \vec{b}u[t] + \vec{l}(\vec{c}^T\hat{\vec{x}}[t] - y[t])$

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- Observer: $\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + \vec{b}u[t] + \underbrace{\vec{l}}_{\text{error feedback vector - TBD}}(\underbrace{\vec{c}^T \hat{\vec{x}}[t] - y[t]}_{\text{error in predicted output (scalar)}})$

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- Define a state prediction error: $\vec{\epsilon}[t] \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$

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error feedback vector - TBD

error in predicted output (**scalar**)

- Define a state prediction error: $\vec{\epsilon}[t] \triangleq \hat{\vec{x}}[t] - \vec{x}[t]$
  - then we can derive (**move to xournal**):
    - ➔ $\vec{\epsilon}[t+1] = (A + \vec{l}\,\vec{c}^T)\vec{\epsilon}[t]$

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    - **choose $\vec{l}$ to make the eigenvalues of $A + \vec{l}\,\vec{c}^T$ stable!**

Observers – Why/How They Work

- Observer: $\hat{\vec{x}}[t+1] = A\hat{\vec{x}}[t] + \vec{b}u[t] + \vec{l}\,(\underbrace{\vec{c}^T\,\hat{\vec{x}}[t] - y[t]}_{\text{error in predicted output (scalar)}})$

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    - evs of $A + \vec{l}\,\vec{c}^T$ = evs of $A^T + \vec{c}\,\vec{l}^T \rightarrow$ $-\vec{c} \mapsto \vec{b}, \quad \vec{l}^T \mapsto \vec{k}^T$

Observers – Why/How They Work

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  error feedback vector - TBD

  error in predicted output (**scalar**)

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    - evs of $A + \vec{l}\,\vec{c}^T$ = evs of $A^T + \vec{c}\,\vec{l}^T \to$ $-\vec{c} \mapsto \vec{b}, \quad \vec{l}^T \mapsto \vec{k}^T$
- **i.e., can always make $A + \vec{l}\,\vec{c}^T$ stable if $(A^T, -\vec{c})$ is controllable** **(using previous controllability + feedback result)**

Observers – Why/How (contd.)

- $(A^T, -\vec{c})$ controllable $\rightarrow$ $-\left[\vec{c} \,|\, A^T\vec{c} \,|\, \cdots \,|\, (A^T)^{n-2}\vec{c} \,|\, (A^T)^{n-1}\vec{c}\right]$ must be full rank

Observers – Why/How (contd.)

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  - $\rightarrow \left[\vec{c} \,|\, A^T\vec{c} \,|\, \cdots \,|\, (A^T)^{n-2}\vec{c} \,|\, (A^T)^{n-1}\vec{c}\right]^T$ must be full rank

Observers – Why/How (contd.)

- $(A^T, -\vec{c})$ controllable $\rightarrow -\left[\vec{c}\,|\, A^T\vec{c}\,|\, \cdots \,|\, (A^T)^{n-2}\vec{c}\,|\, (A^T)^{n-1}\vec{c}\right]$ must be full rank
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  - $\rightarrow \begin{bmatrix} \longleftarrow & \vec{c}^T & \longrightarrow \\ \longleftarrow & \vec{c}^T A & \longrightarrow \\ \longleftarrow & \vec{c}^T A^2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{c}^T A^{n-1} & \longrightarrow \end{bmatrix}$ must be full rank

Observers – Why/How (contd.)

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just the OBSERVABILITY matrix

Observers – Why/How (contd.)

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just the **OBSERVABILITY matrix**

- **Conclusion: if a system is observable, we can build an observer for it whose estimate $\hat{\vec{x}}[t]$ will approximate $\vec{x}[t]$ more and more closely with t**

Observer: Rotation Matrix Example

- $\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{A} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$, $y[t] = x_1[t] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\vec{c}^T} \vec{x}[t]$

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- example: $\theta = \dfrac{\pi}{2} \longrightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longrightarrow O = \begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

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full rank

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  - side note: eigenvalues of A: $\pm \jmath \longrightarrow$ BIBO unstable

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  - let $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, then $A + \vec{l}\vec{c}^T = \begin{bmatrix} l_1 & -1 \\ 1 + l_2 & 0 \end{bmatrix}$

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    - ➔ eigenvalues (see the notes): $\lambda_{1,2} = \frac{l_1}{2} \pm \frac{l_1^2 - 4(1 + l_2)}{2}$
      - and can easily show: $l_1 = \lambda_1 + \lambda_2, \quad l_2 = \lambda_1 \lambda_2 - 1$

Observer: Rotation Matrix Example

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  - side note: eigenvalues of A: $\pm \jmath \longrightarrow$ BIBO unstable
  - let $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, then $A + \vec{l}\vec{c}^T = \begin{bmatrix} l_1 & -1 \\ 1 + l_2 & 0 \end{bmatrix}$
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  - warning: if complex, **ensure evs are complex conjugates**
    - **what will happen if you don't?**

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cannot be changed/stabilized using $\vec{l}$

Observability: The Continuous Case

- Observability for C.T. state-space systems
  - and implications for placing observer eigenvalues
- **EXACTLY THE SAME CRITERIA**
  - $$\begin{bmatrix} \longleftarrow & \vec{c}^T & \longrightarrow \\ \longleftarrow & \vec{c}^T A & \longrightarrow \\ \longleftarrow & \vec{c}^T A^2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{c}^T A^{n-1} & \longrightarrow \end{bmatrix}$$ **must be full rank**
- Stability for C.T. means Re(eigenvalues) < 0

Observers: Accurate Positioning

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  - due to the relationship between position, velocity and acceleration
    - ➔ $\dot{x} = v, \quad \dot{v} = a$

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  - **NOT A VERY PRACTICALLY USEFUL WAY TO LOCATE YOURSELF**

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  - **this is the famous KALMAN FILTER**
    - **used in all rockets, drones, autonomous cars, ships, ...**

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- stability, controllability, observability and implications
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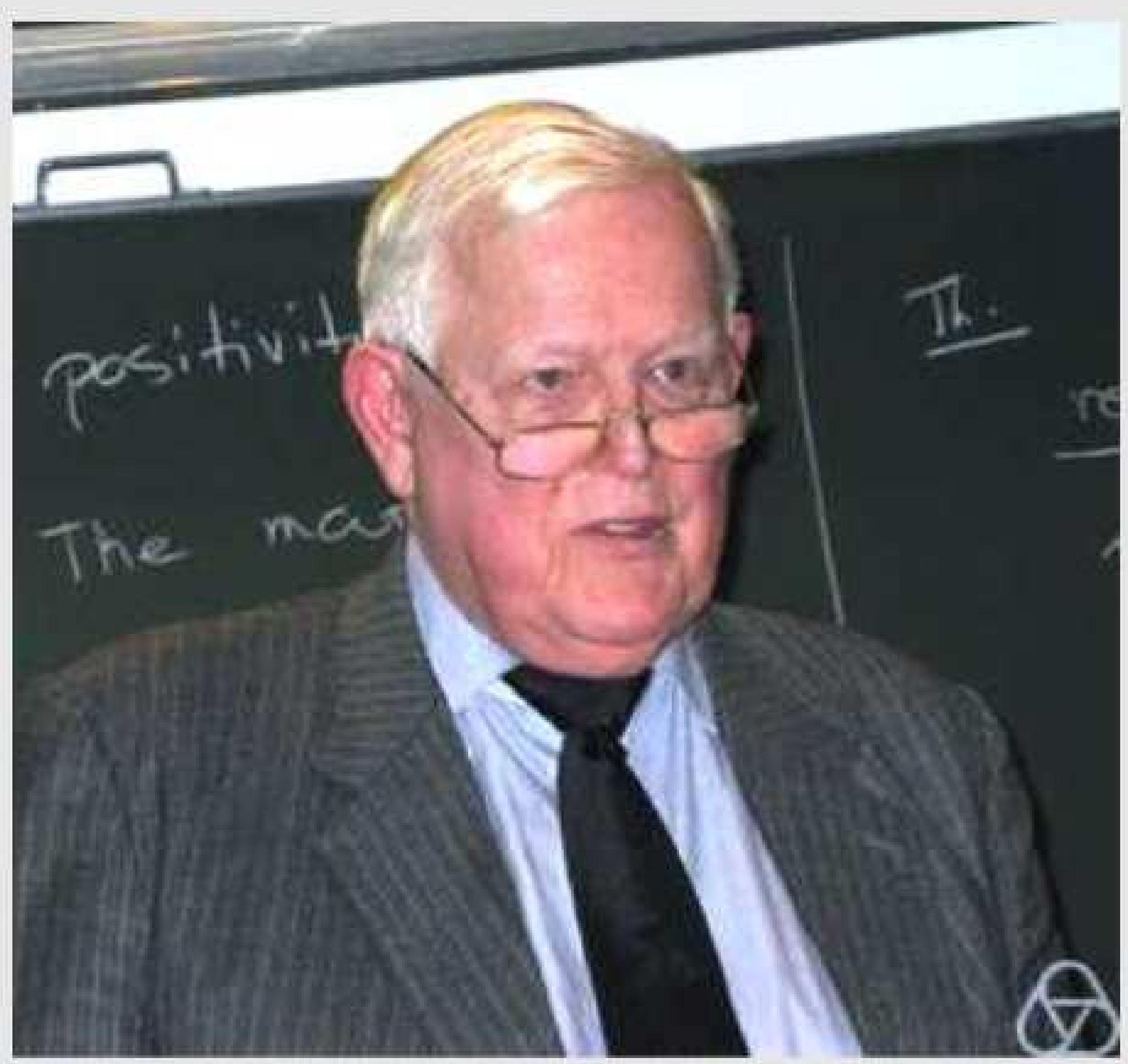


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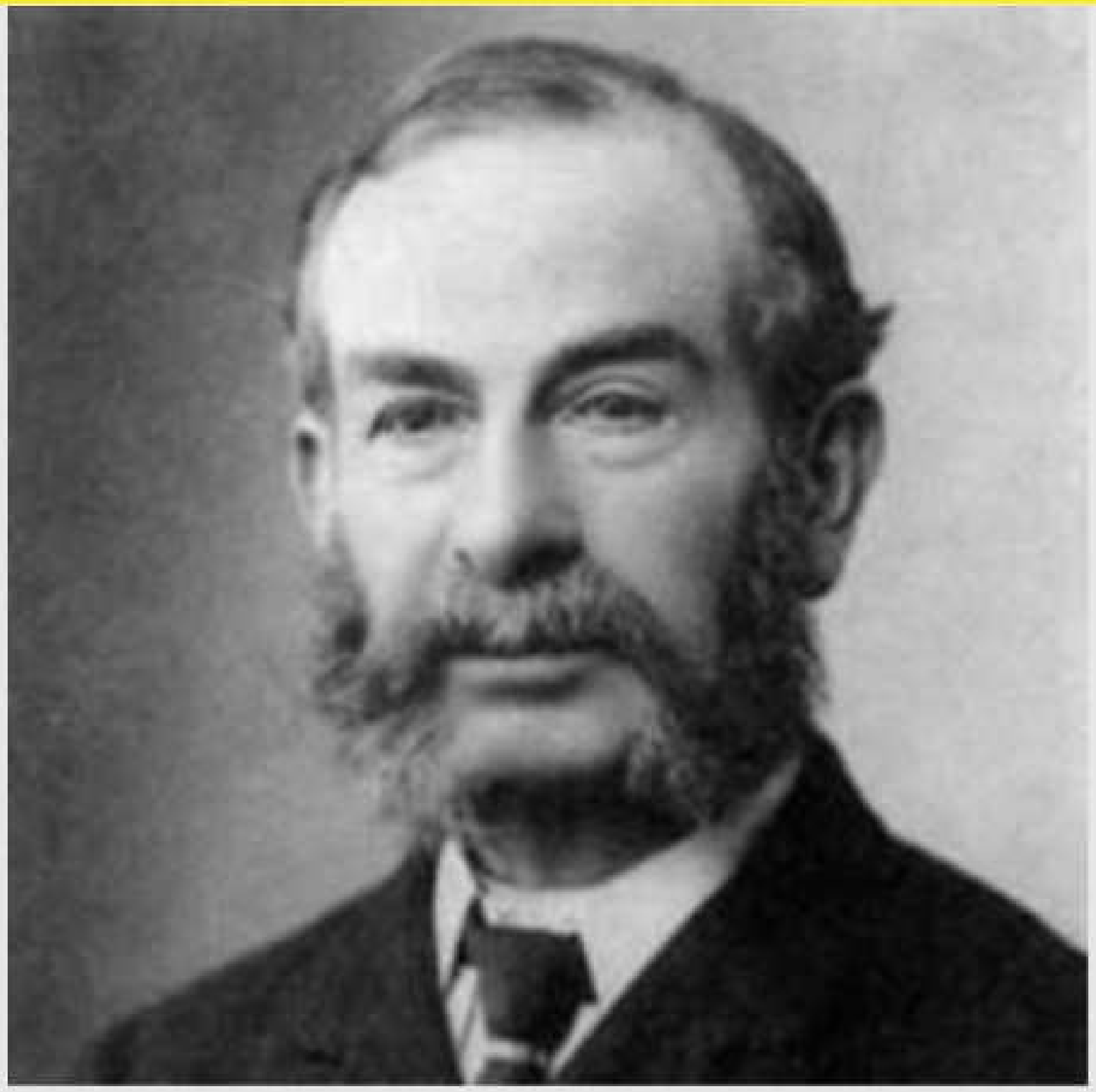

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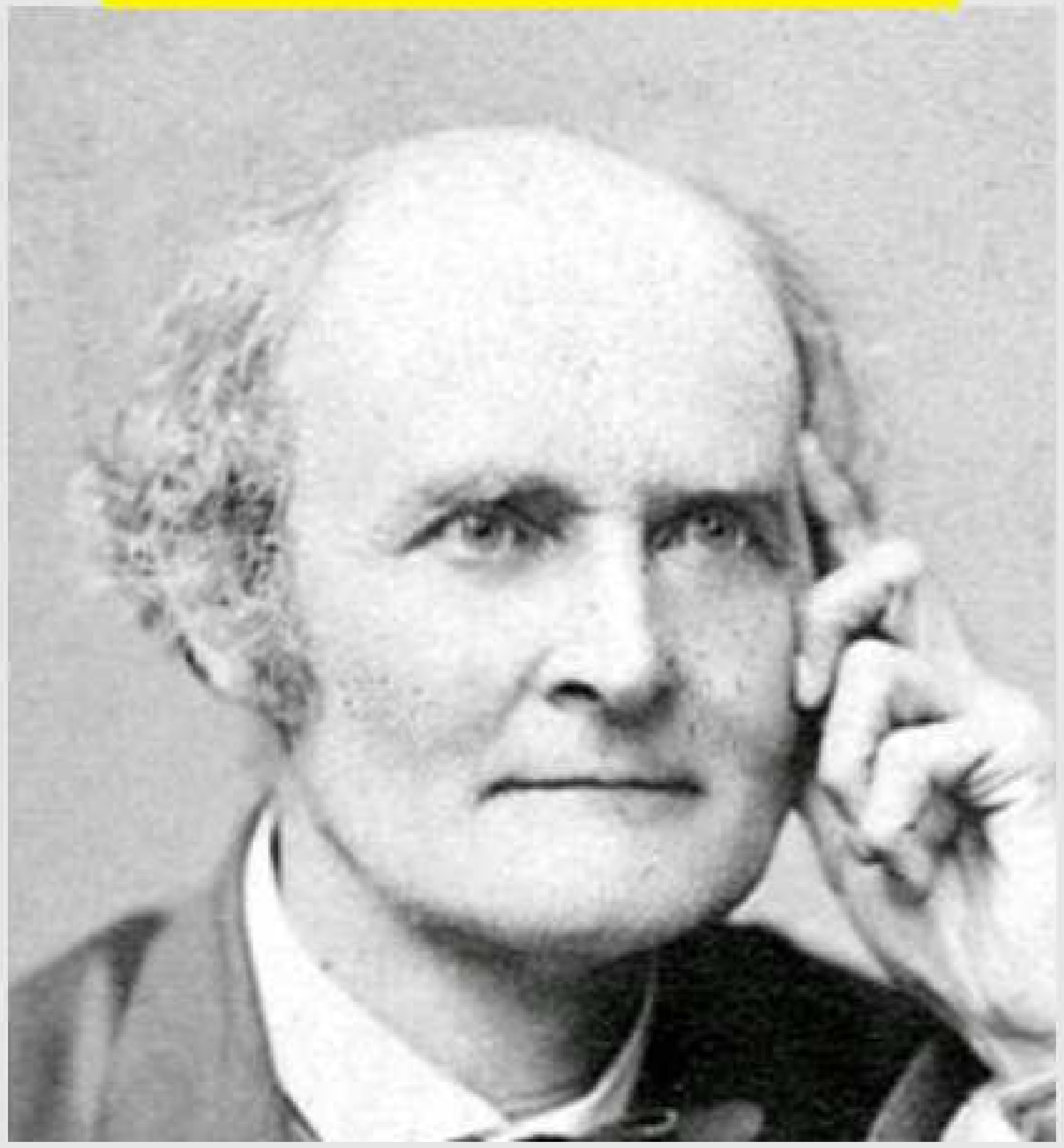

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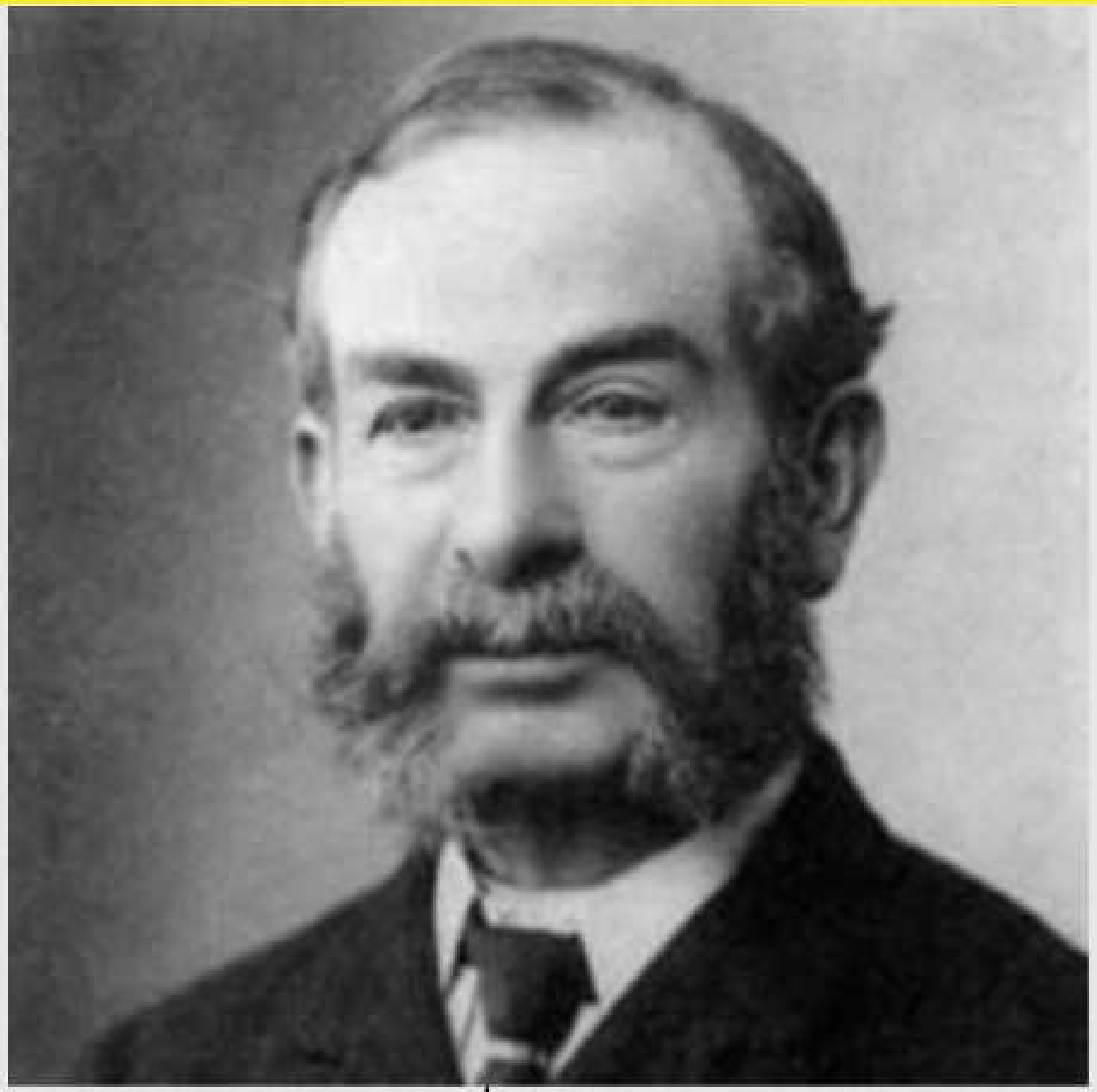

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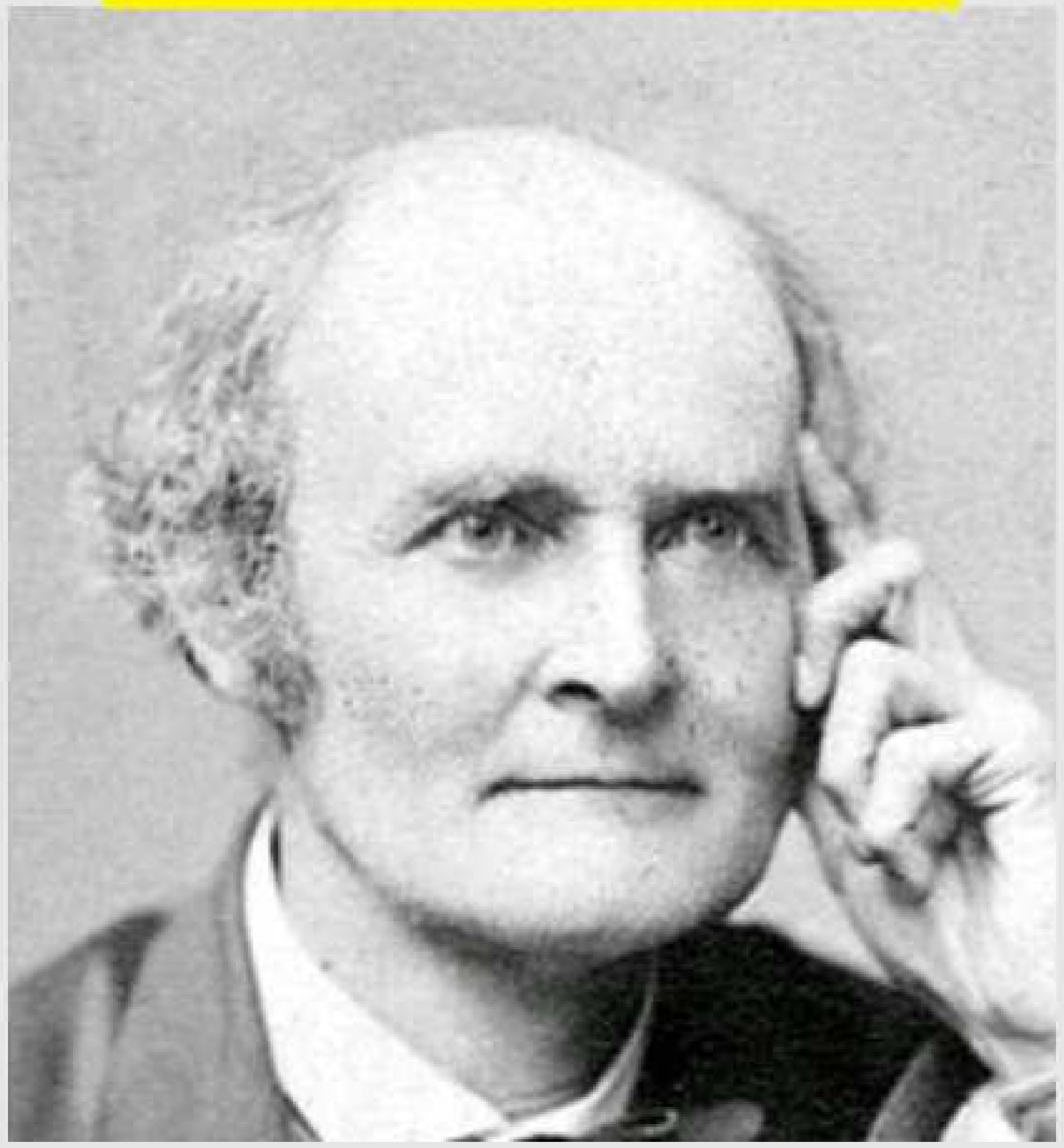

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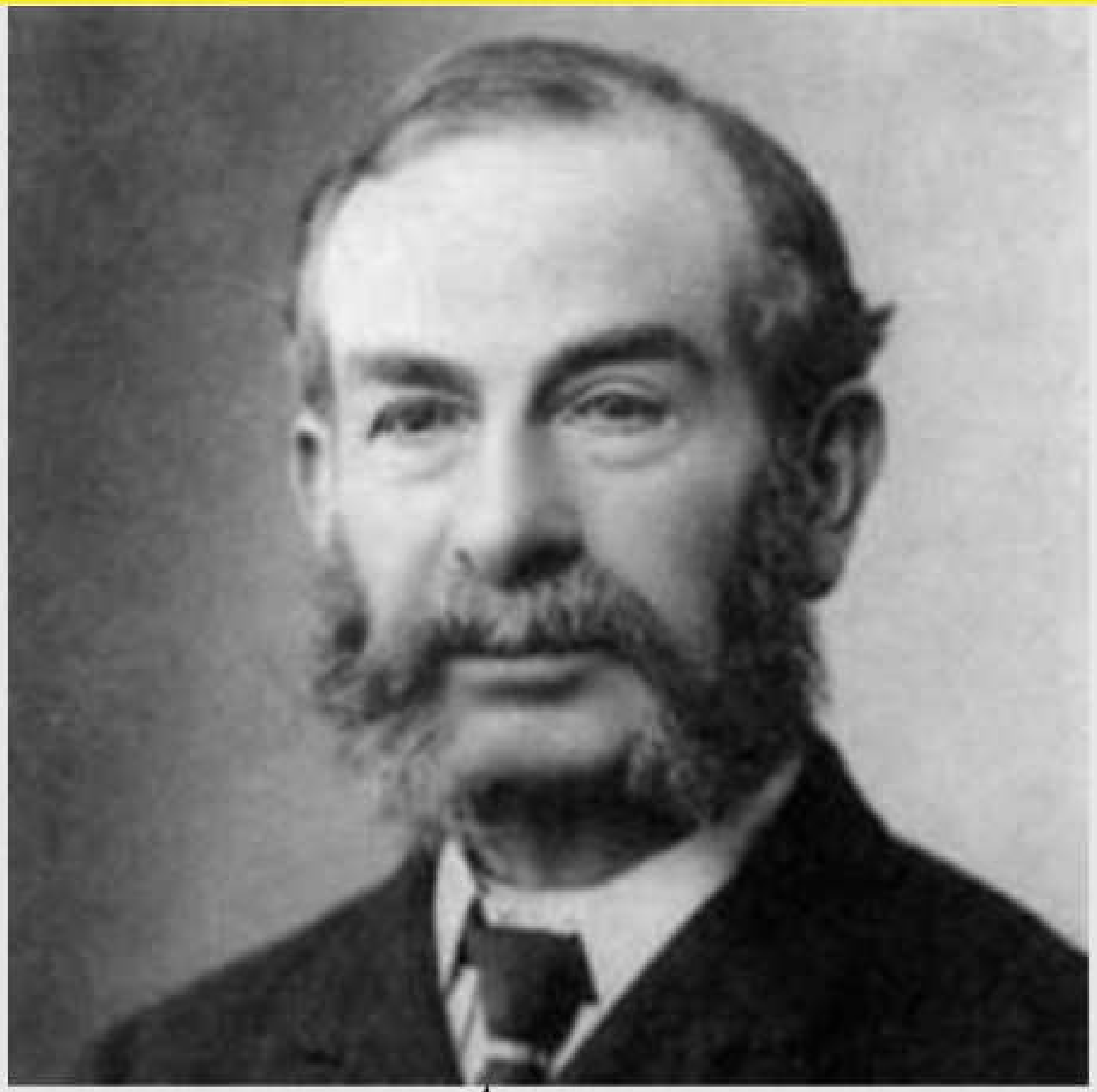

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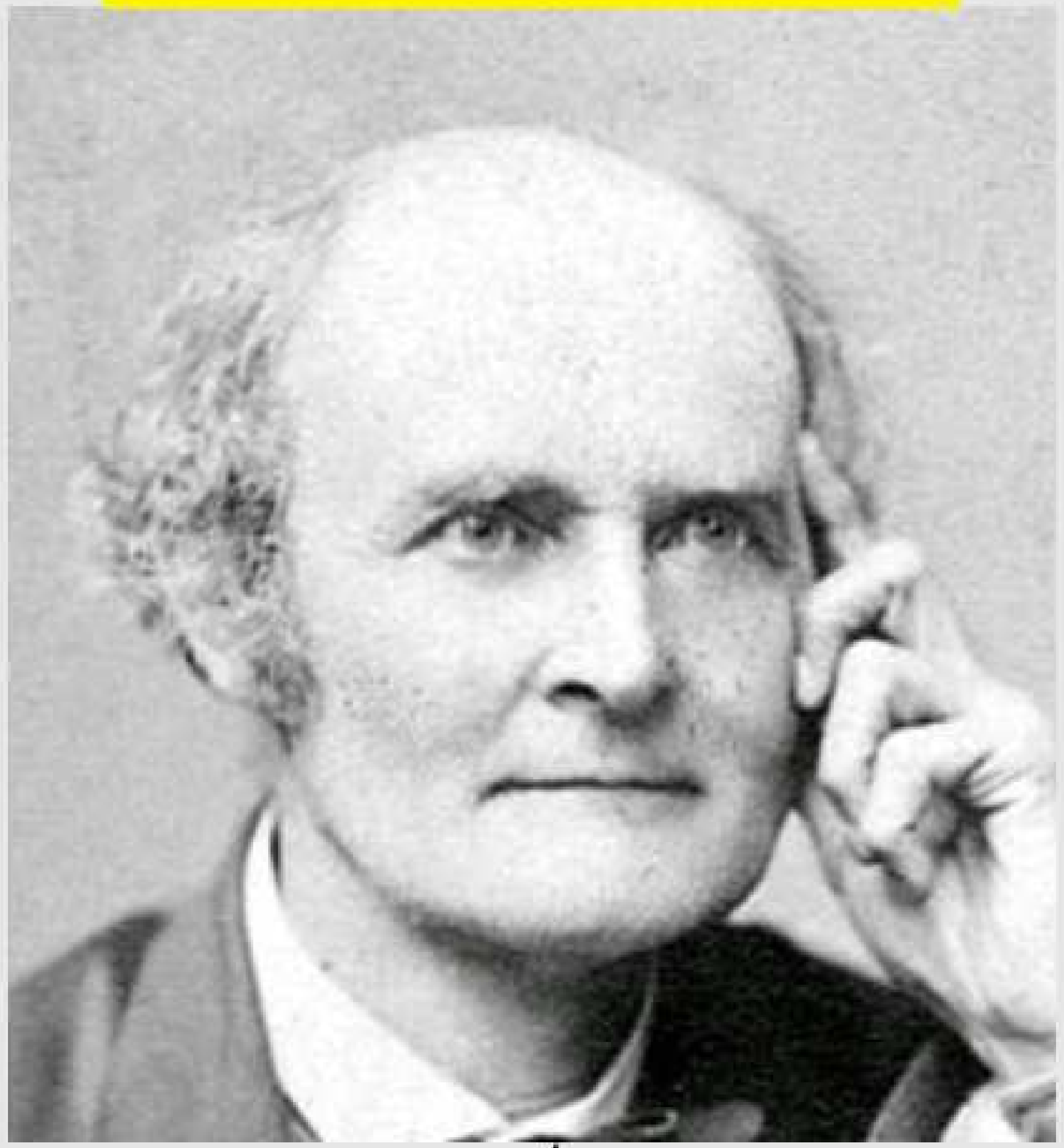

Cayley-Hamilton Theorem

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- 1683: **Seki** ("Japan's Newton") used matrices
- developed in Europe by **Gauss** and many others
  - finally, into its modern form by **Cayley** (mid 1800s)

Cardano

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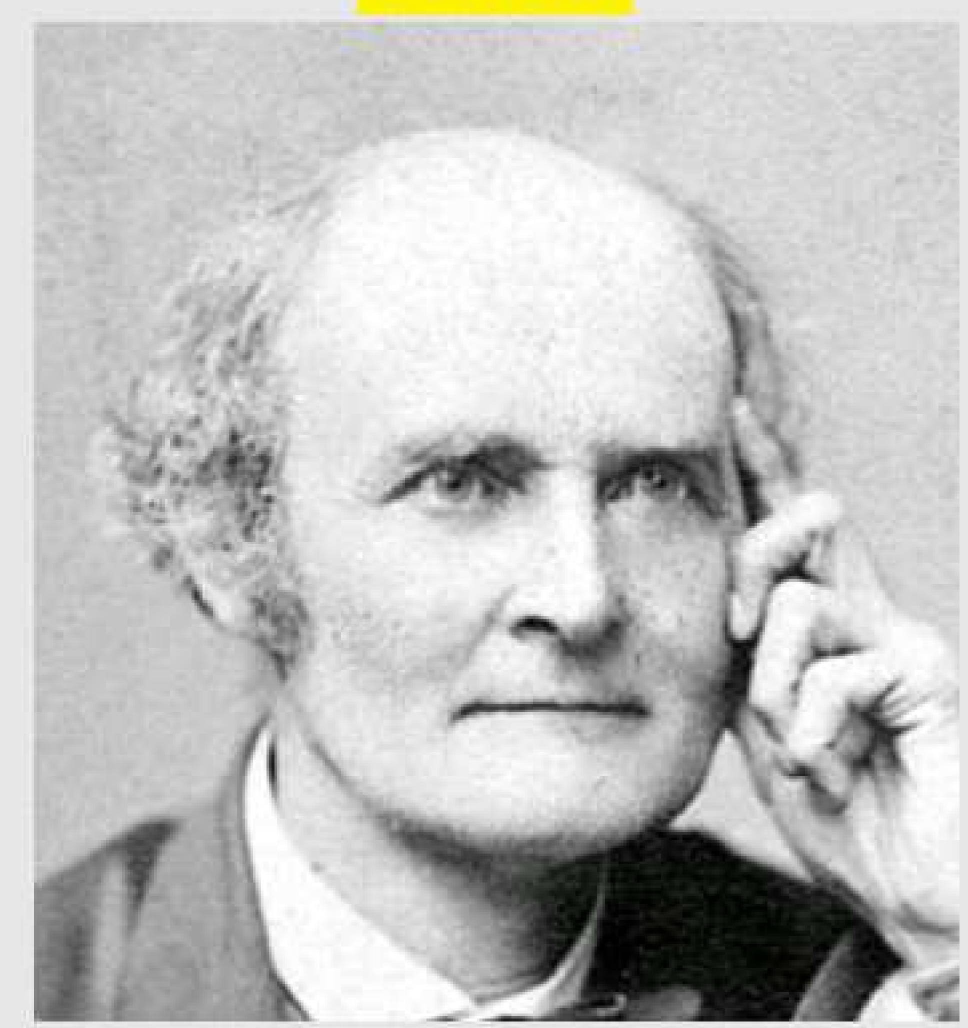

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