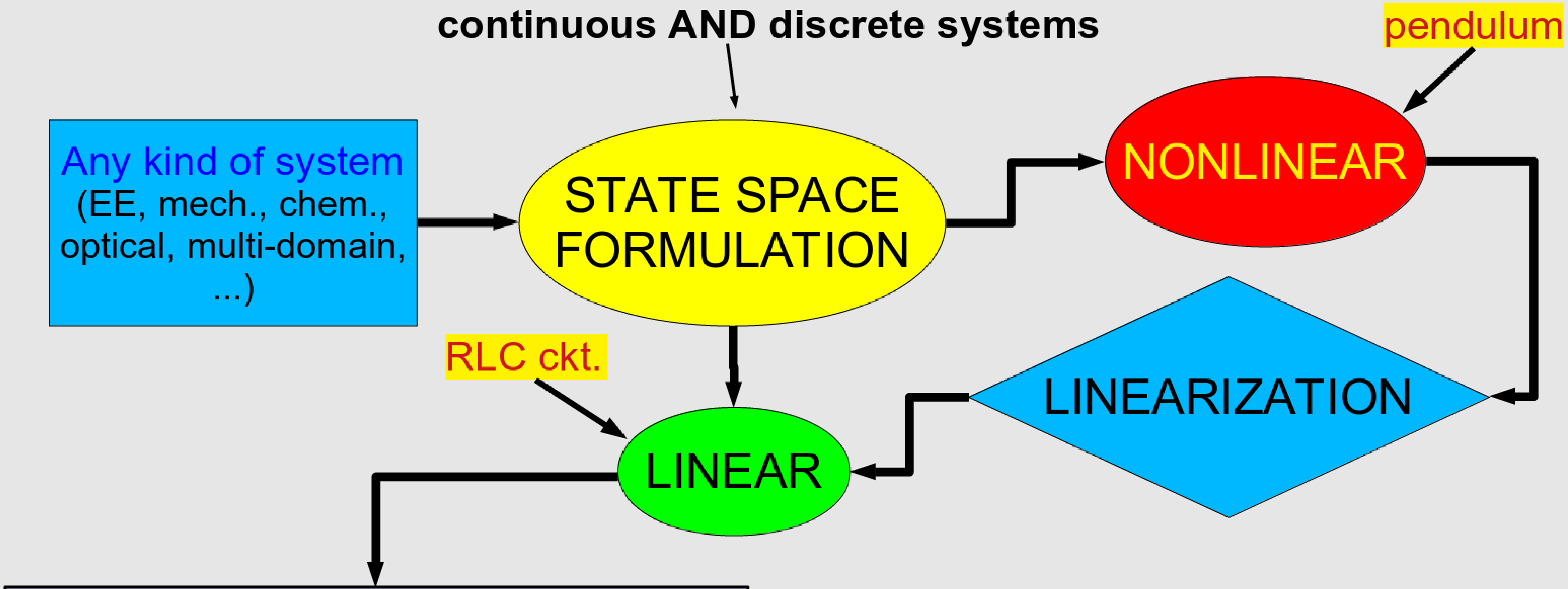
EE16B, Spring 2018
UC Berkeley EECS

Maharbiz and Roychowdhury

Lectures 6A & 6B: Overview Slides

Controllability and Feedback

Where We Were Before



continuous AND discrete systems

pendulum

Any kind of system (EE, mech., chem., optical, multi-domain, ...)

STATE SPACE FORMULATION

NONLINEAR

LINEARIZATION

RLC ckt.

LINEAR

Stability

Bounded Input Bounded Output

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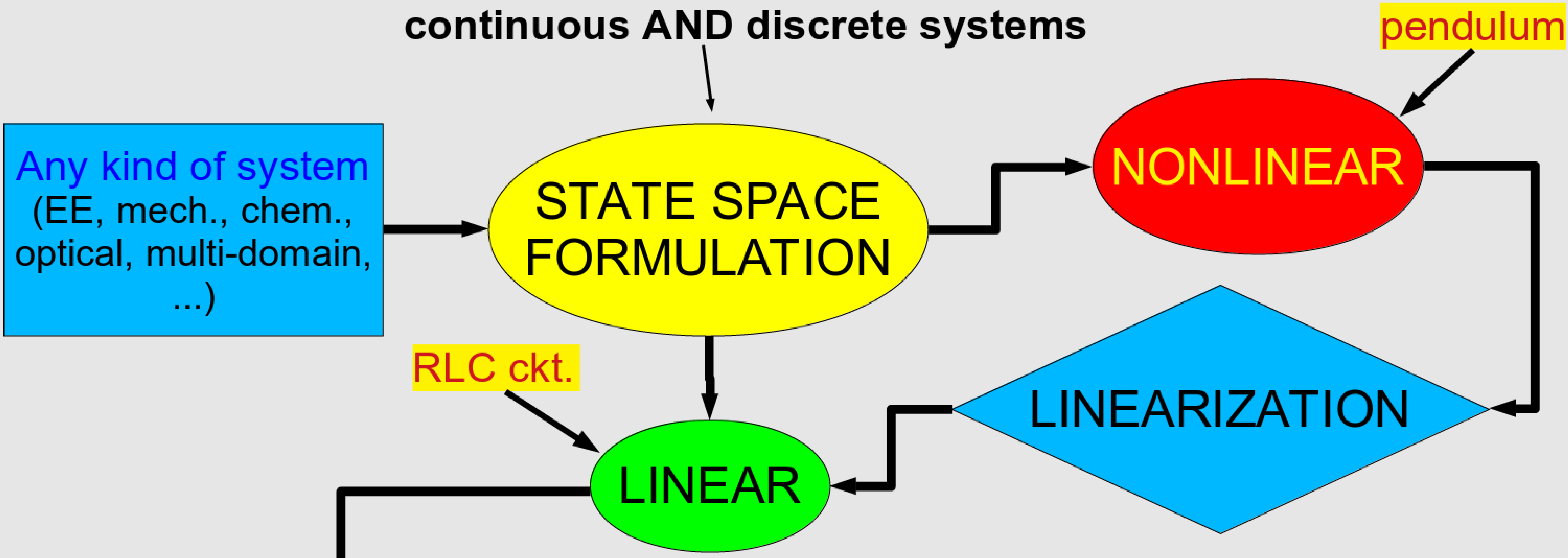

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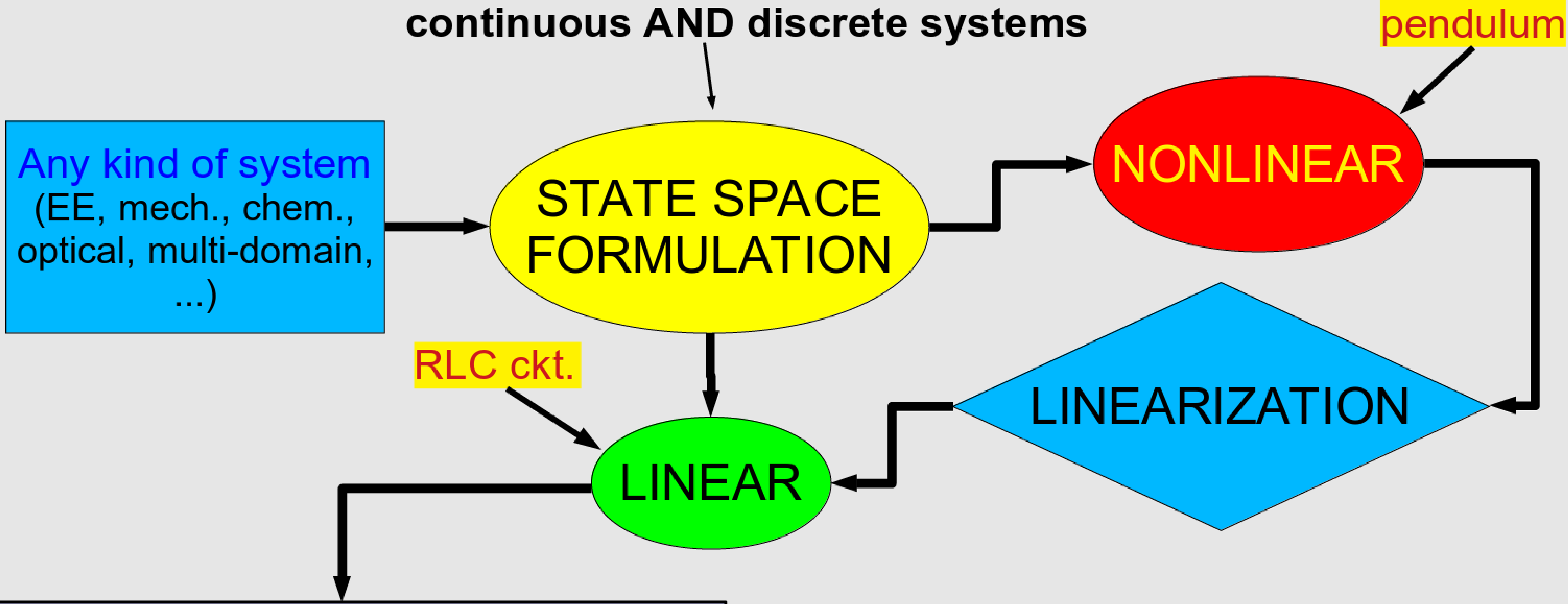

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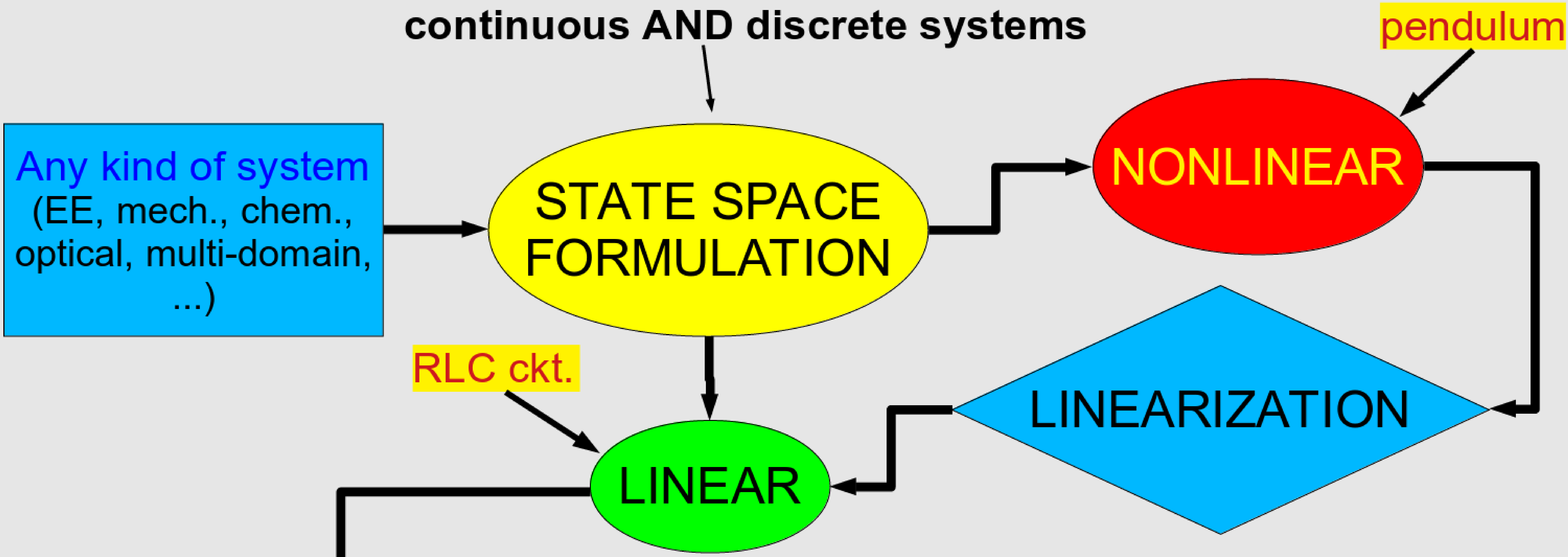
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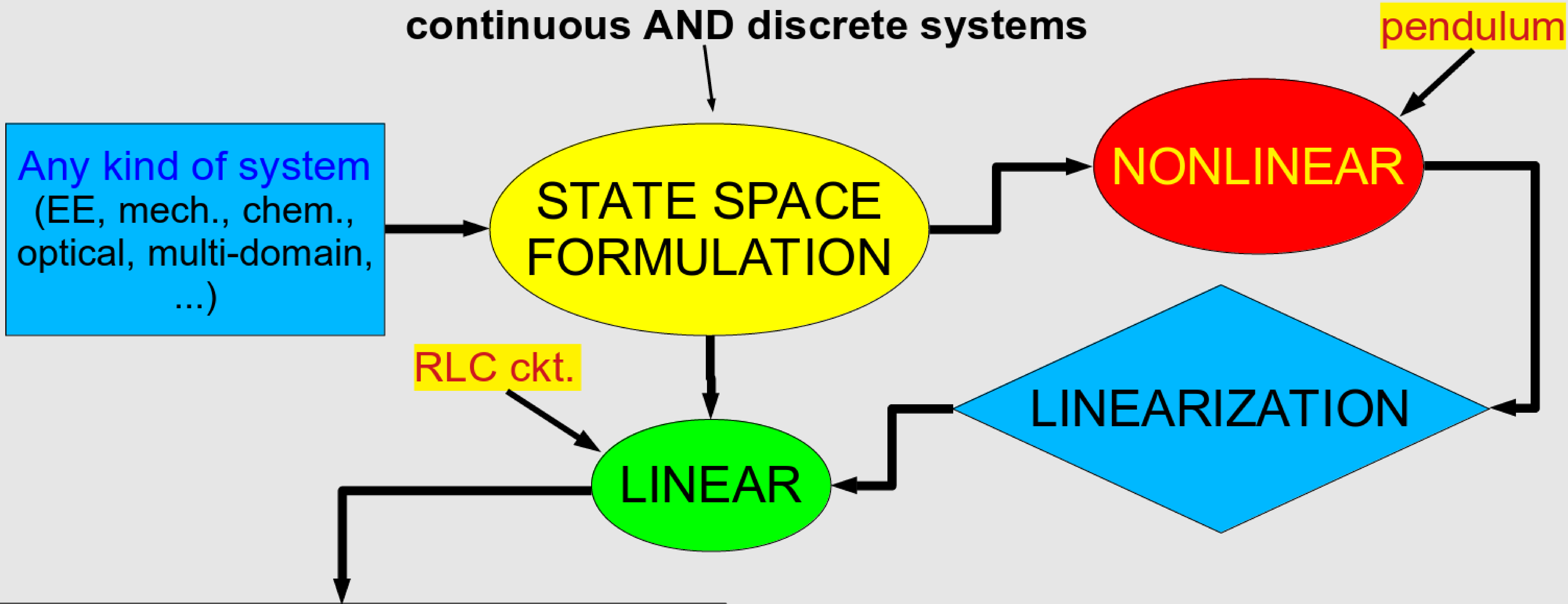

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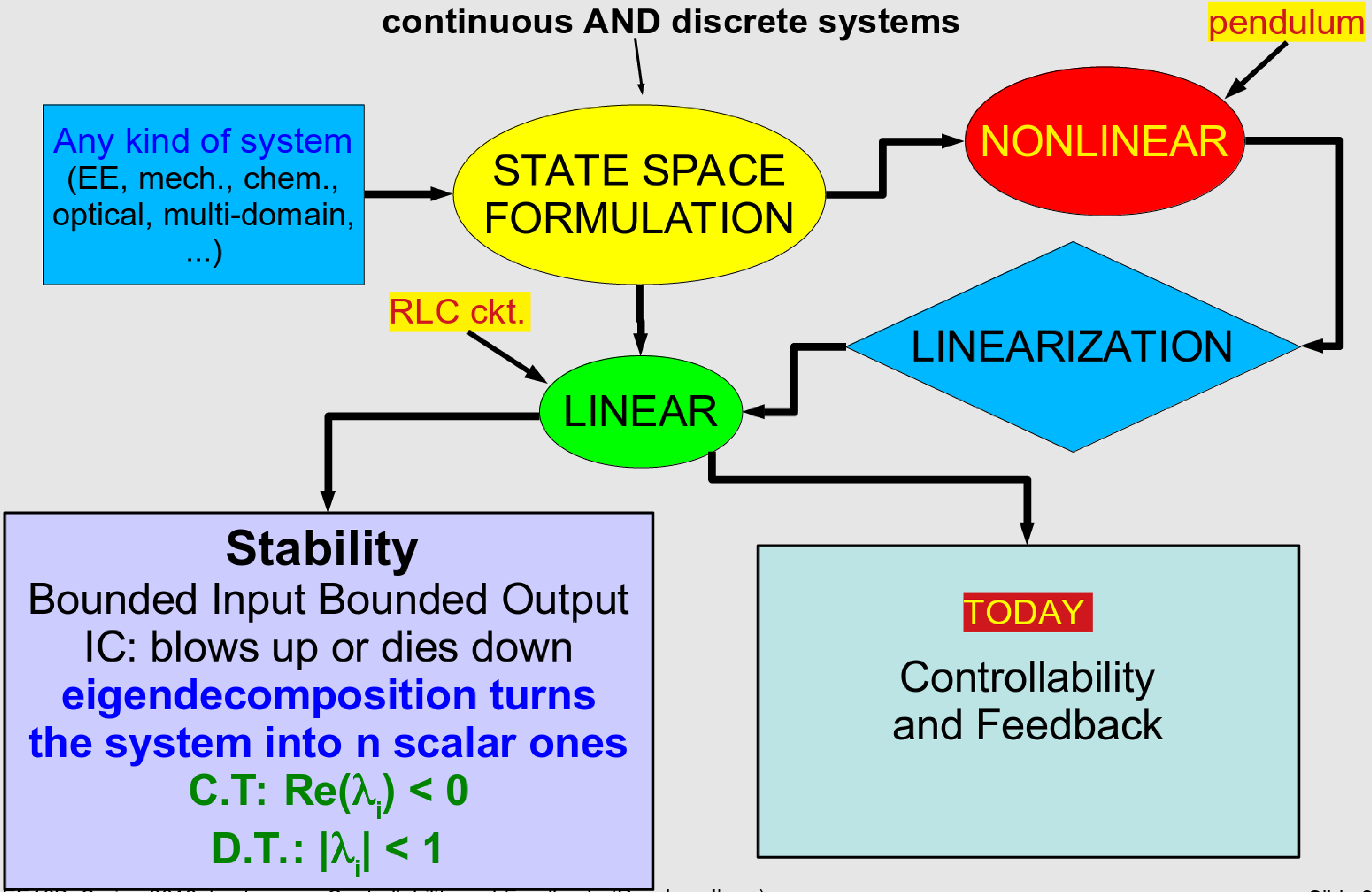

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TODAY

Controllability and Feedback

Controllability

- Given (linearized) S.S.R: $\Delta\vec{x}[t+1] = \overset{\text{nxn matrix}}{A}\Delta\vec{x}[t] + \overset{\text{nx1 vector}}{\vec{b}}\Delta u[t]$
  - can you drive $\Delta\vec{x}[t]$ to any value you want (using $\Delta\vec{u}[t]$)?
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nxn matrix nx1 vector

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linear comb. of [b, Ab, A^2b, …, A^{k-1}b]

    - ie, A^kb, A^{k+1}b, … will not contribute new linearly indep. columns

Cayley-Hamilton Theorem

- Every matrix A satisfies its own characteristic polynomial!
  - char. poly.: $p_A(\lambda) \triangleq det(A - \lambda I)$

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

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→ $A^n\vec{b} = \underbrace{-a_{n-1}A^{n-1}\vec{b} - a_{n-2}A^{n-2}\vec{b} - \cdots - a_1 A\vec{b} - a_0\vec{b}}_{\text{linear comb. of } [b,\ Ab,\ A^2b,\ \ldots,\ A^{n-1}b]}$

- ie, A^n b, A^{n+1} b, … will not contribute new linearly indep. columns

Cayley-Hamilton Theorem

- Every matrix A satisfies its own characteristic polynomial!
  - char. poly.: $p_A(\lambda) \triangleq det(A - \lambda I)$

scalar

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

matrix

  - C-H Thm: $p_A(A) = 0$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 = 0$$

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$$A^n\vec{b} = \underbrace{-a_{n-1}A^{n-1}\vec{b} - a_{n-2}A^{n-2}\vec{b} - \cdots - a_1 A\vec{b} - a_0\vec{b}}_{\text{linear comb. of } [b,\ Ab,\ A^2b,\ \ldots,\ A^{n-1}b]}$$

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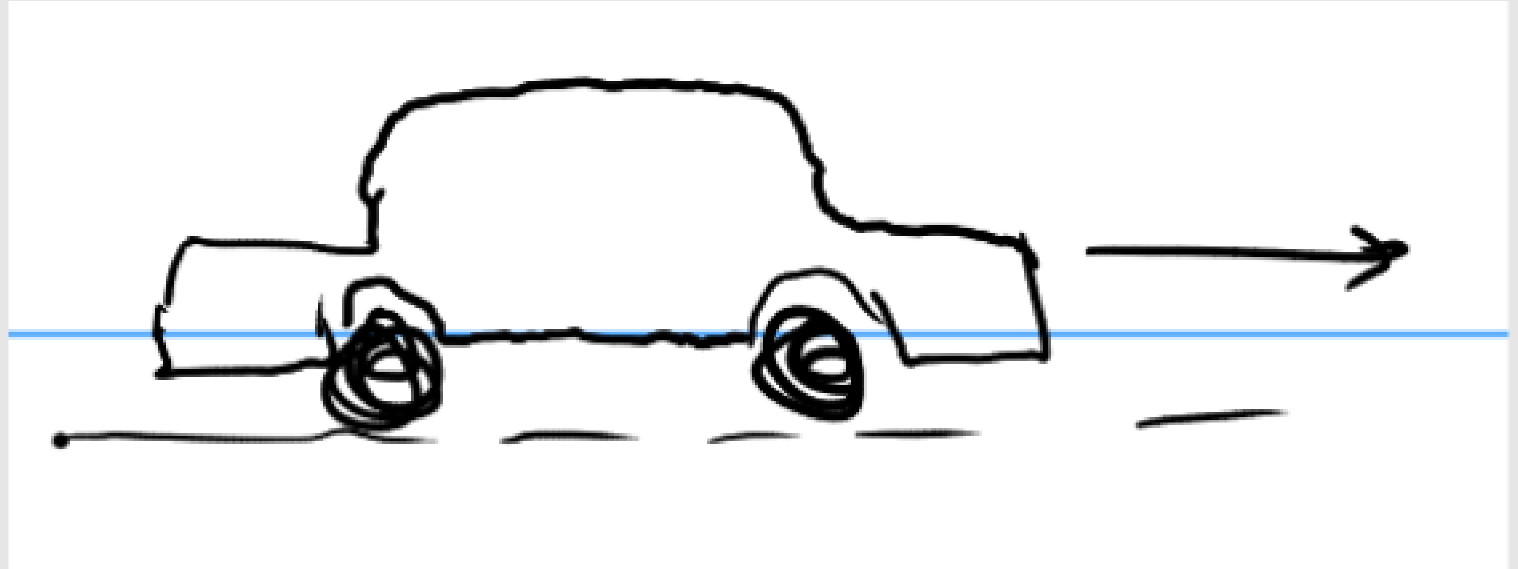
if no eigenvalues repeated, then n is the degree of the minimal polynomial (ie, k=n)

Example: Accelerating Car

- control input: **acceleration**



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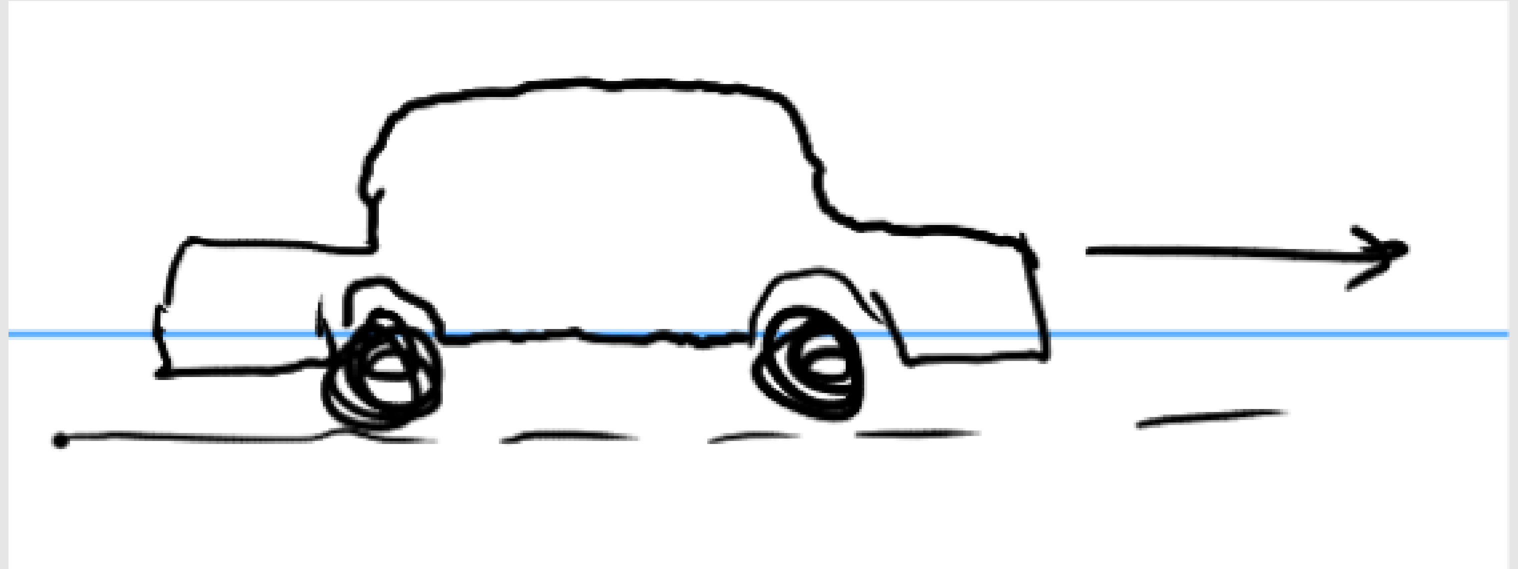

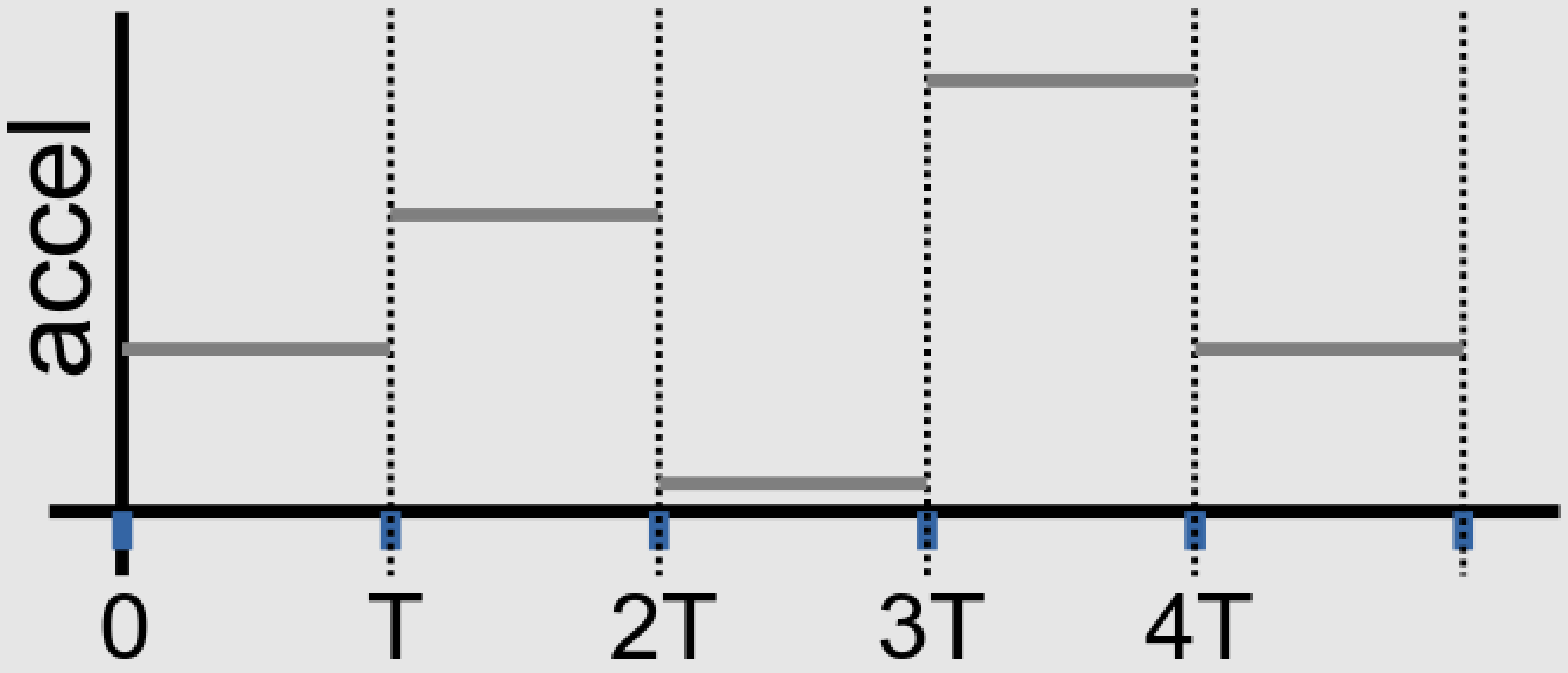

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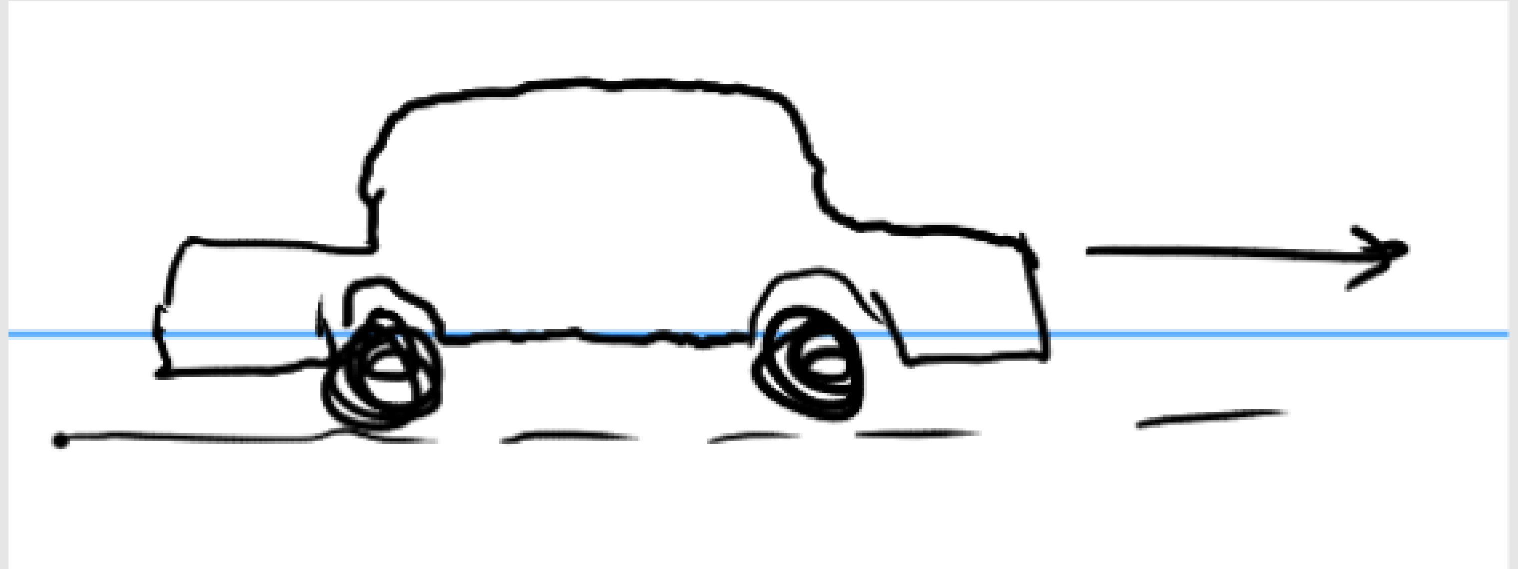

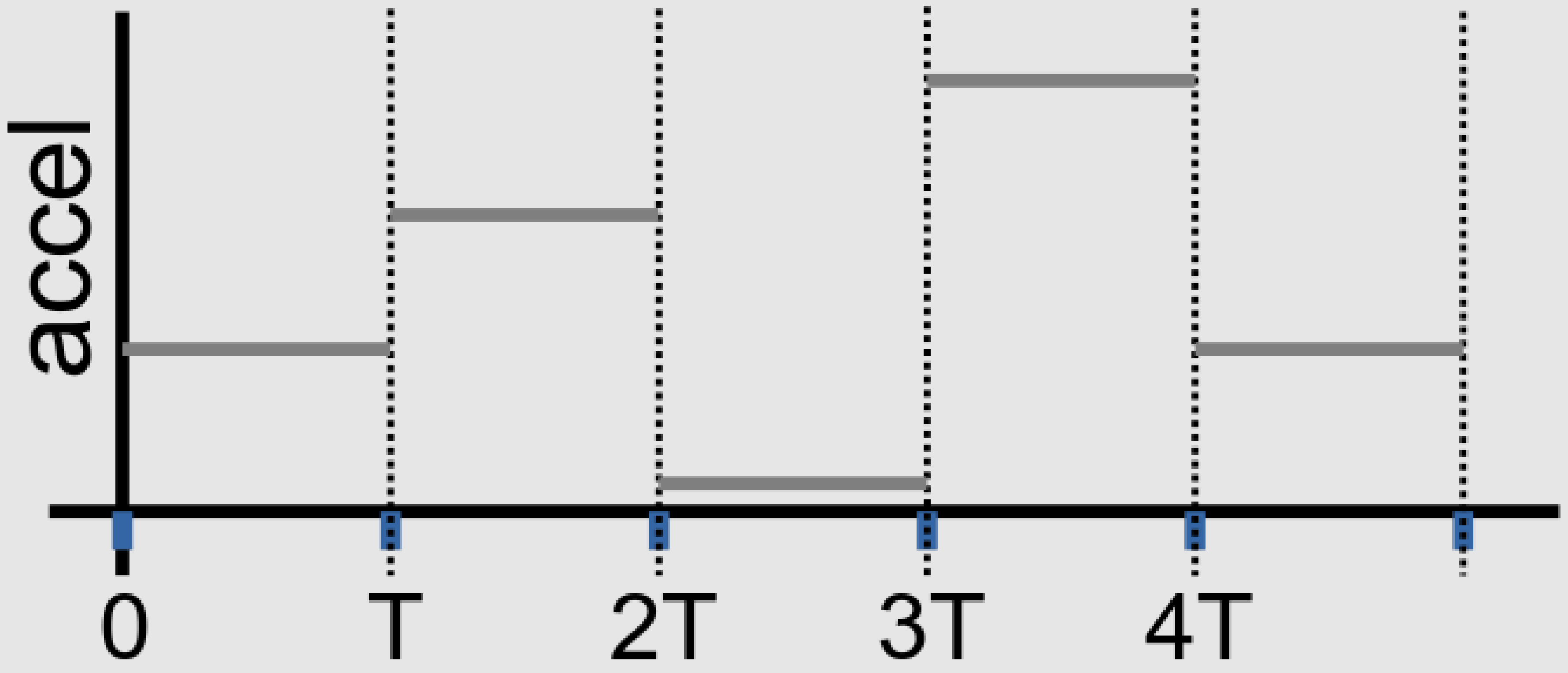

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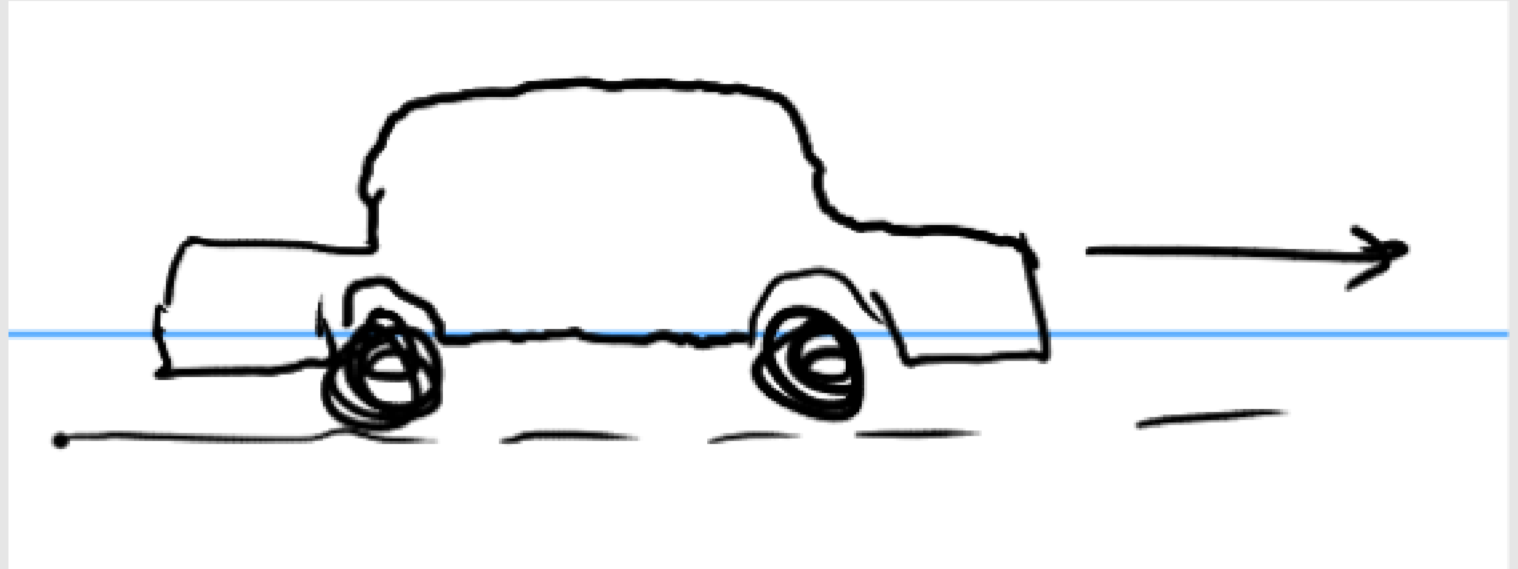

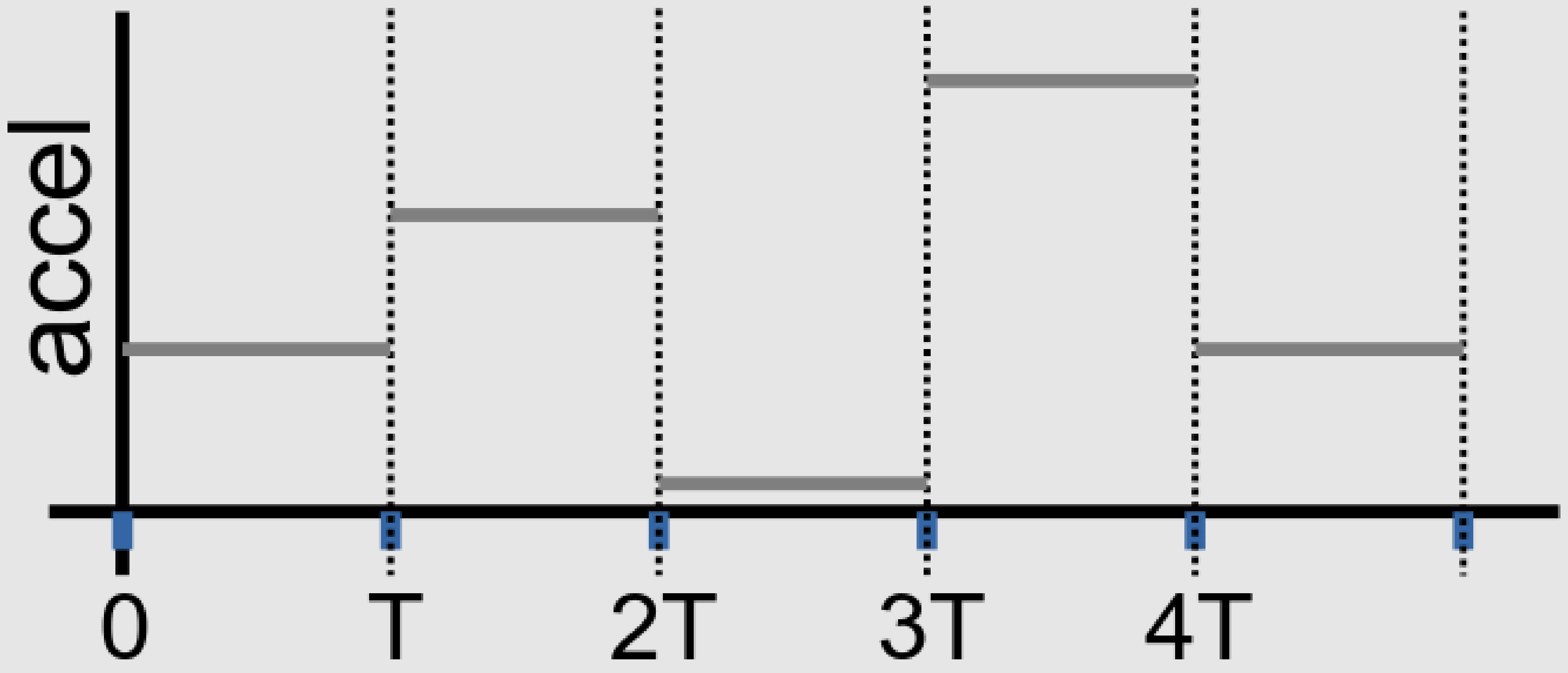

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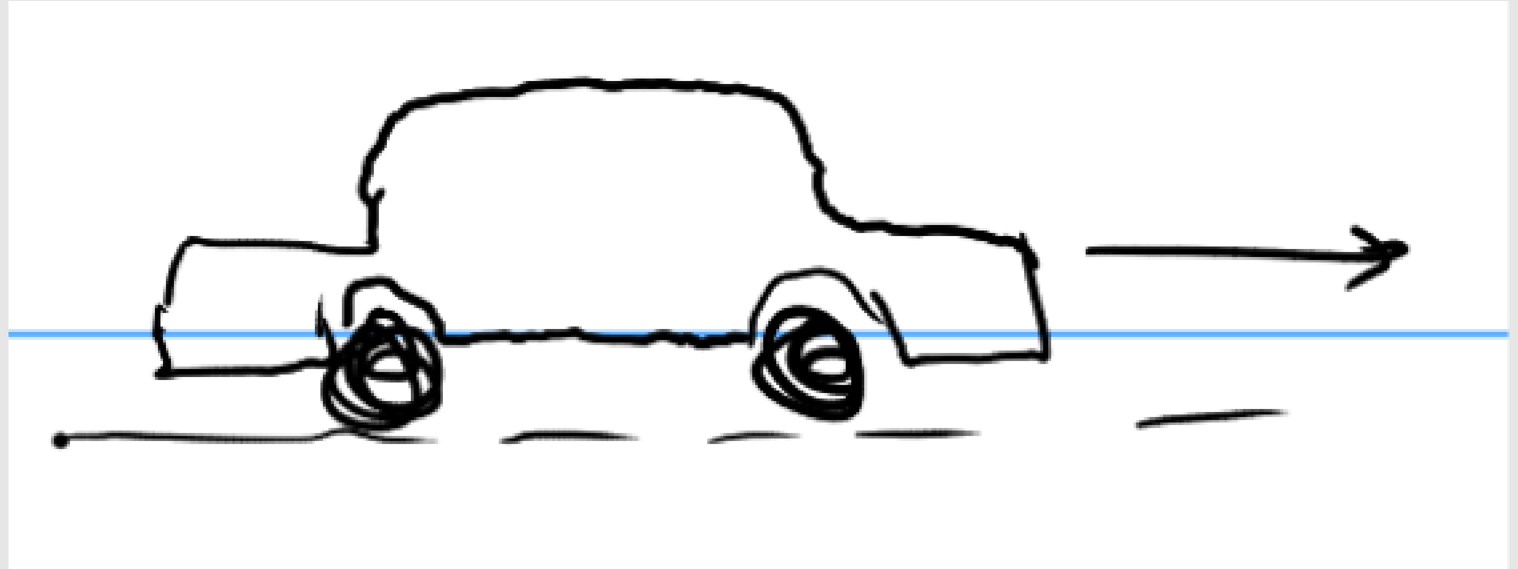

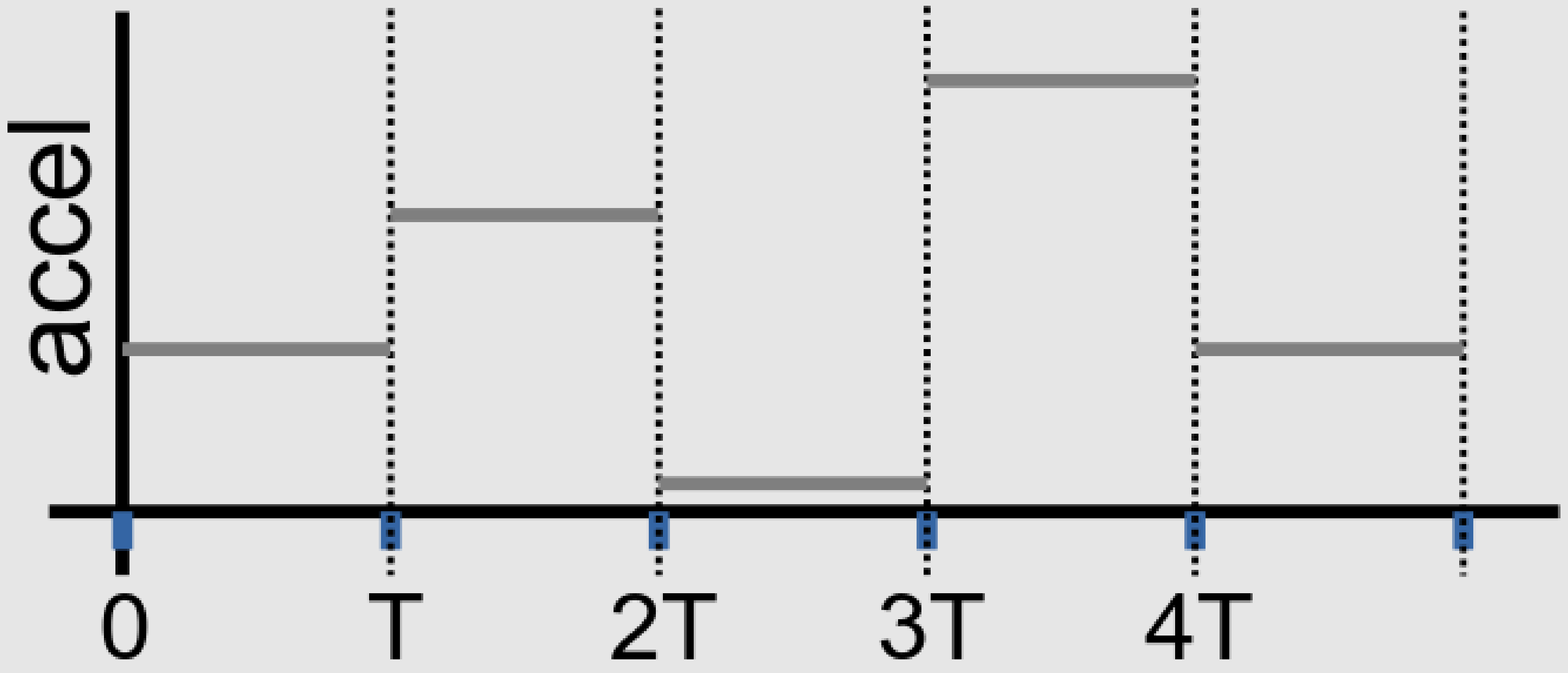

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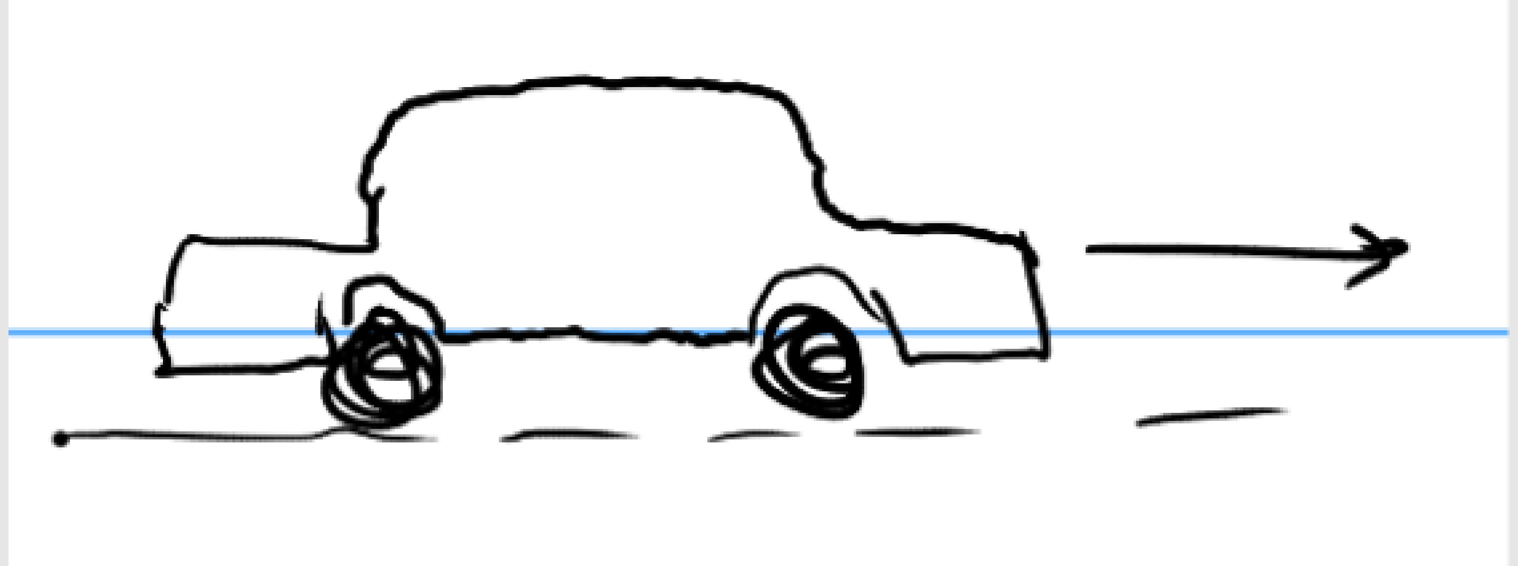

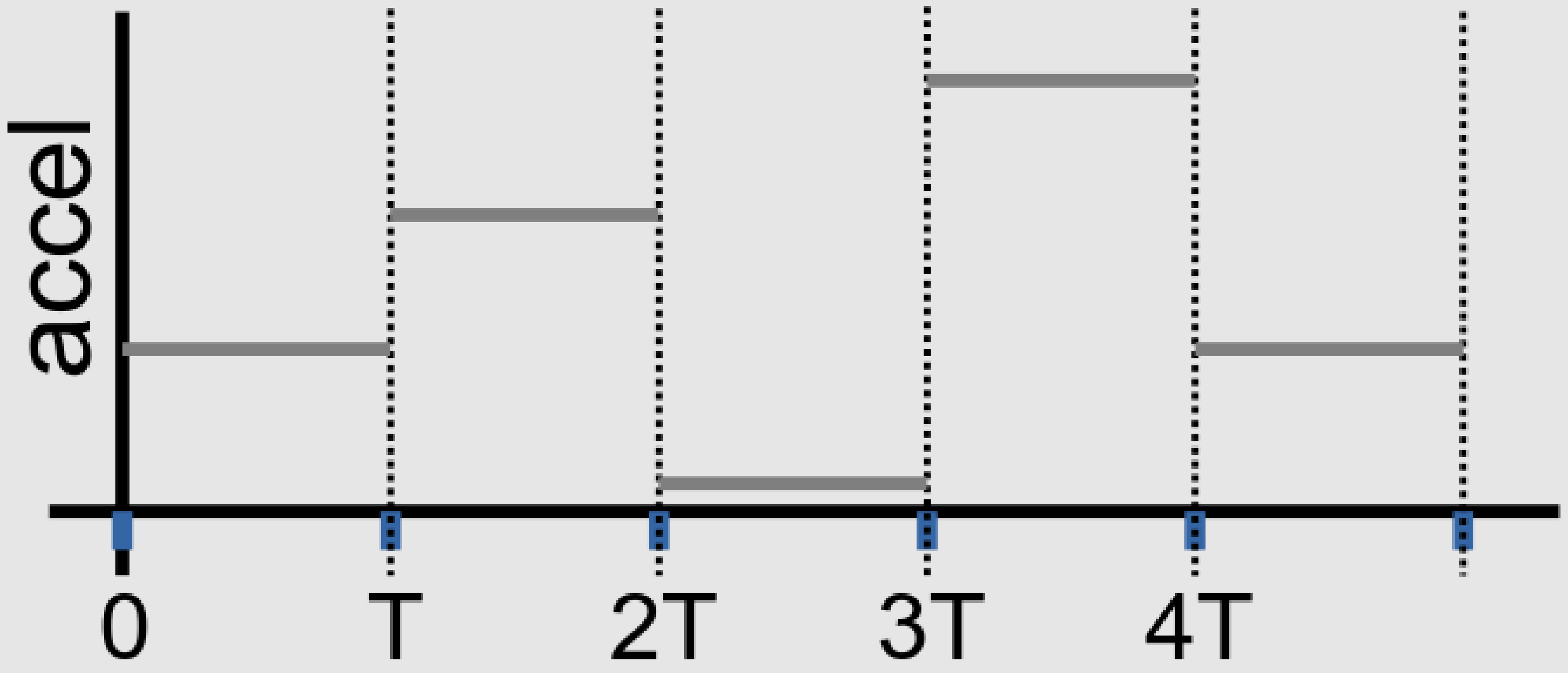

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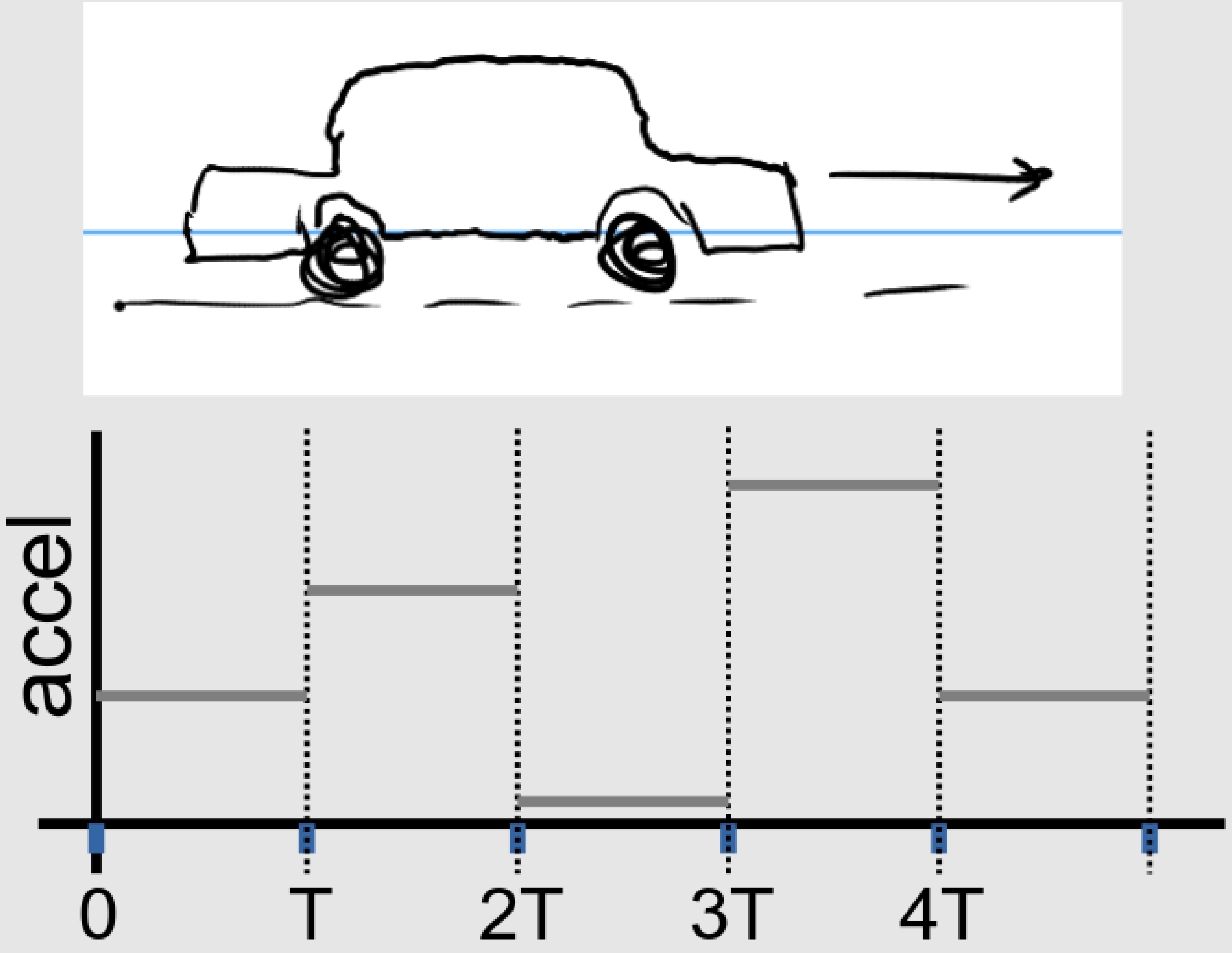

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$tT \leq \tau \leq (t+1)T$

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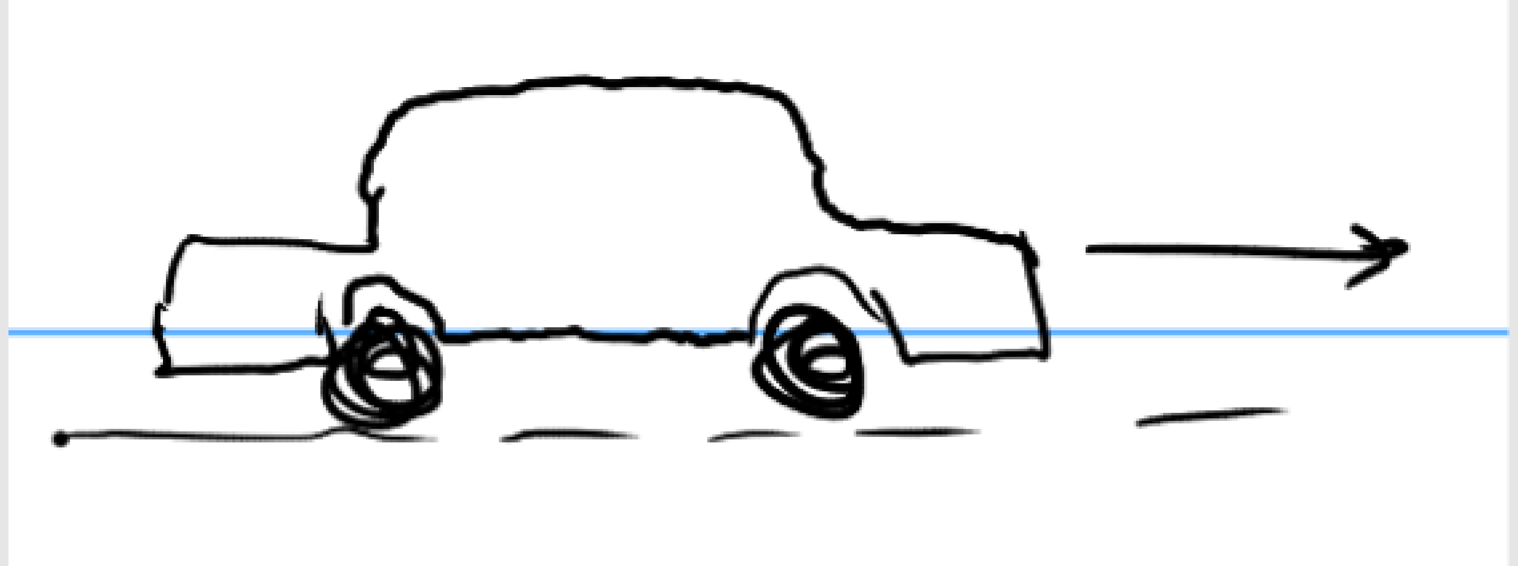

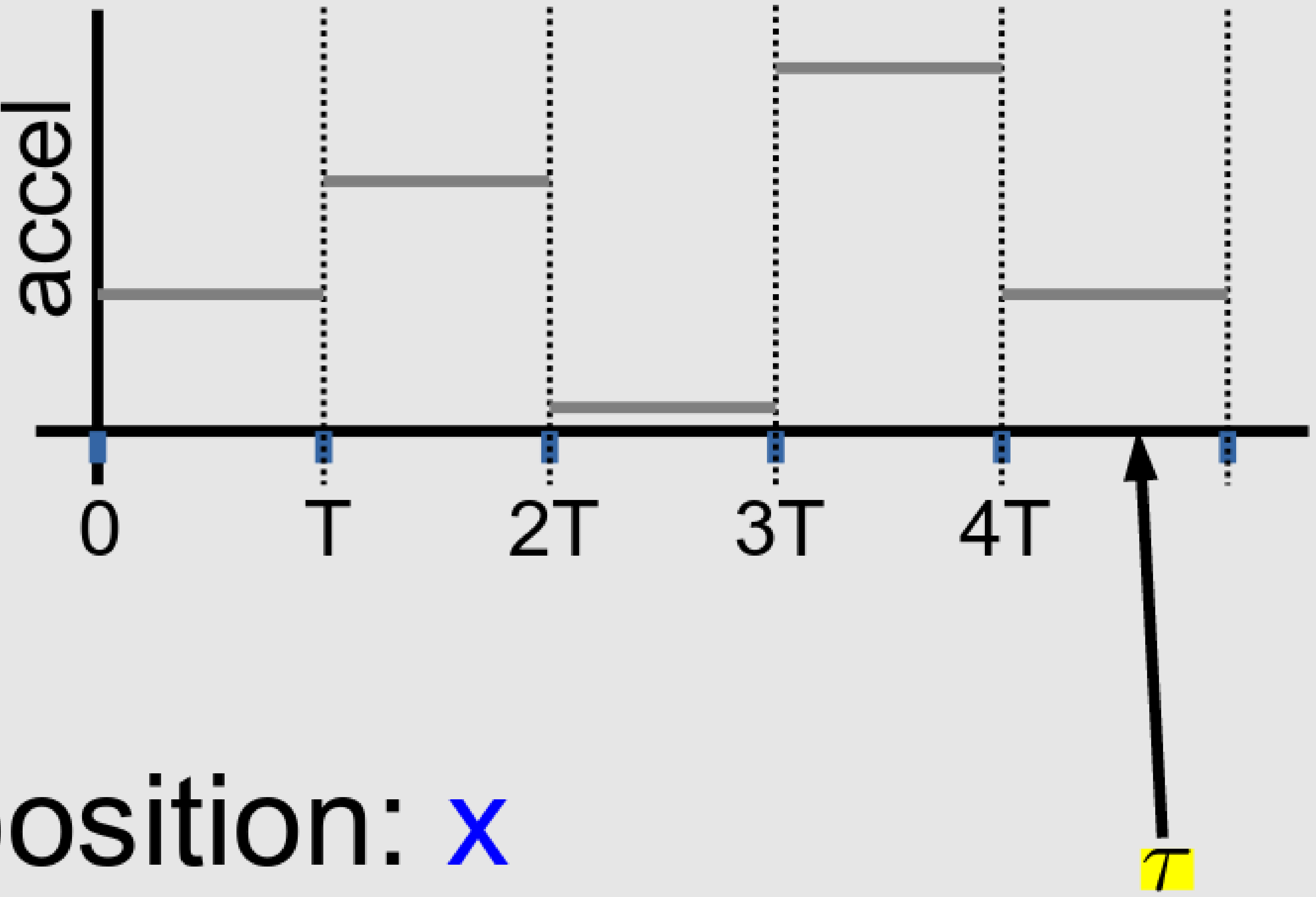

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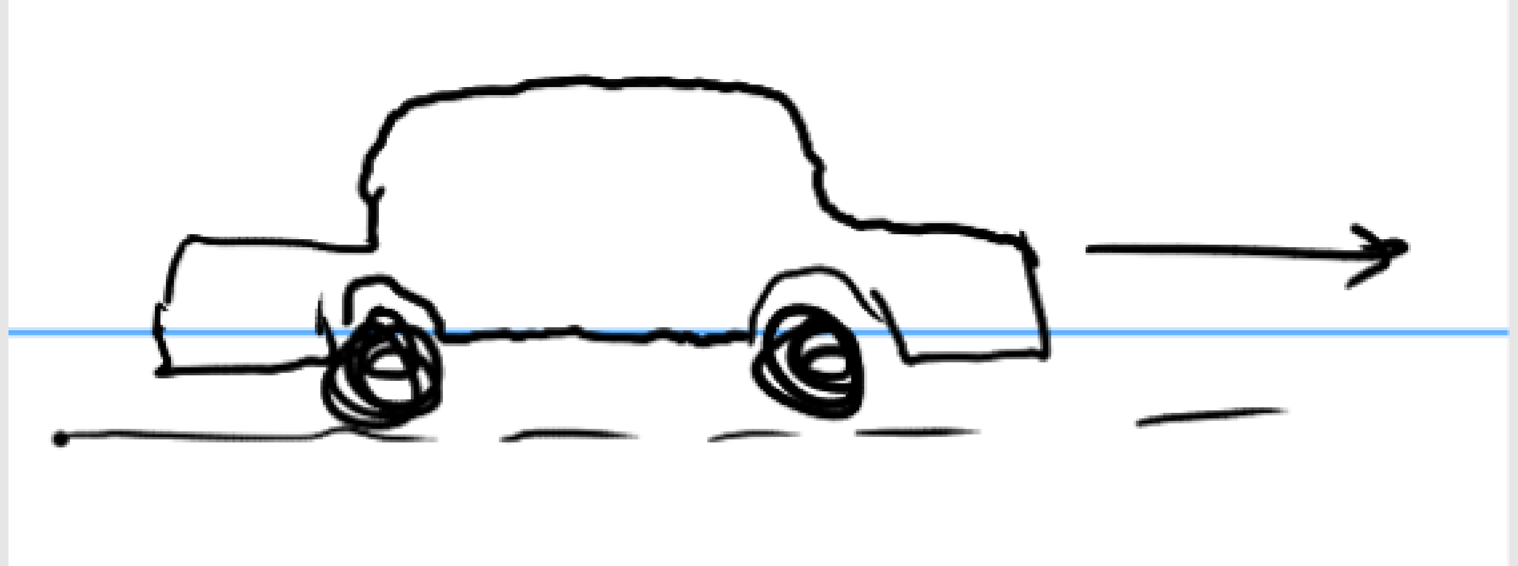

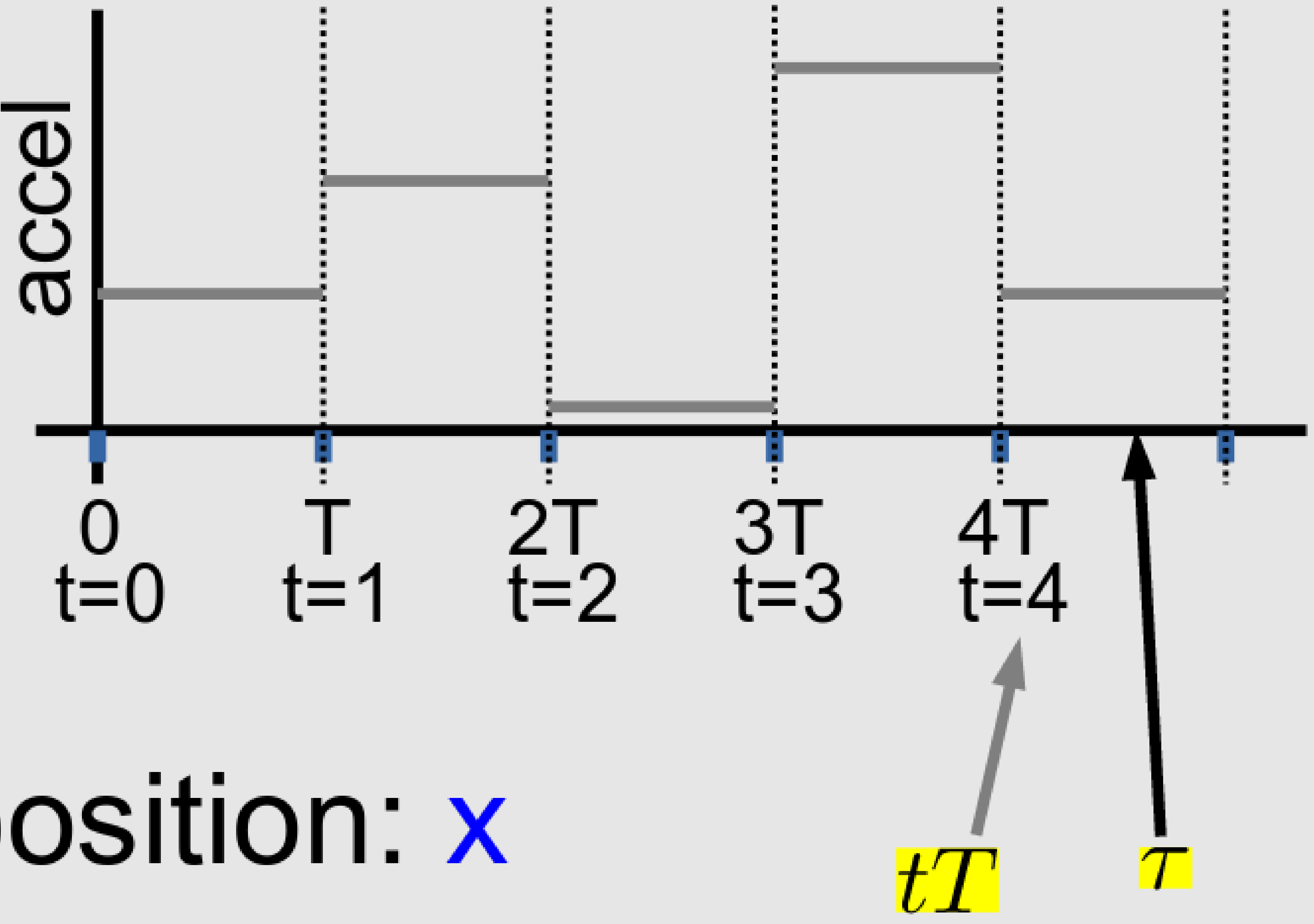

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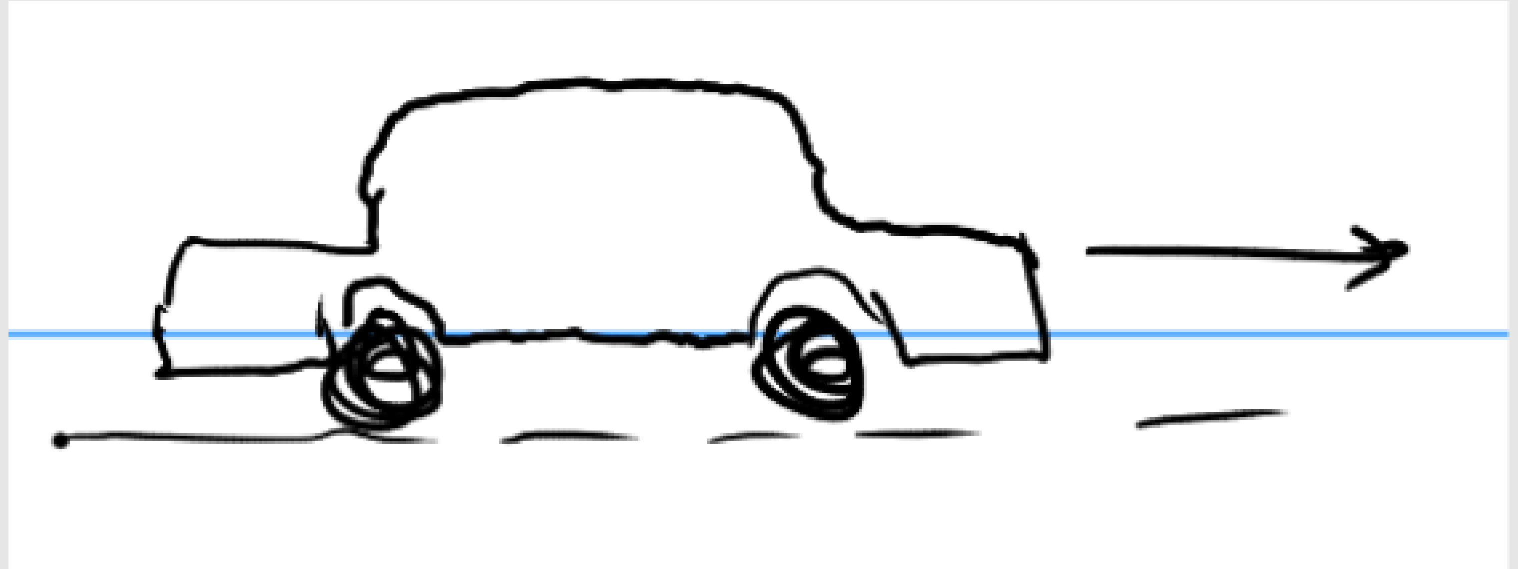

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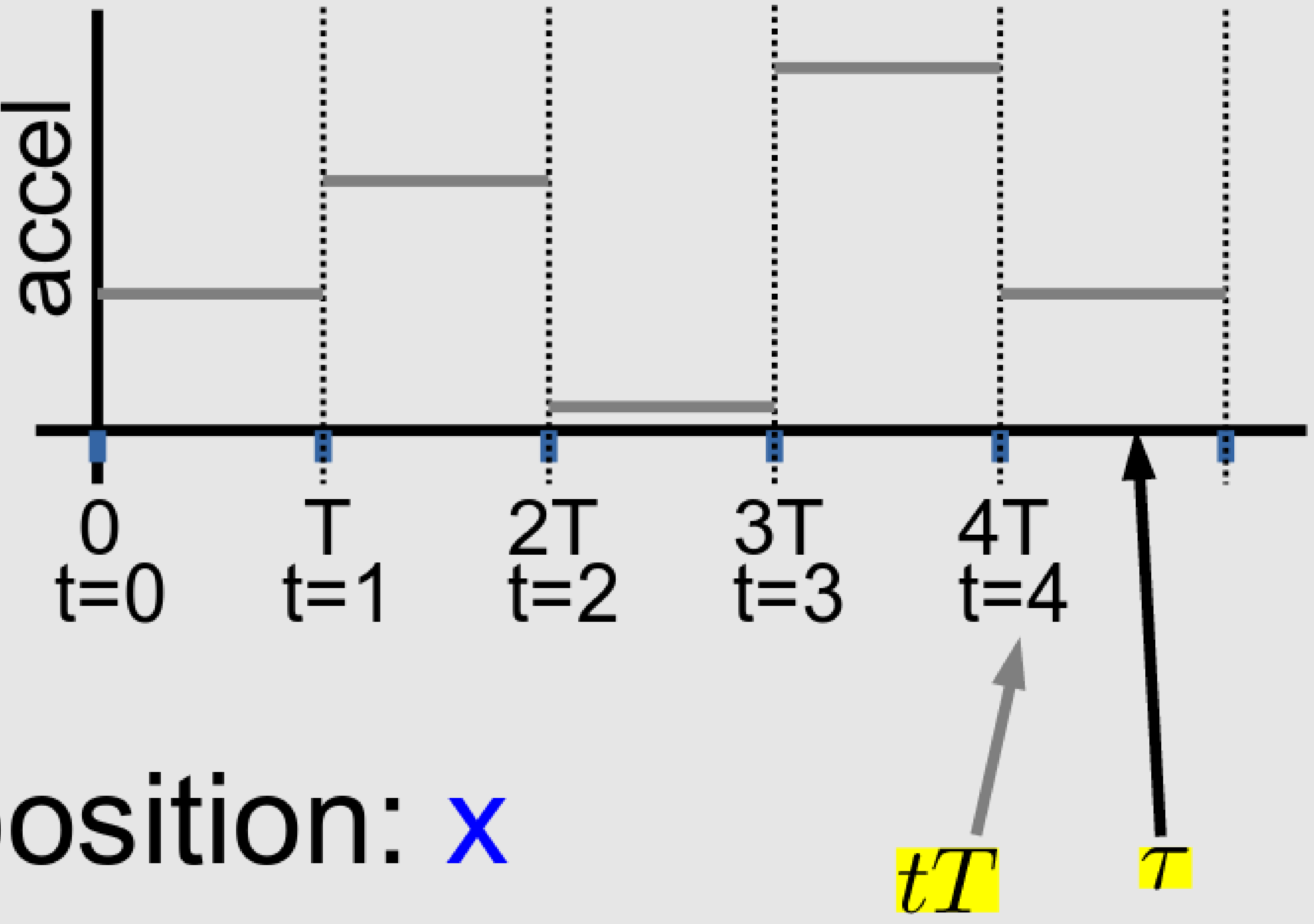

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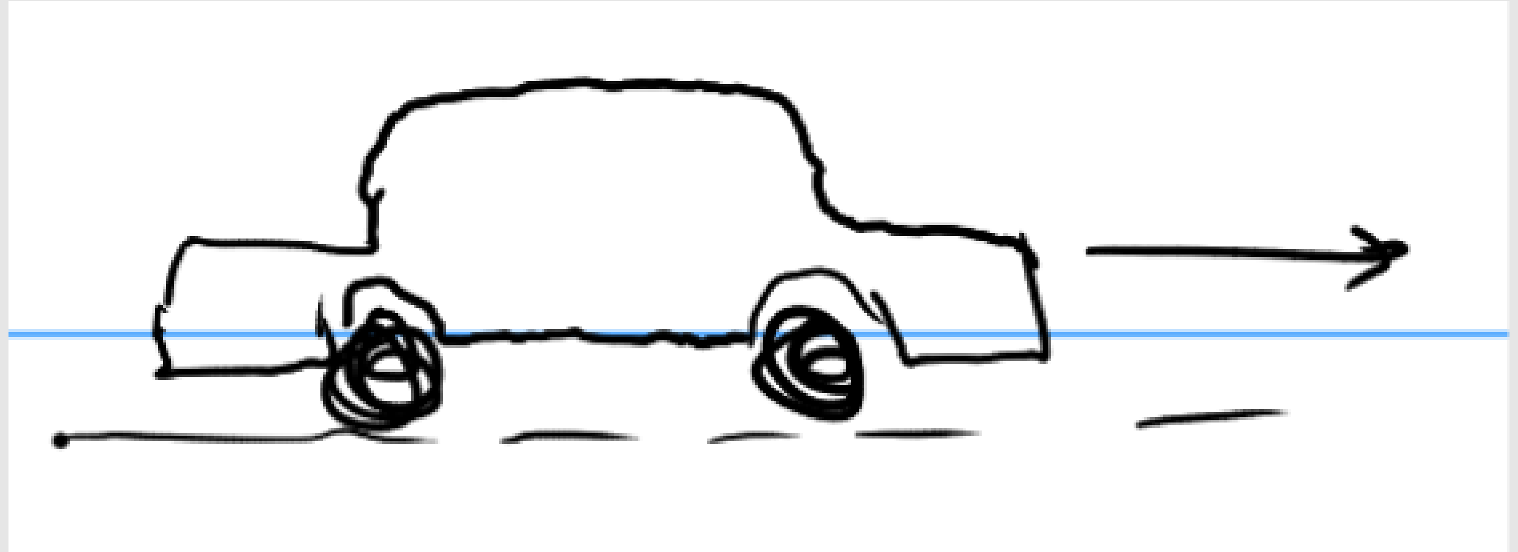

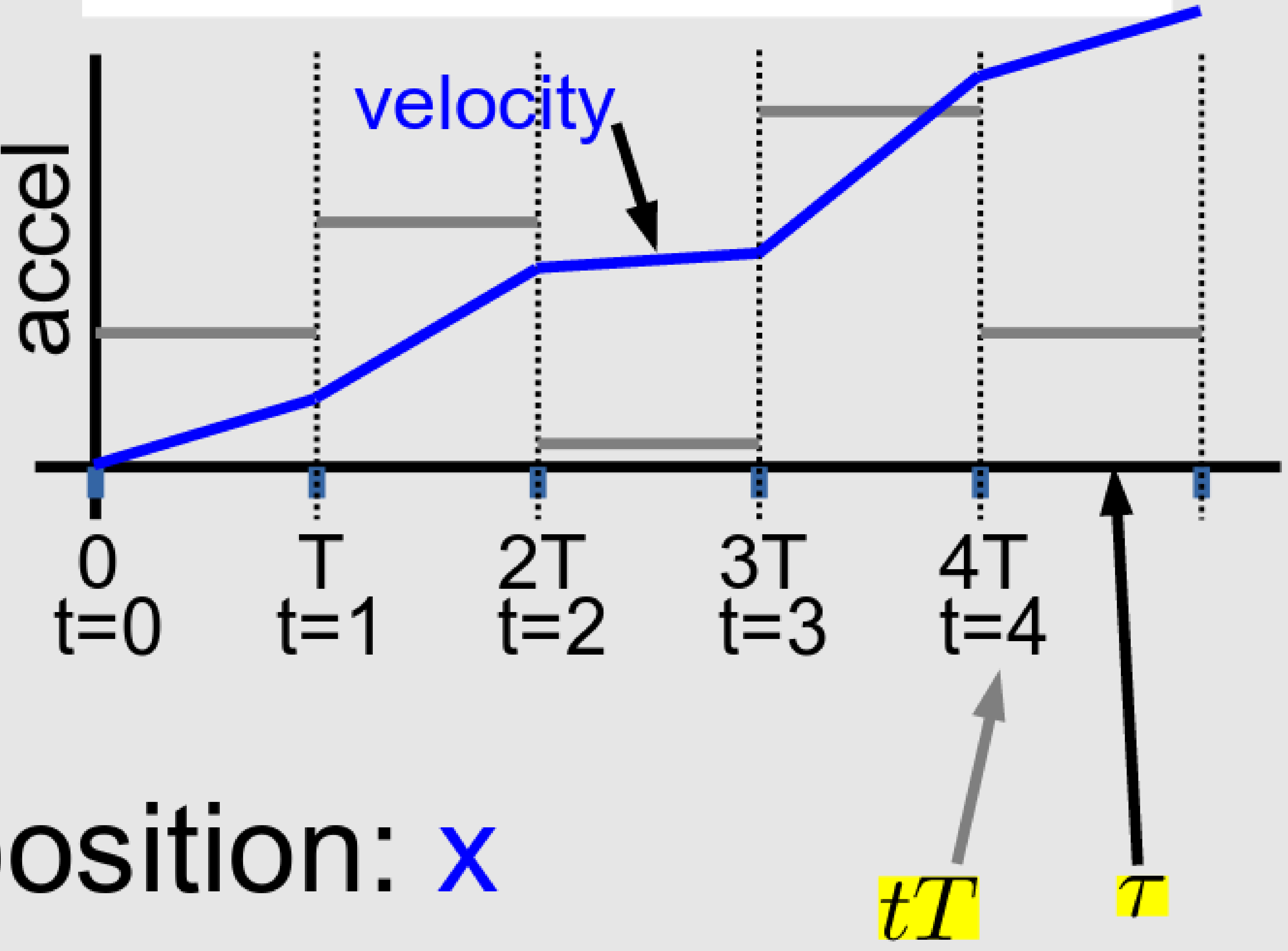

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    $$= (\tau - tT) v(tT) + a(tT) \frac{(\tau - tT)^2}{2}$$

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    $= (\tau - tT) v(tT) + a(tT) \frac{(\tau - tT)^2}{2}$ $\quad tT \leq \tau \leq (t+1)T$
  - set $\tau = (t+1)T$; the above become:
  - $$x((t+1)T) = x(tT) + Tv(tT) + \frac{T^2 a(tT)}{2}$$
    $$v((t+1)T) = v(tT) + Ta(tT)$$

Accelerating car (contd. - 2)

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  - S.S.R in matrix-vector form:
  - $\begin{bmatrix} x((t+1)T) \\ v((t+1)T) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(tT) \\ v(tT) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} a(t)$

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- A: YES, we **can** drive the car's **position** AND **velocity** to whatever values we want (at every τ=tT for t≥2)

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- System: $\frac{d}{dt}\Delta\vec{x}(t) = \overset{\text{nxn matrix}}{A}\Delta\vec{x}(t) + \overset{\text{nxm matrix}}{B}\Delta\vec{u}(t)$

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$$rank\left(\left[B \,|\, AB \,|\, \cdots \,|\, A^{t-1}B\right]\right) = n$$

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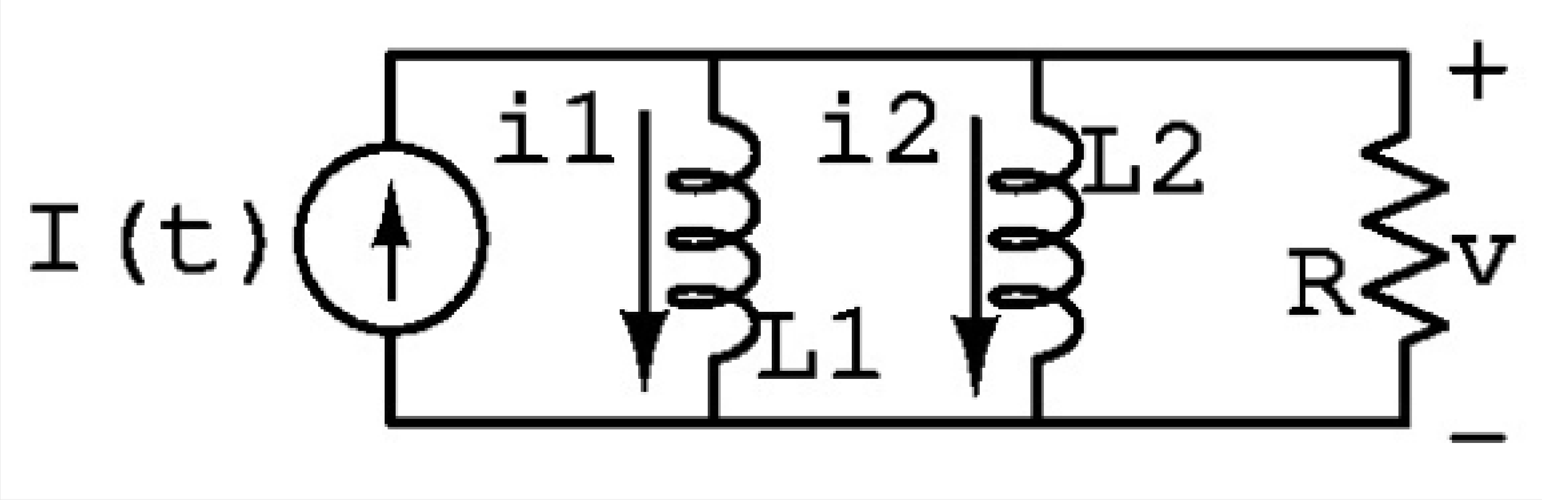

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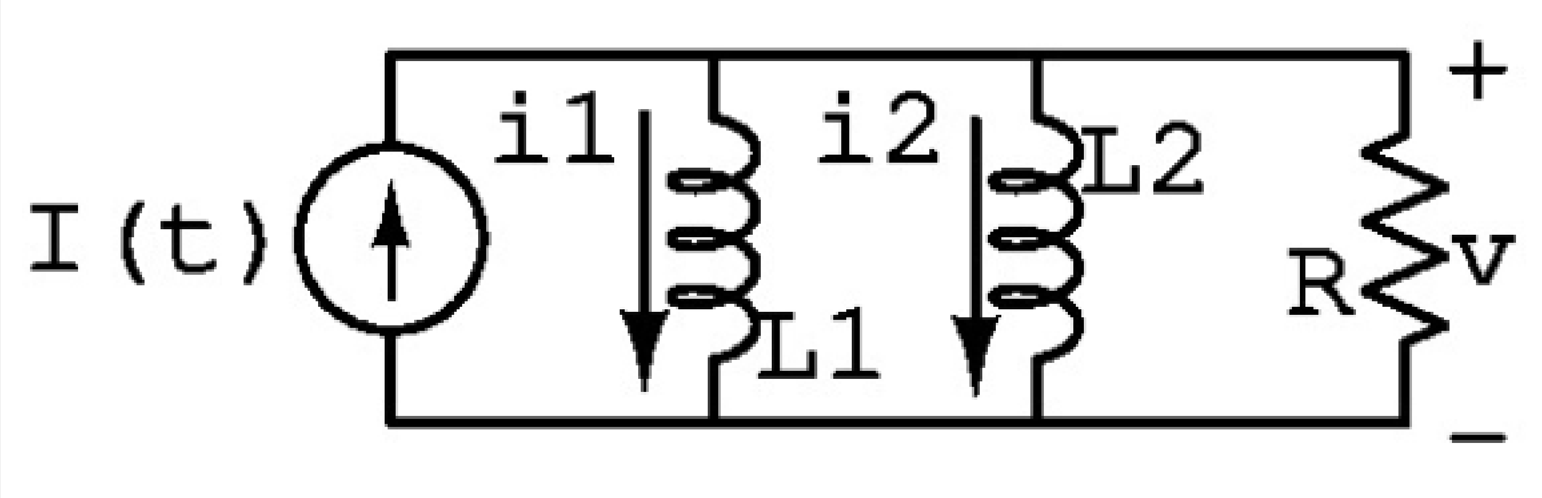

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- $i_1 + i_2 + \frac{v}{R} = I_1(t)$, $\frac{di_1}{dt} = \frac{v}{L_1}$, $\frac{di_2}{dt} = \frac{v}{L_2}$

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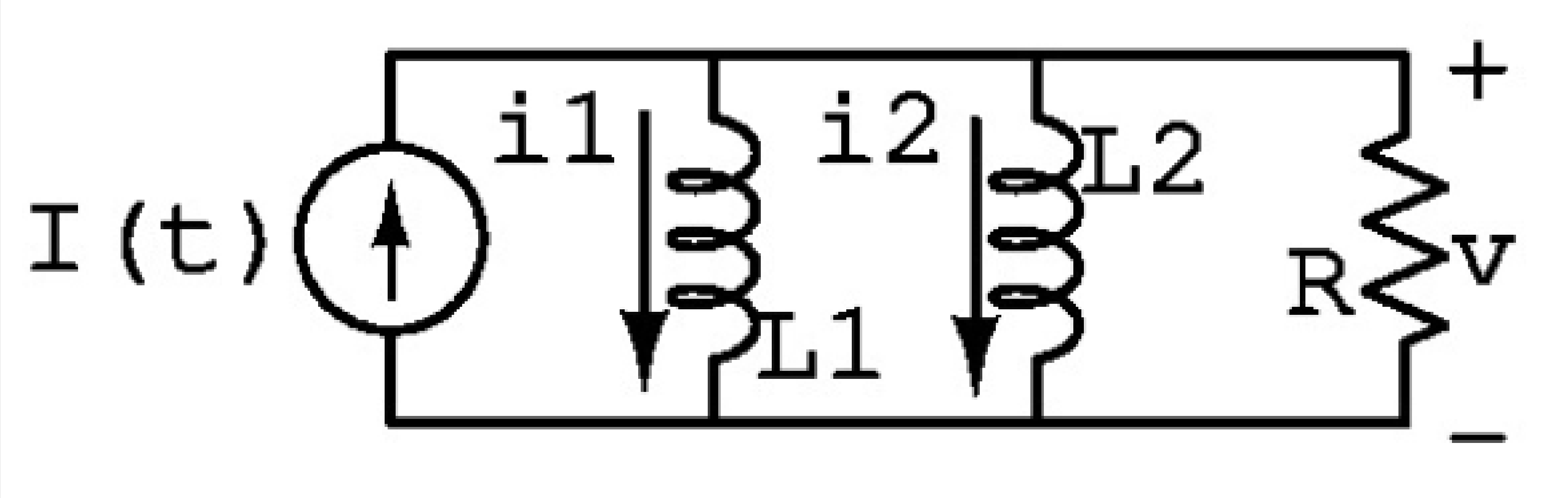

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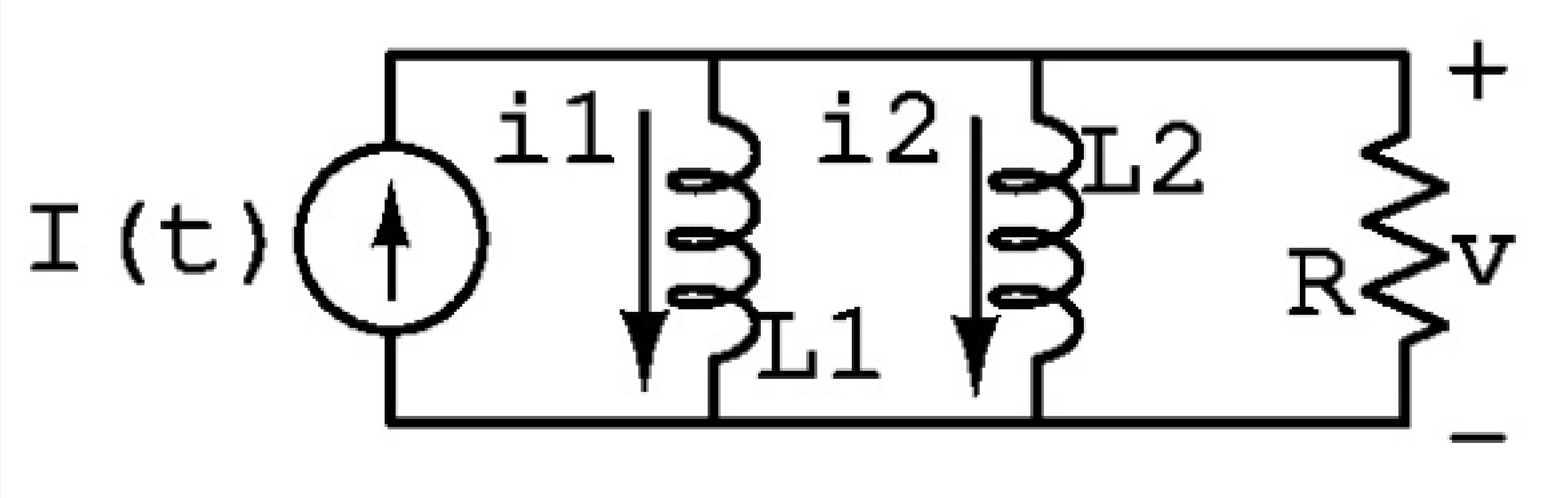

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- $\frac{di_1}{dt} = \frac{R(I_1(t) - i_1(t) - i_2(t))}{L_1}$, $\frac{di_2}{dt} = \frac{R(I_1(t) - i_1(t) - i_2(t))}{L_2}$
- $$\frac{d}{dt}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix} I(t)$$

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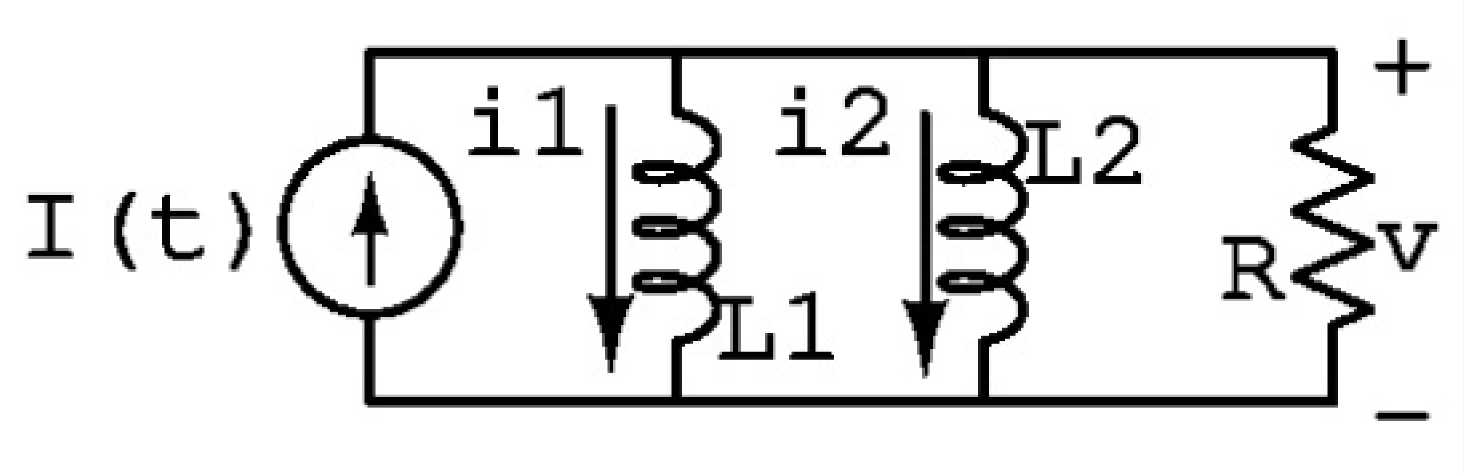

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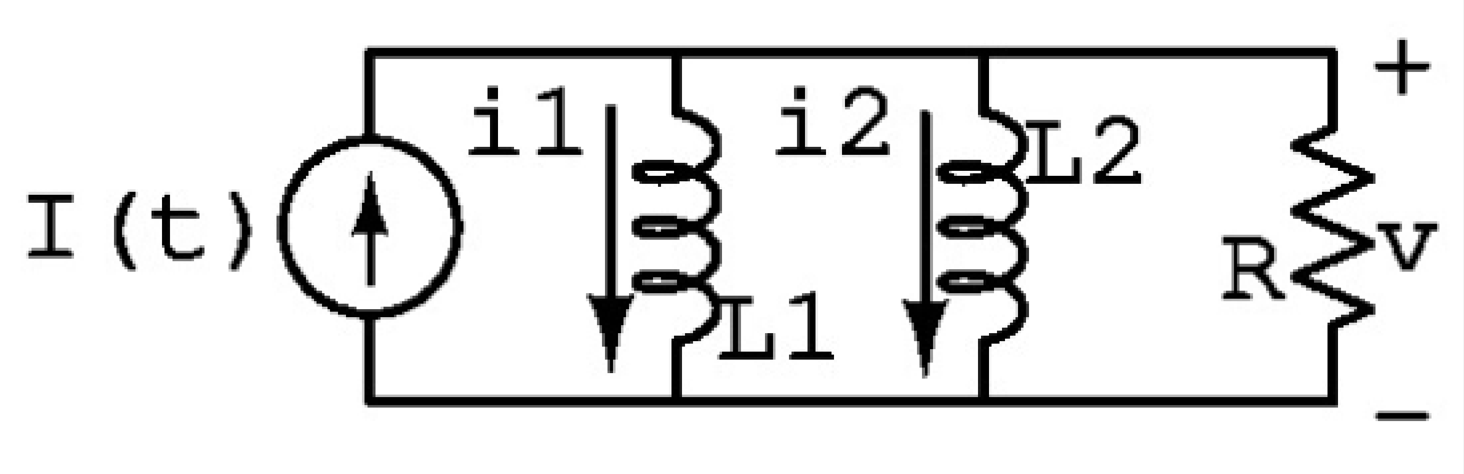

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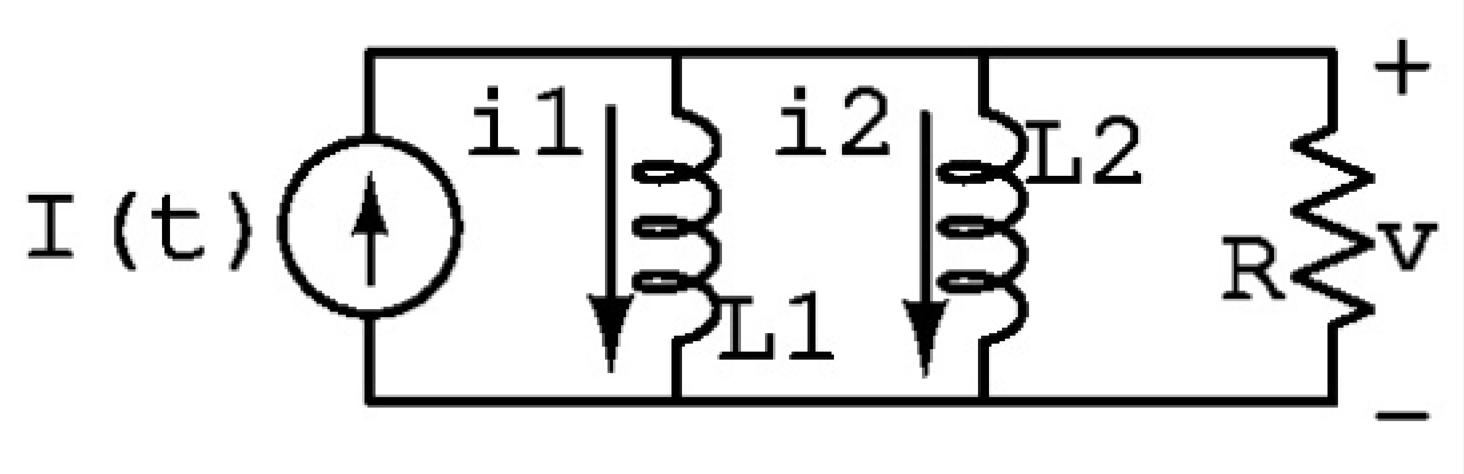

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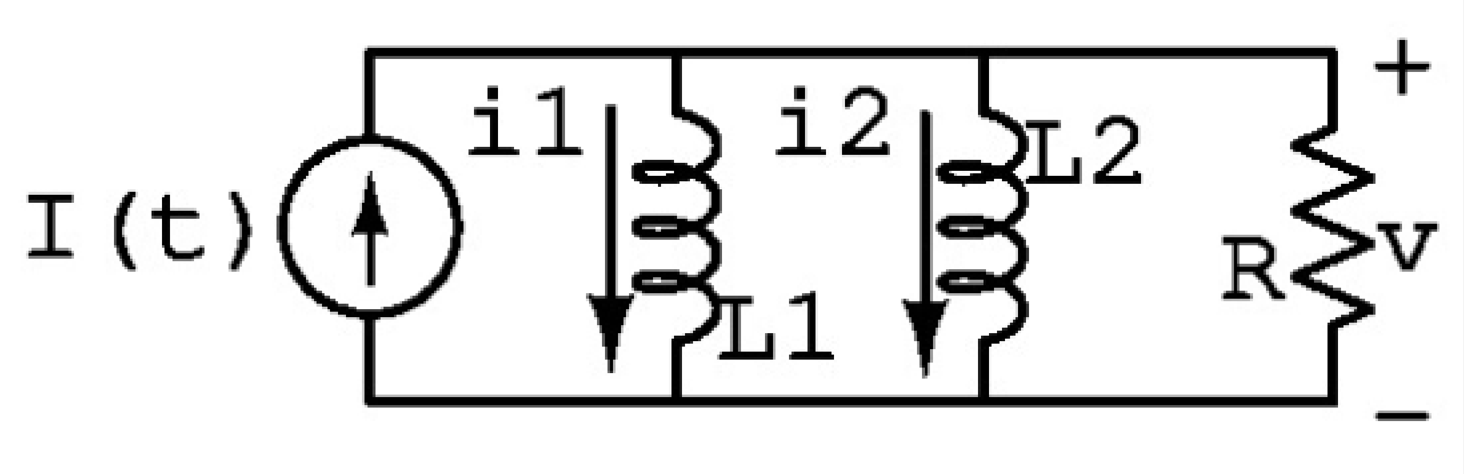

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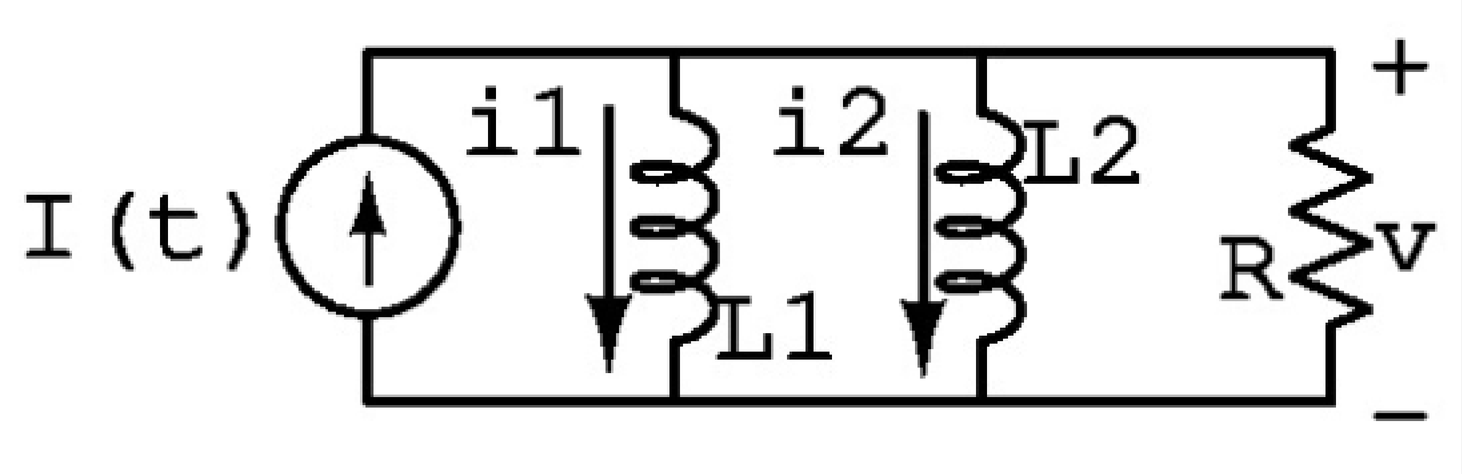

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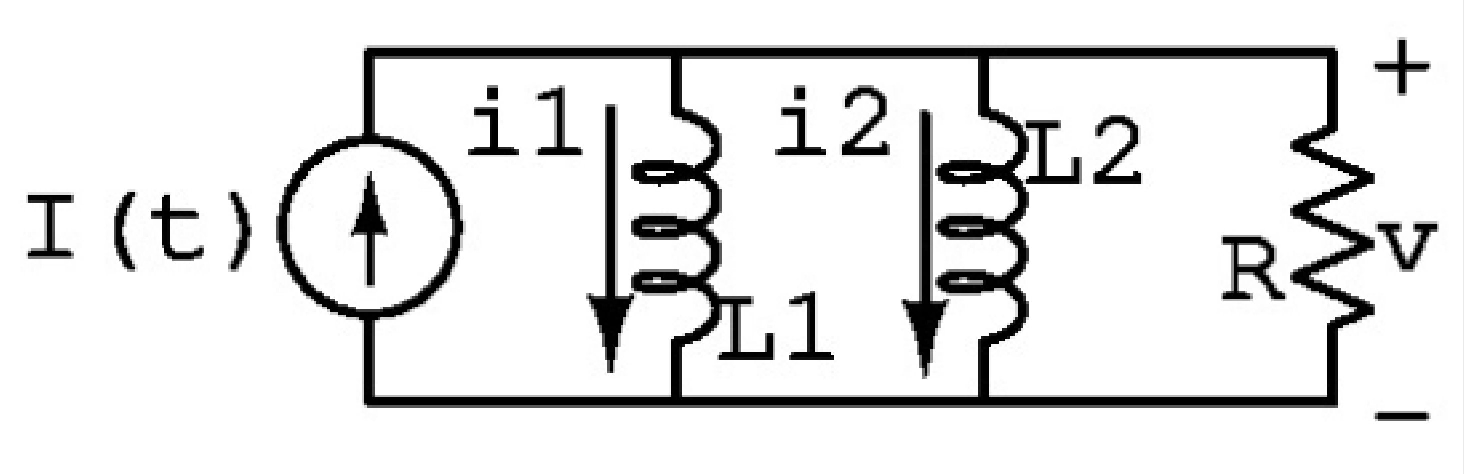

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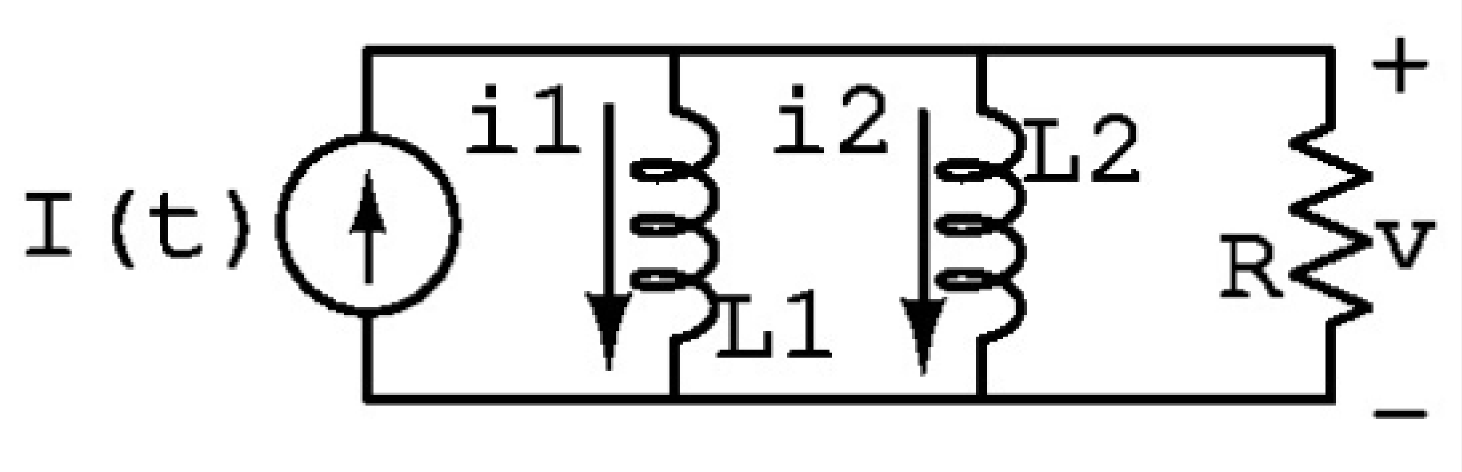

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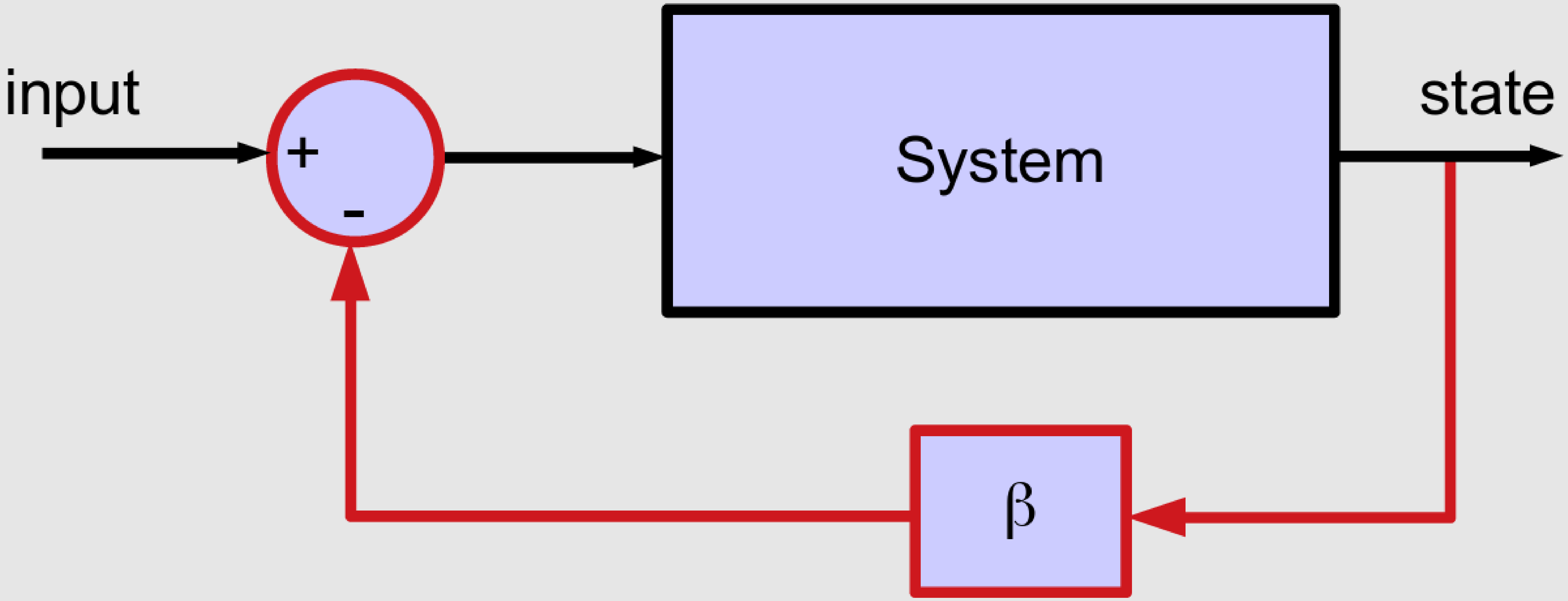

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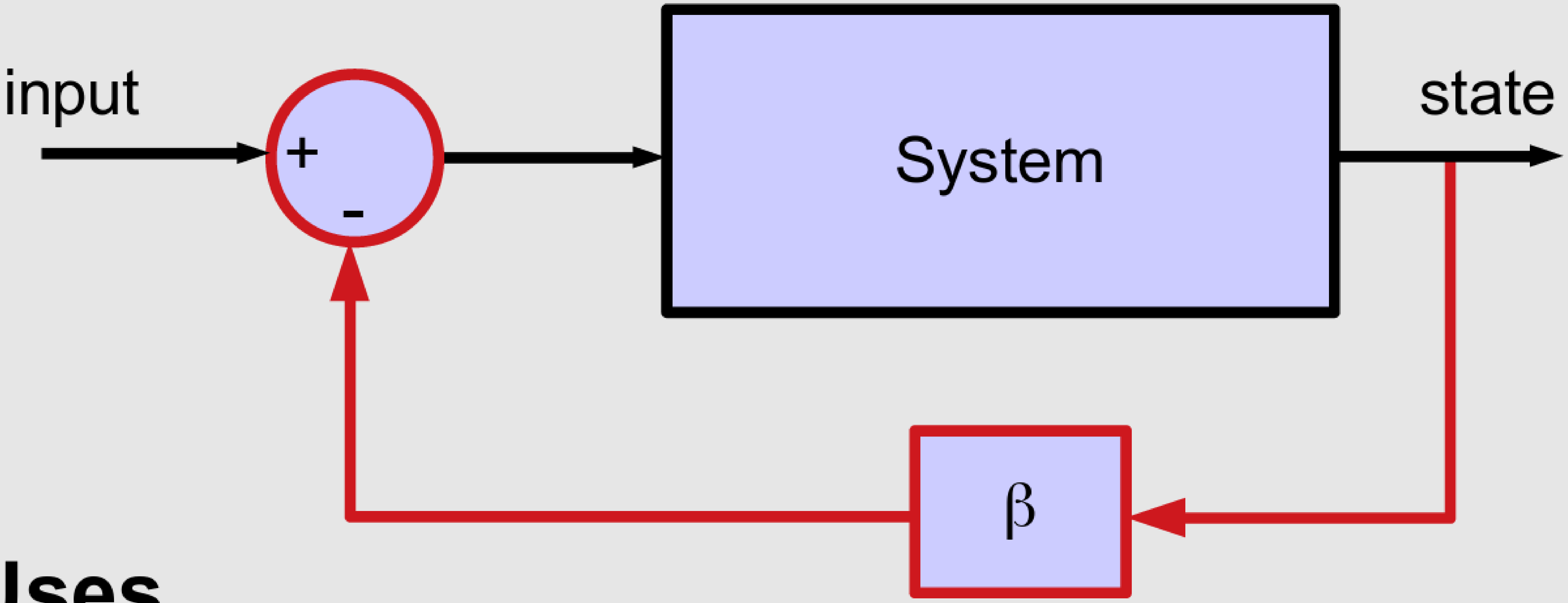

- **Uses**
  - making systems **less sensitive** to undesired noise and uncertainties (ALWAYS PRESENT in practical systems)
  - **stabilizing** unstable systems (if they are controllable)
    - ➔ thus making them practically usable

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- How will this change if a = -1?

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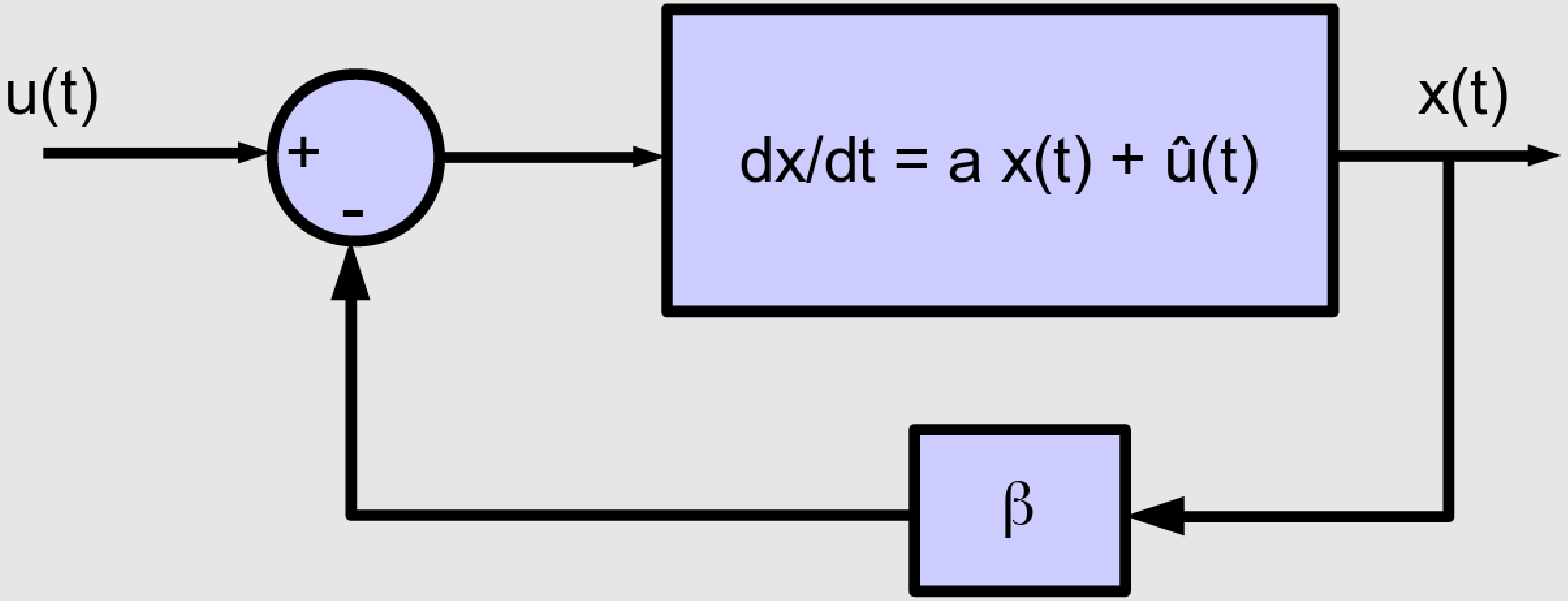
Stabilization via Feedback (Scalar)



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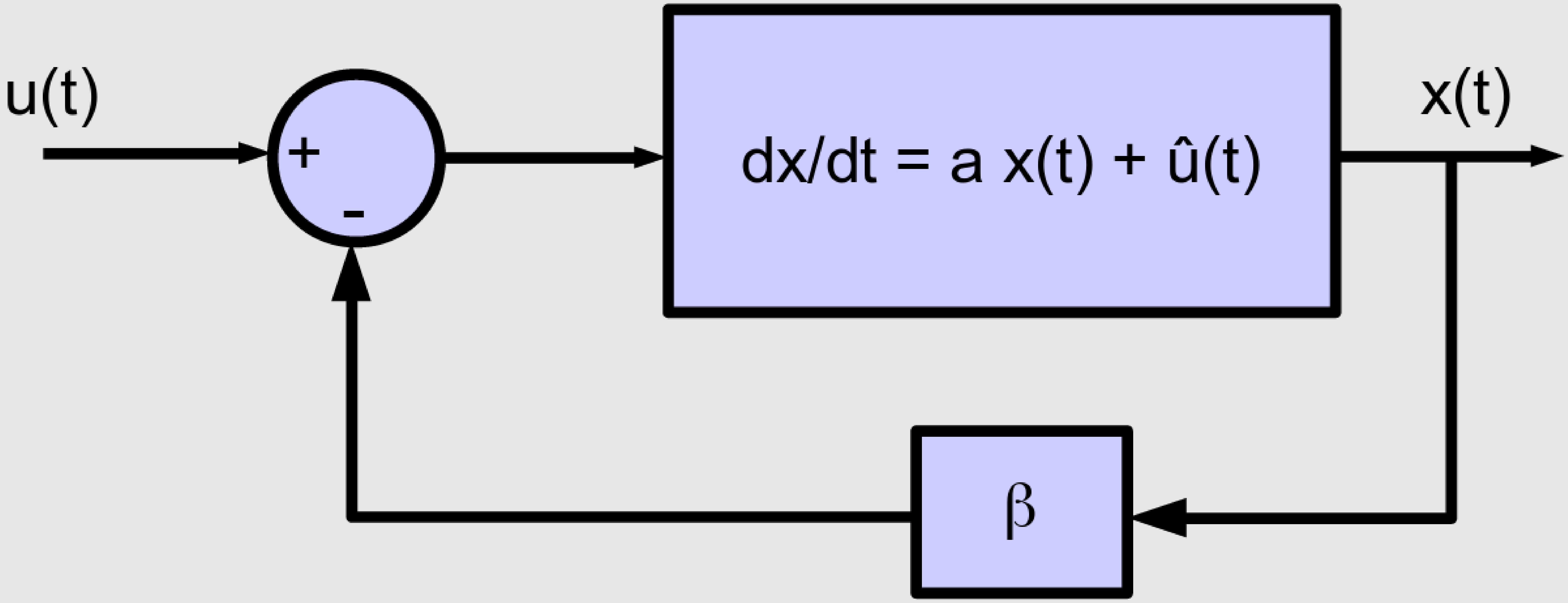

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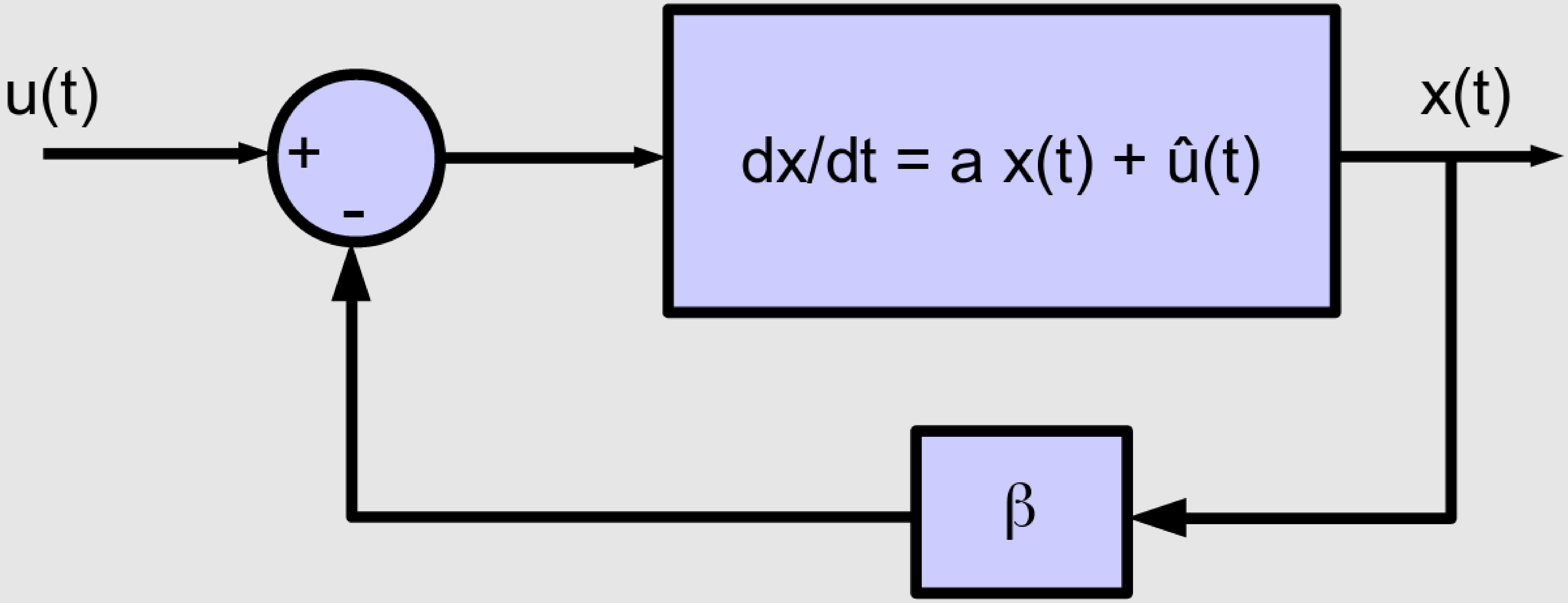

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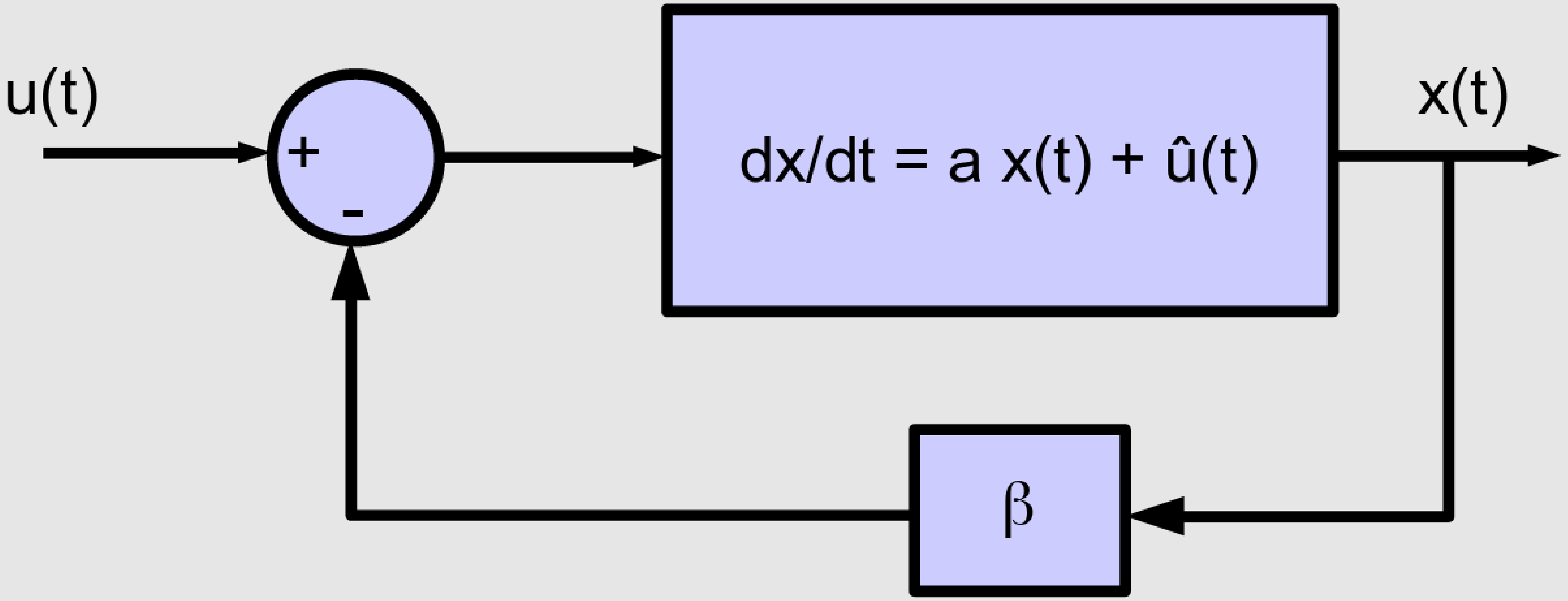

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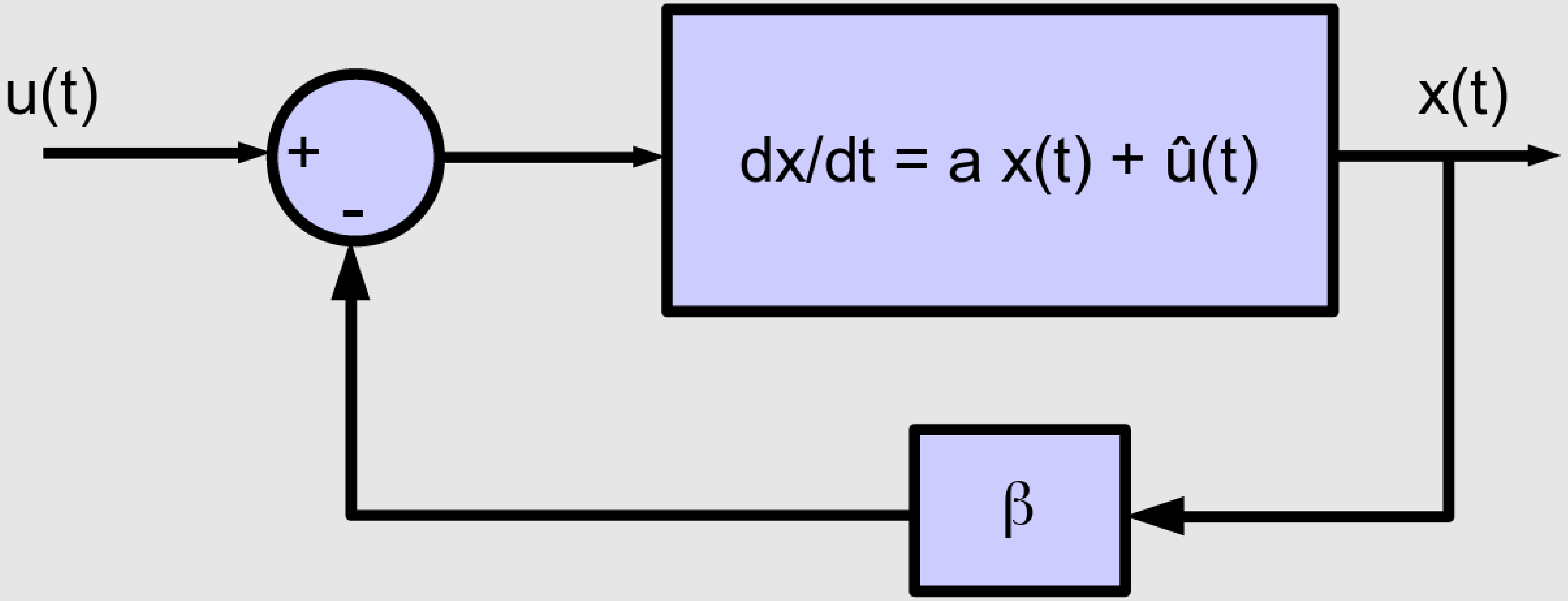
  - $\dot{x}(t) = ax(t) + \hat{u}(t) \mapsto \boxed{\dot{x}(t) = ax(t) + u(t) - \beta x(t), \quad a = 1 > 0}$

Stabilization via Feedback (Scalar)



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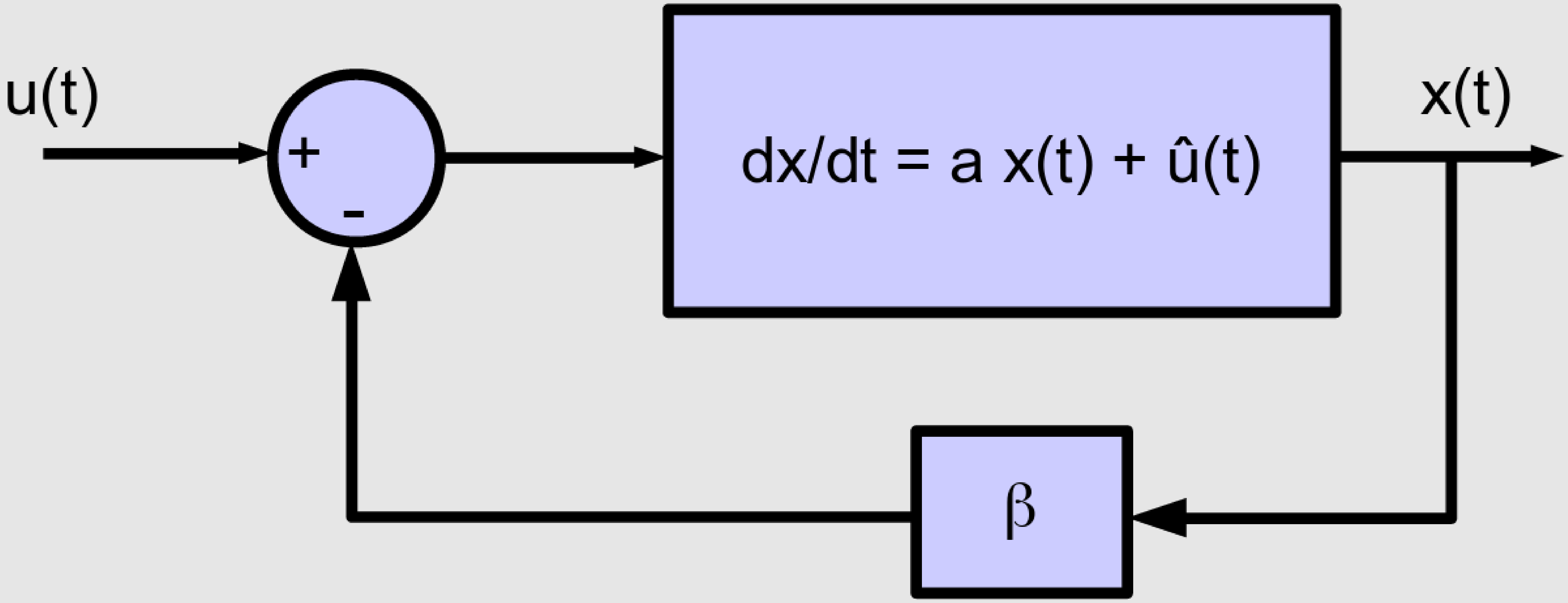

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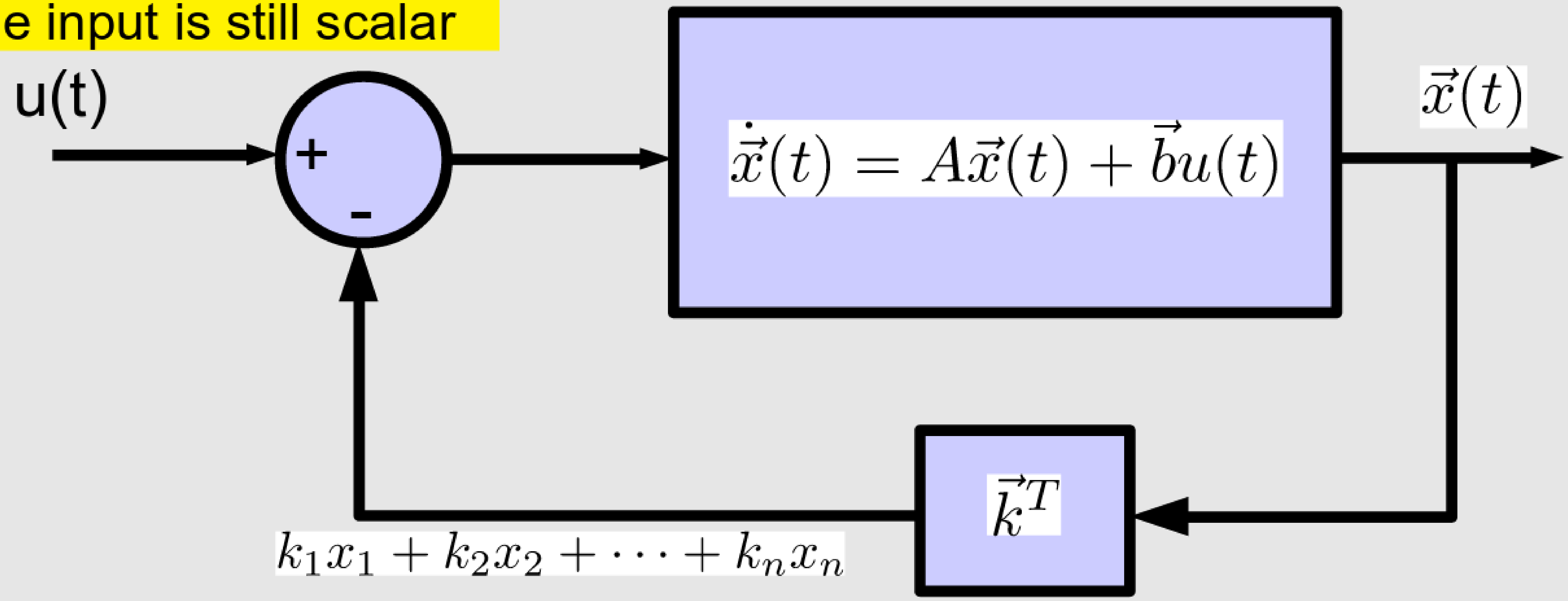
- $\dot{x}(t) = \boxed{(a - \beta)} x(t) + u(t), \quad a = 1 > 0$

choose β>a → **system is stabilized**

Feedback for Vector S.S. Systems

assumption for simplicity:
the input is still scalar



Feedback for Vector S.S. Systems

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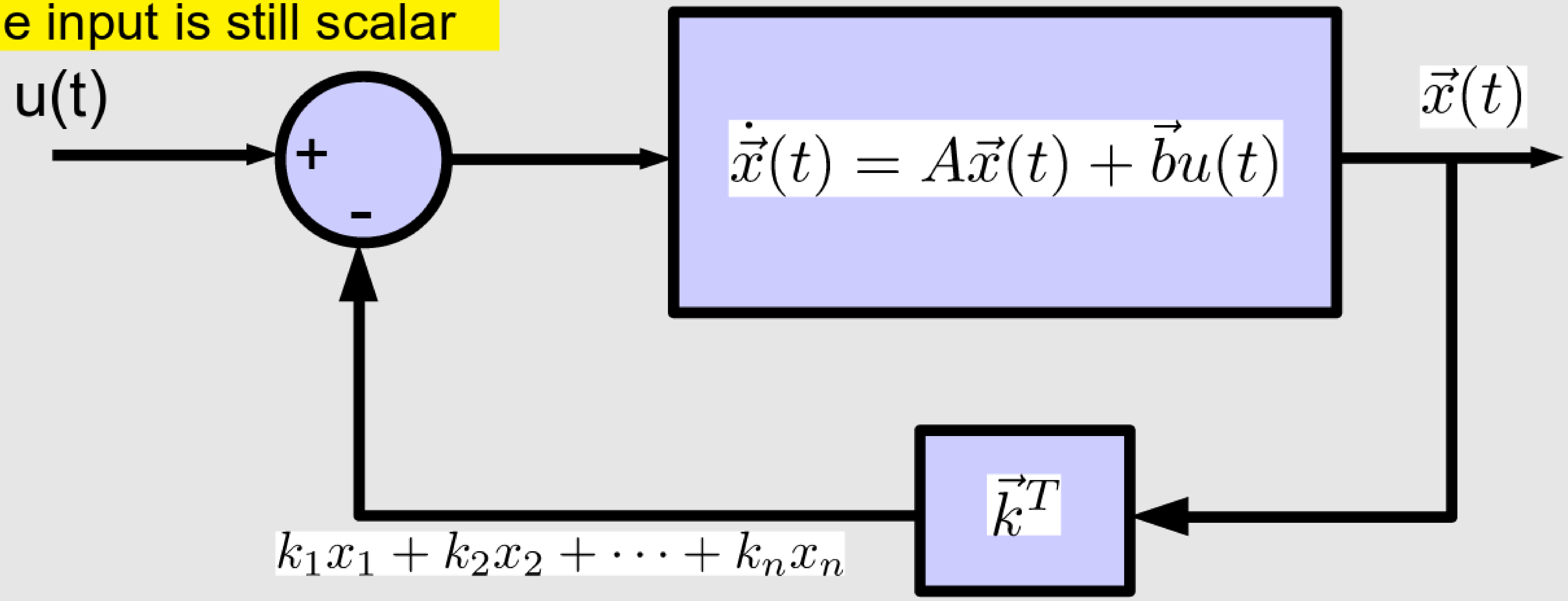
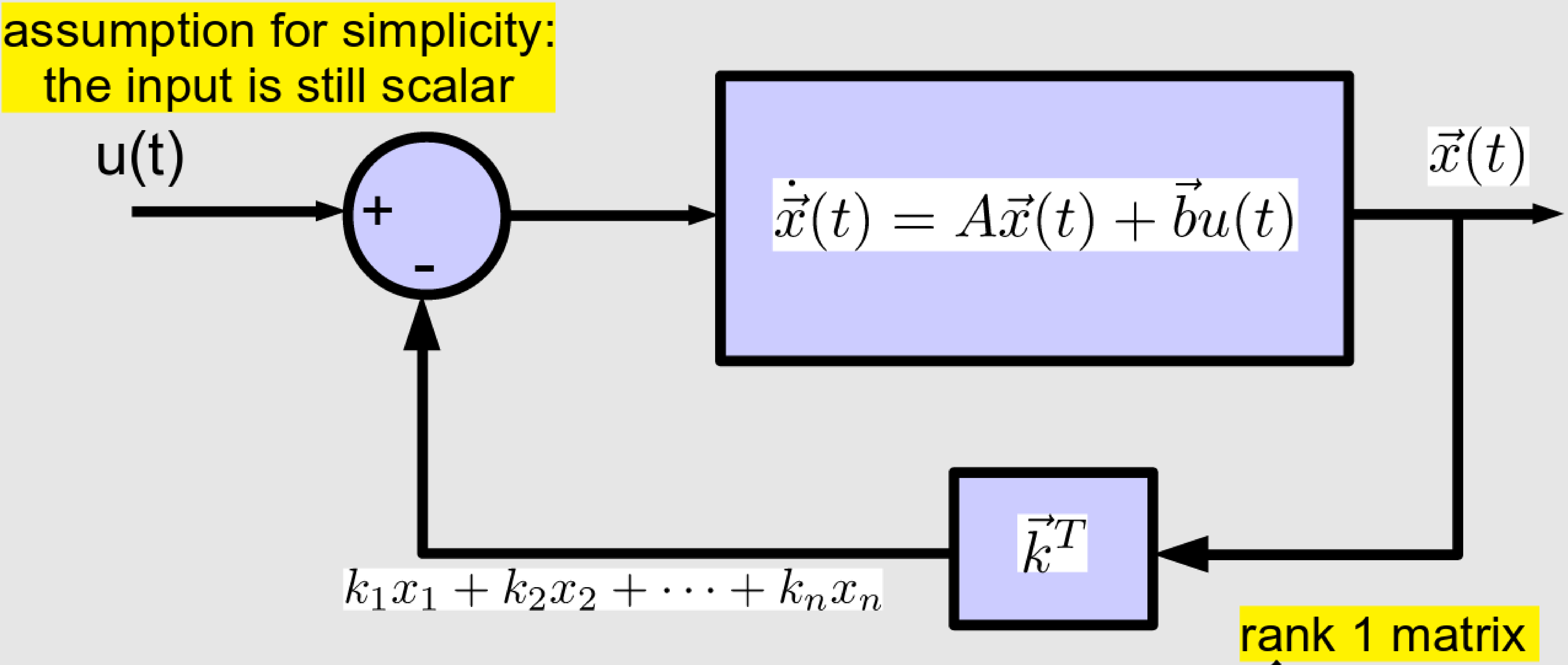


- system w feedback: $\dot{\vec{x}}(t) = (A - \vec{b}\vec{k}^T)\vec{x}(t) + \vec{b}u(t)$

Feedback for Vector S.S. Systems



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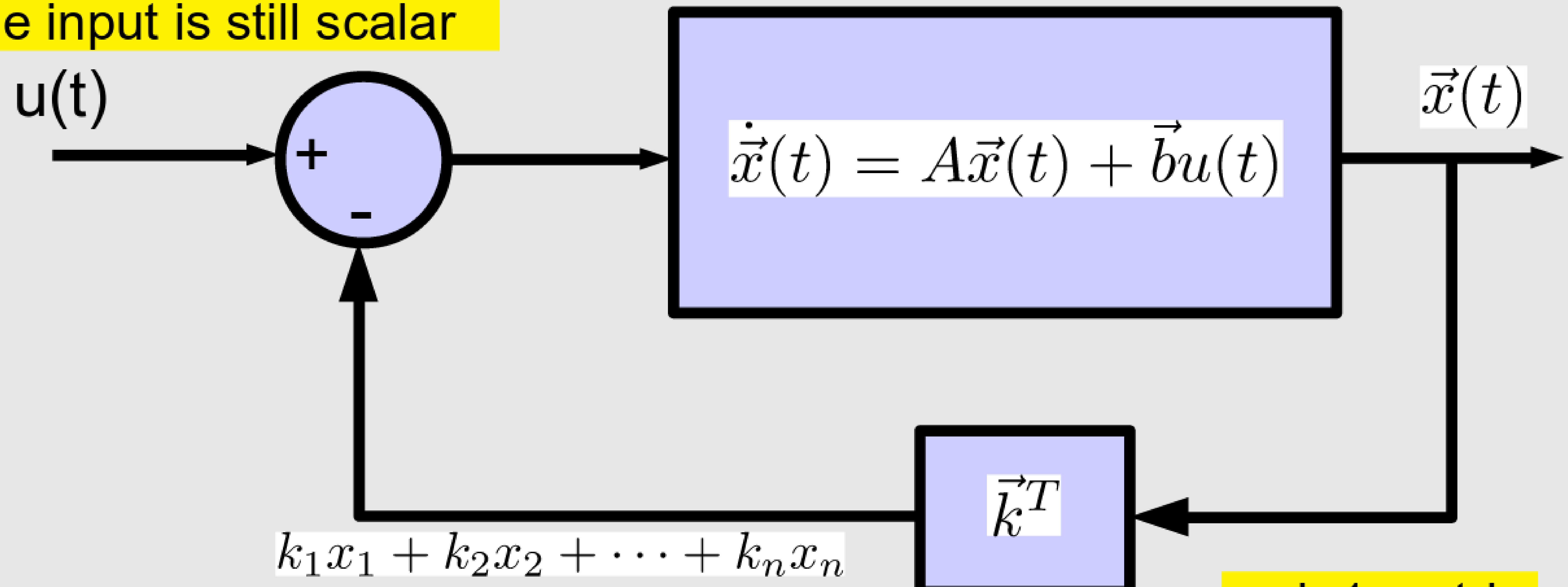
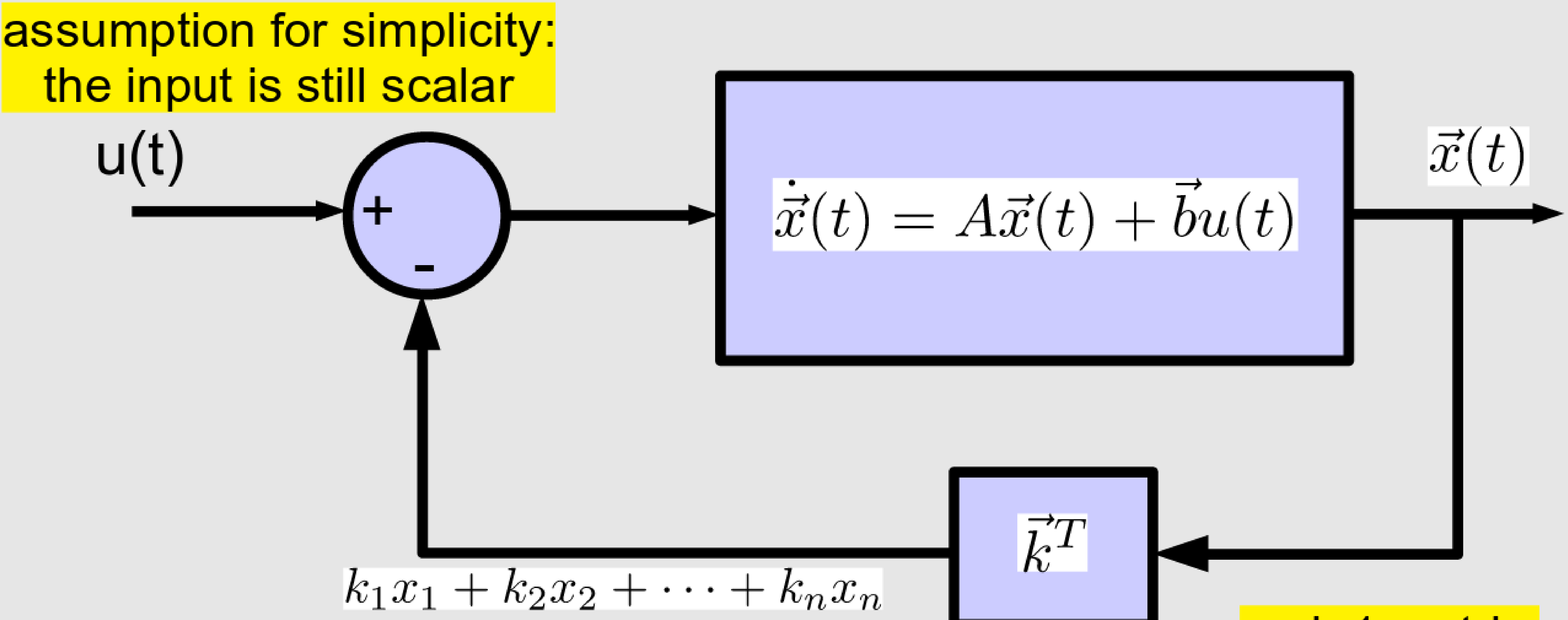


rank 1 matrix

- system w feedback: $\dot{\vec{x}}(t) = (A - \boxed{\vec{b}\vec{k}^T})\vec{x}(t) + \vec{b}u(t)$
  - stability governed by eigenvalues of $A - \vec{b}\vec{k}^T$

Feedback for Vector S.S. Systems



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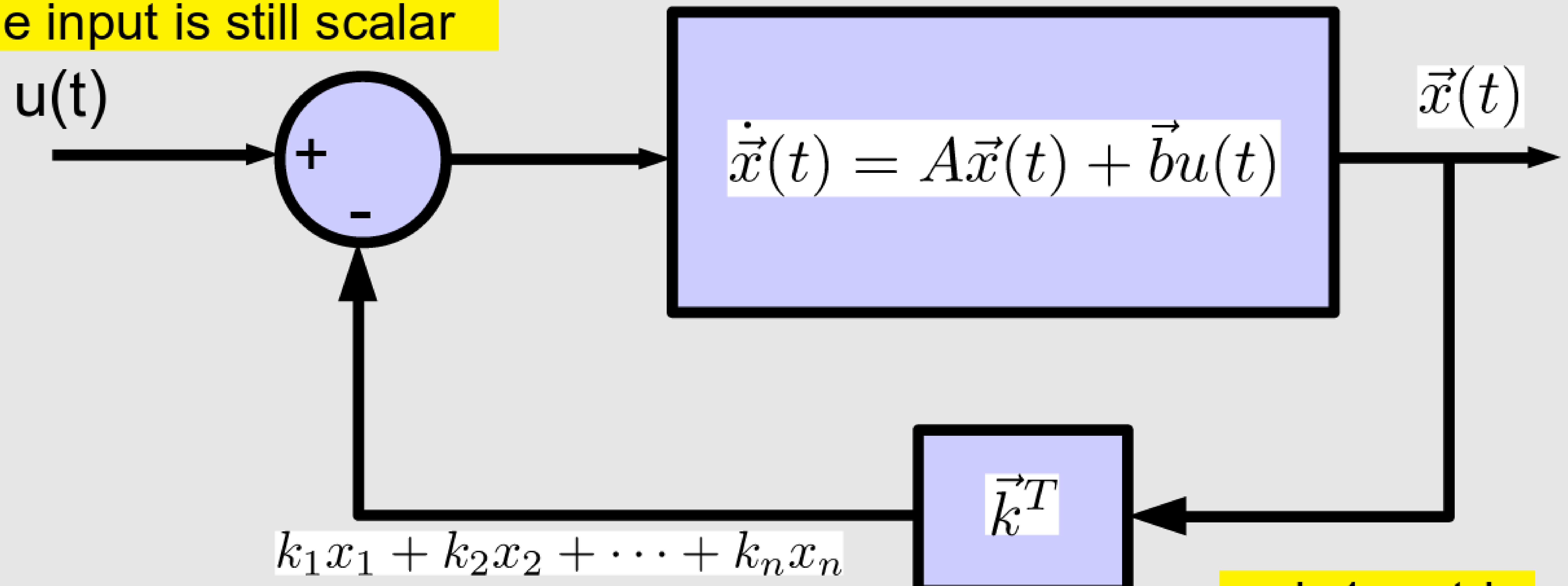
- Q: how do the e.values of A change due to

Feedback for Vector S.S. Systems



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rank 1 matrix

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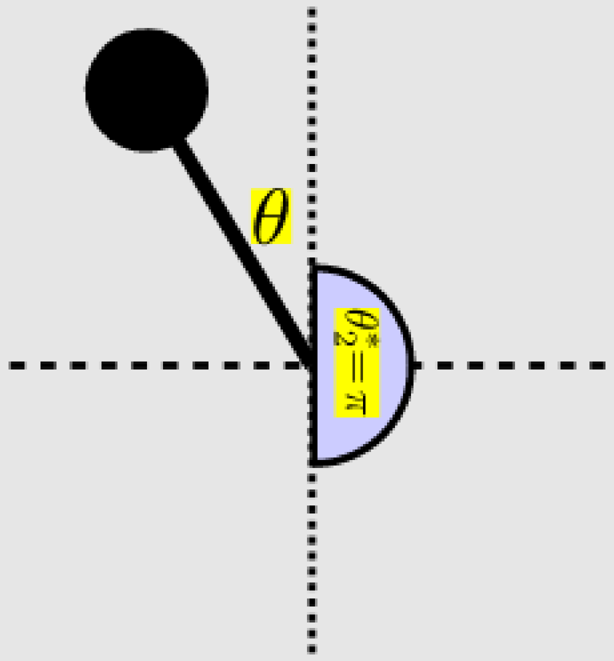
  - stability governed by eigenvalues of $A \boxed{- \vec{b}\vec{k}^T}$
- Q: how do the e.values of A change due to
  - very difficult to figure out analytically!
    - ➔ can do simple examples; otherwise, numerically

Example: stabilizing an inverted pendulum using feedback

- i.p.: $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ +\frac{g}{l} & -\frac{k}{m}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}u(t)$



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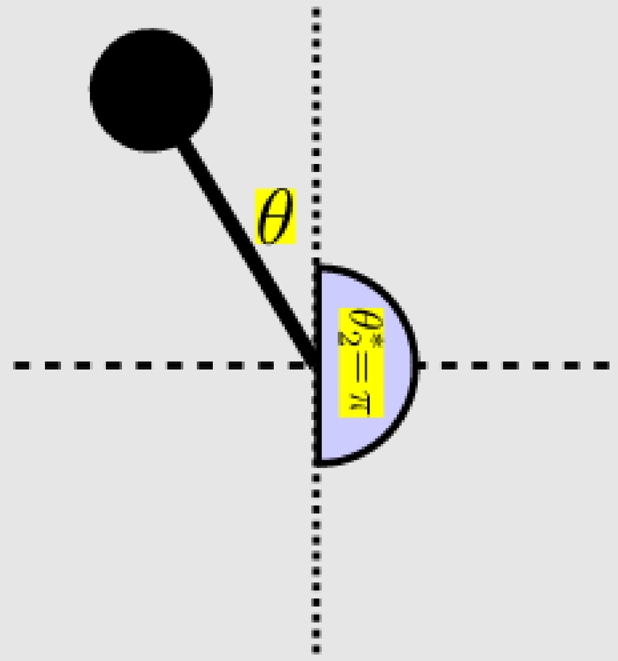
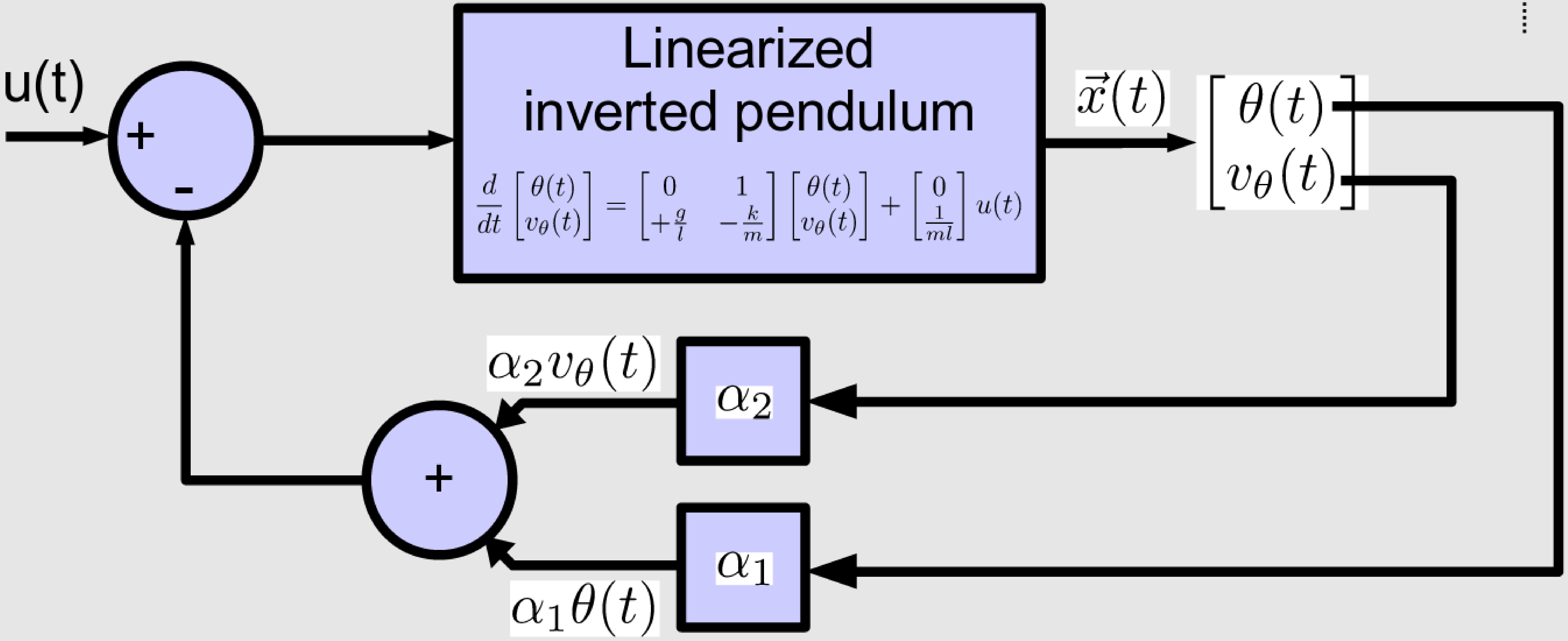
- Closed loop system (ie, with feedback)





u(t)

+

-

Linearized inverted pendulum

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ +\frac{g}{l} & -\frac{k}{m}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix} u(t)$$

$\vec{x}(t)$

$\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix}$

$\alpha_2 v_\theta(t)$

α_2

+

α_1

$\alpha_1\theta(t)$

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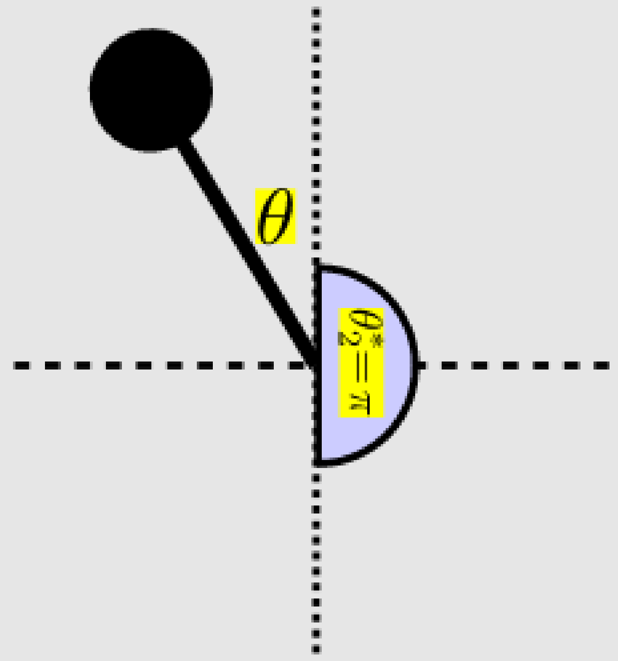

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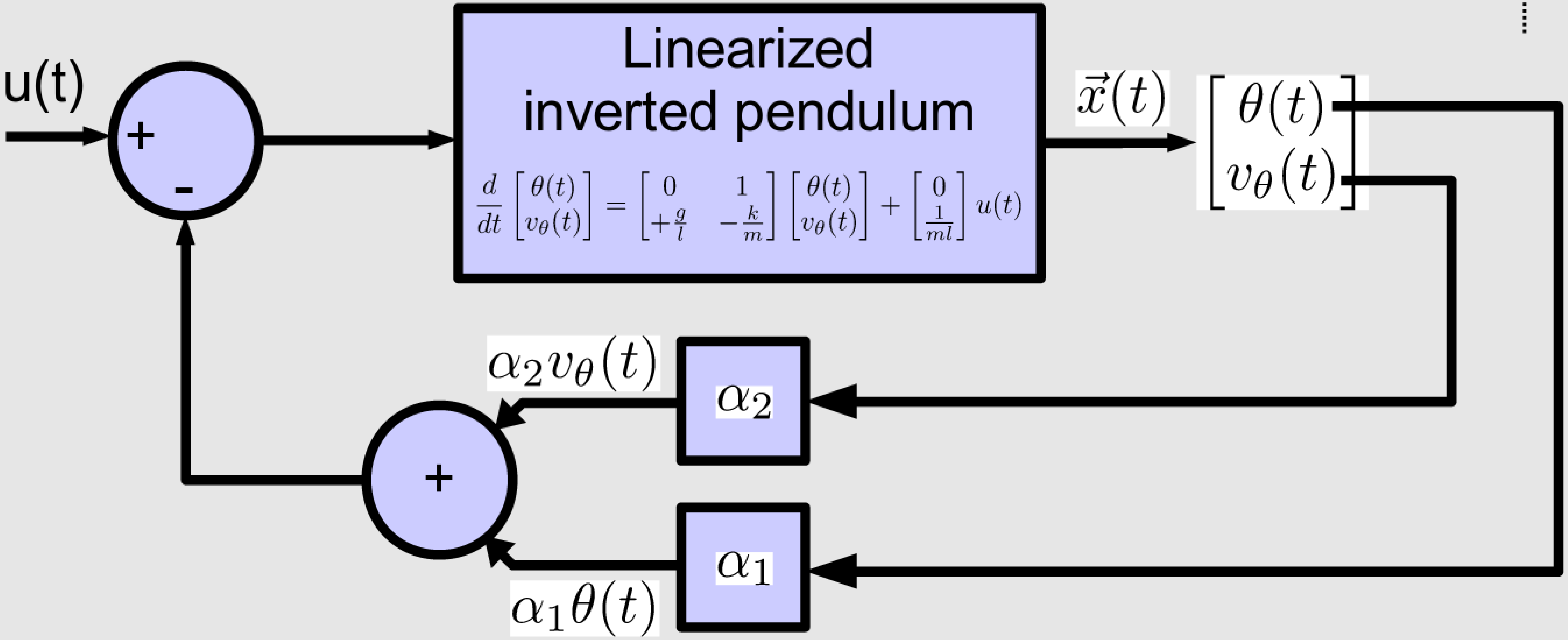

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- $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix} u(t)$

Stabilizing I.P. via feedback (contd.)

- I.P. w F.: $\frac{d}{dt}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{mg-\alpha_1}{ml} & \frac{-kl-\alpha_2}{ml}\end{bmatrix}\begin{bmatrix}\theta(t)\\ v_\theta(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix} u(t)$

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➔ $\lambda_{1,2} = \frac{-(kl+\alpha_2)}{2ml} \pm \frac{\sqrt{(kl+\alpha_2)^2 + 4ml(mg-\alpha_1)}}{2ml}$

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- to stabilize: make both evs -ve (real part)

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make this negative

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make this negative

make this smaller than $|kl+\alpha_2|$

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- to stabilize: make both evs -ve (real part)
  - ➔ choose any $\alpha_2 > -kl, \quad \alpha_1 > mg$

Stabilizing I.P. via feedback (contd.)

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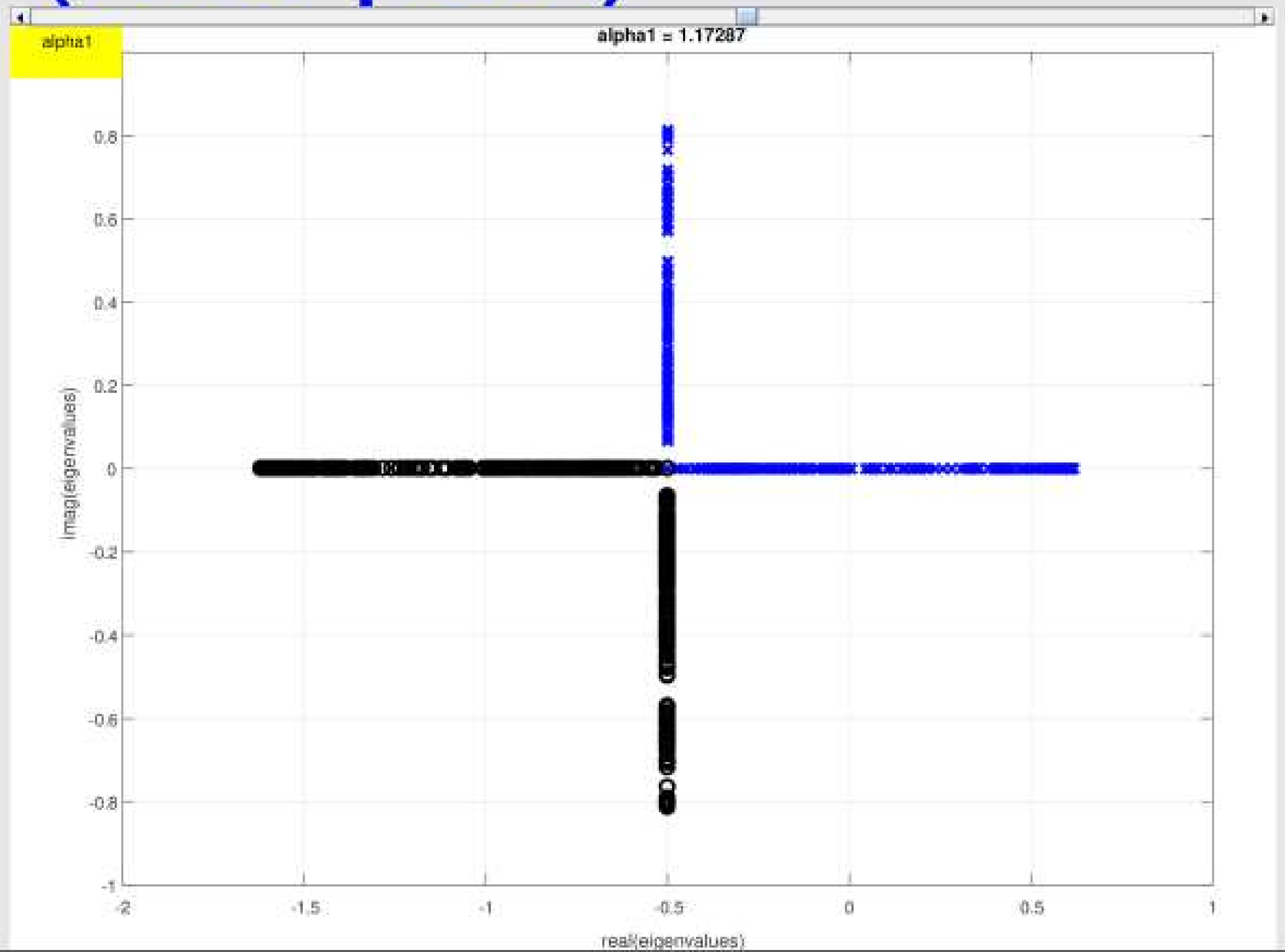
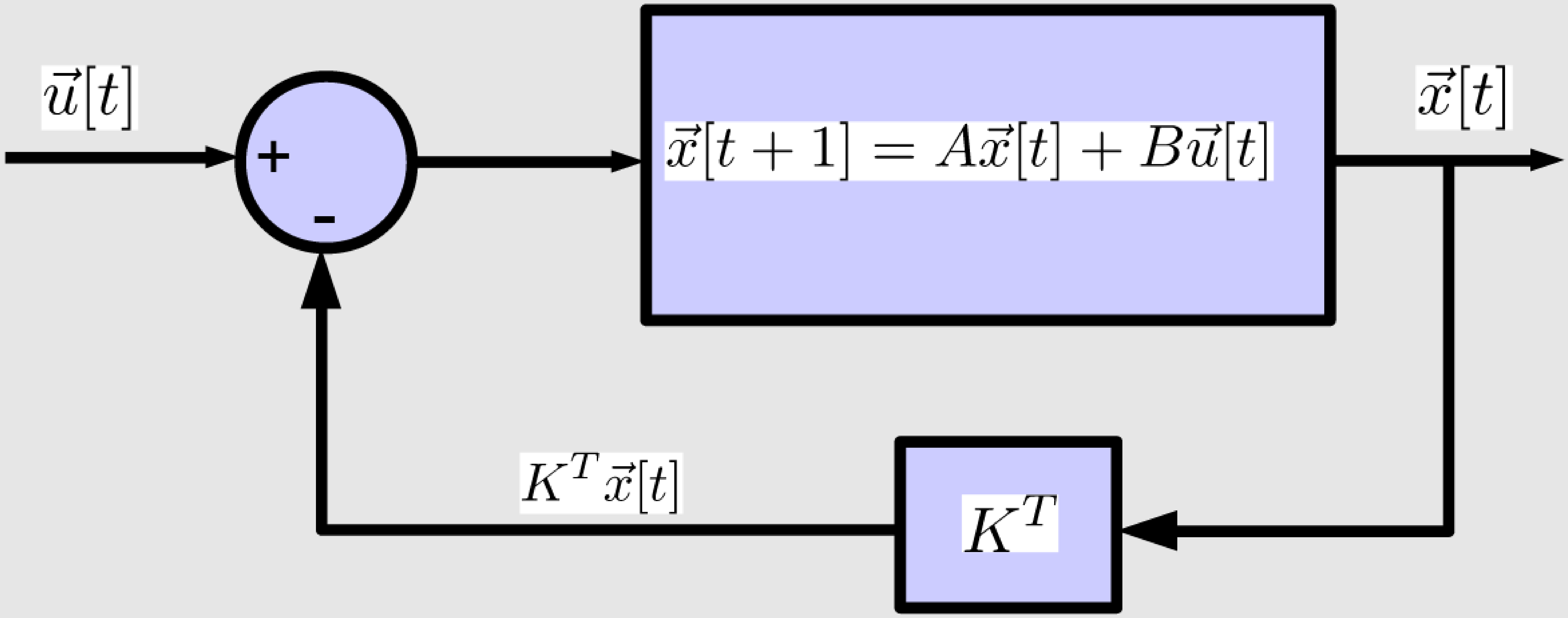
- to stabilize: make both evs -ve (real part)
  - choose any $\alpha_2 > -kl, \quad \alpha_1 > mg$

run MATLAB demo
inverted_pendulum_w_feedback_root_locus.m



Feedback for Discrete-Time S.S.Rs



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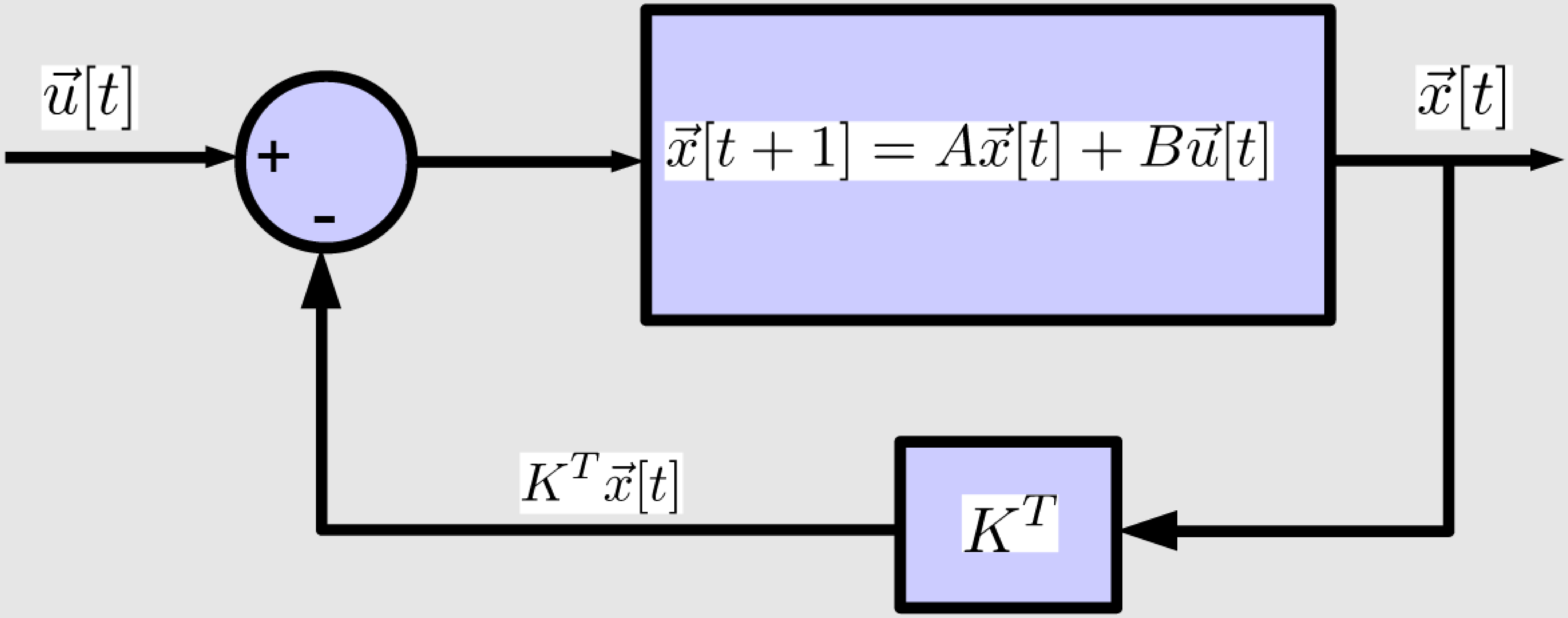
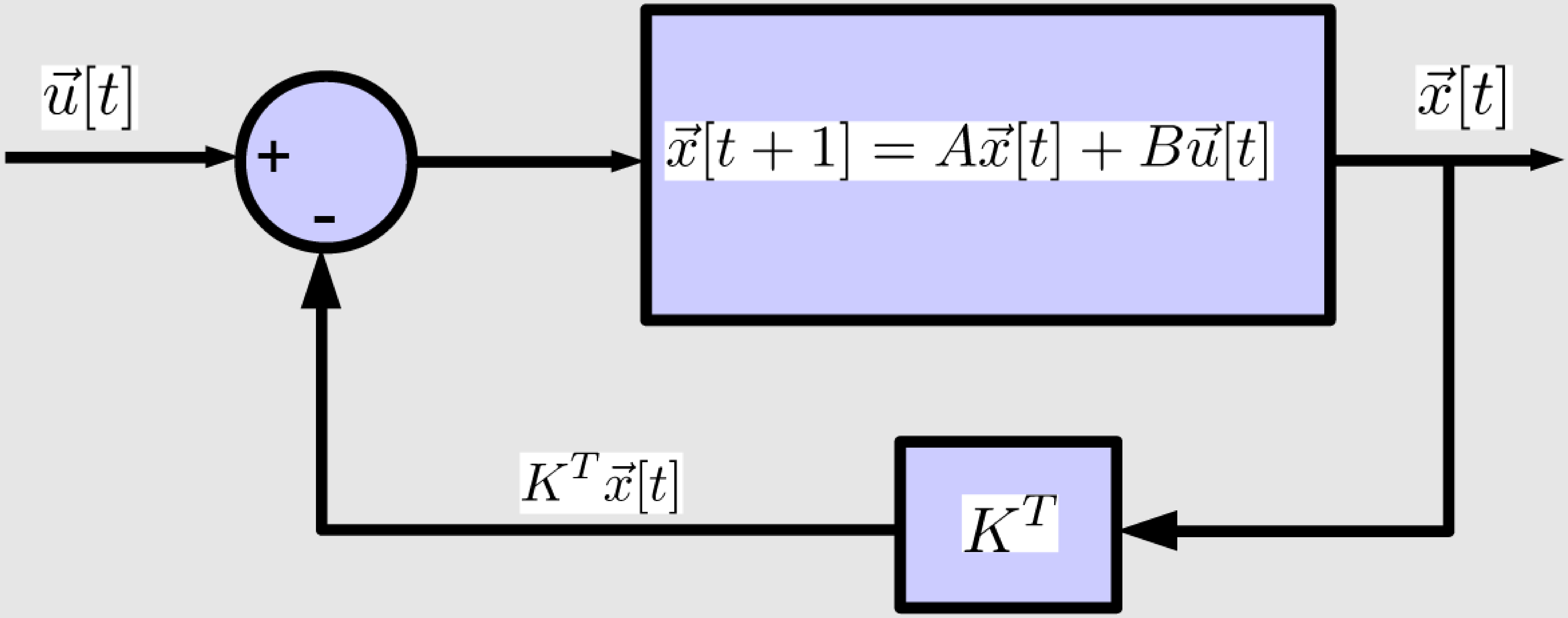


- system w feedback: $\vec{x}[t+1] = (A - B\vec{K}^T)\vec{x}[t] + B\vec{u}[t]$

Feedback for Discrete-Time S.S.Rs



- system w feedback: $\vec{x}[t+1] = (A - B\vec{K}^T)\vec{x}[t] + B\vec{u}[t]$
  - stability still governed by the eigenvalues of $A - BK^T$

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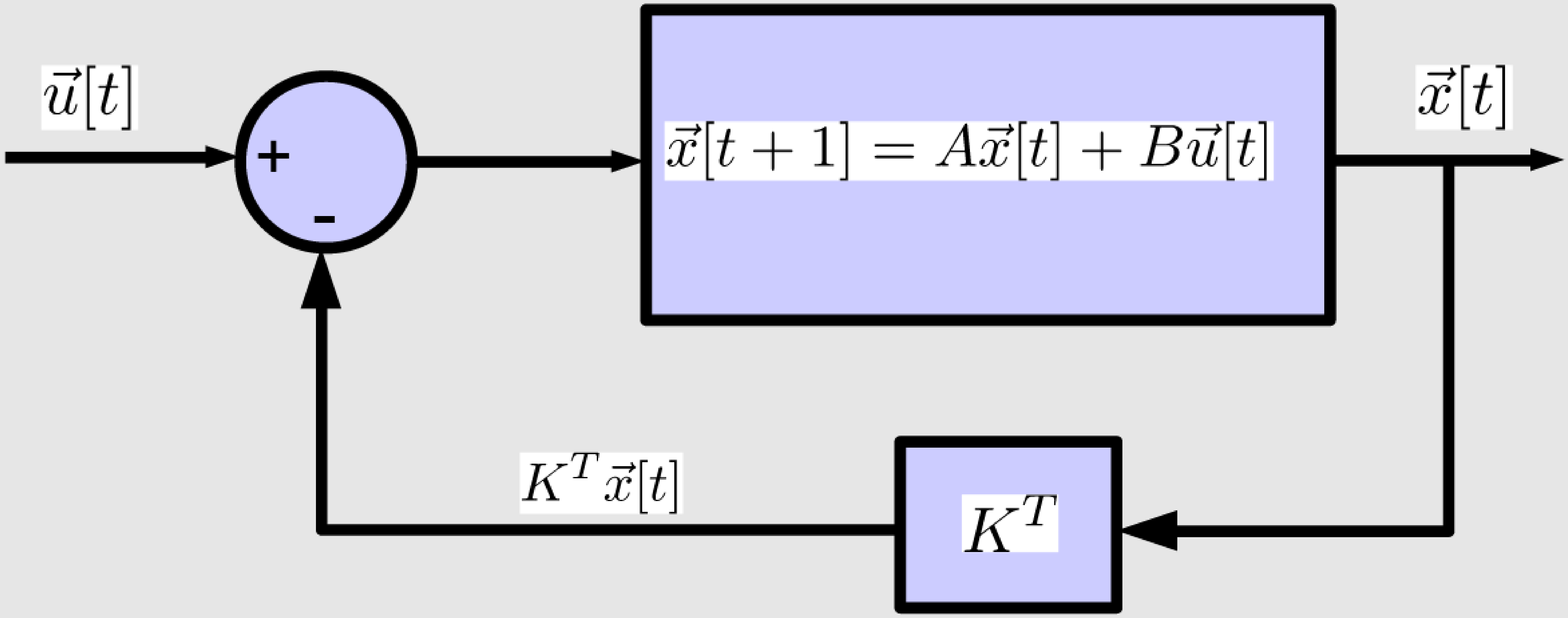

- system w feedback: $\vec{x}[t+1] = (A - B\vec{K}^T)\vec{x}[t] + B\vec{u}[t]$
  - stability still governed by the eigenvalues of $A - BK^T$
- stability (discr.) → <u>magnitude</u> of eigenvalues < 1
  - different from the continuous case

Example: Discrete-Time Feedback

- $\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$

Example: Discrete-Time Feedback

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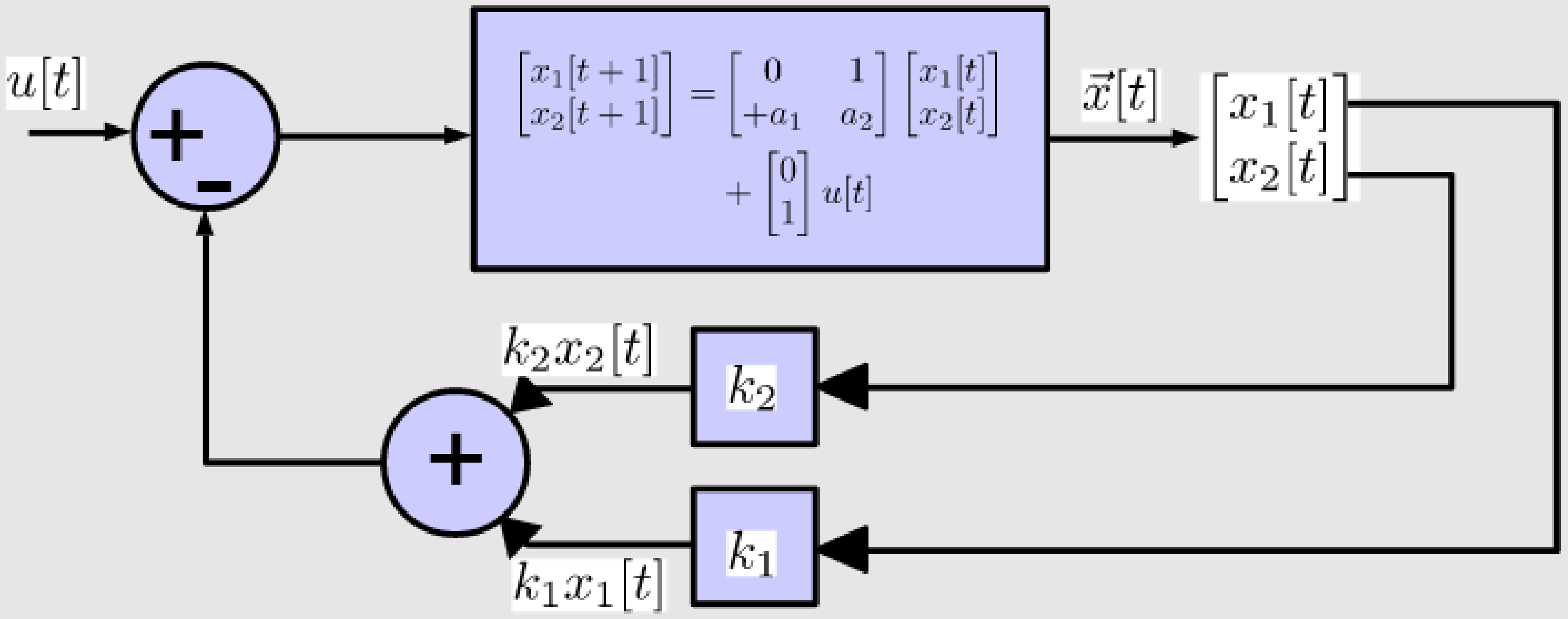

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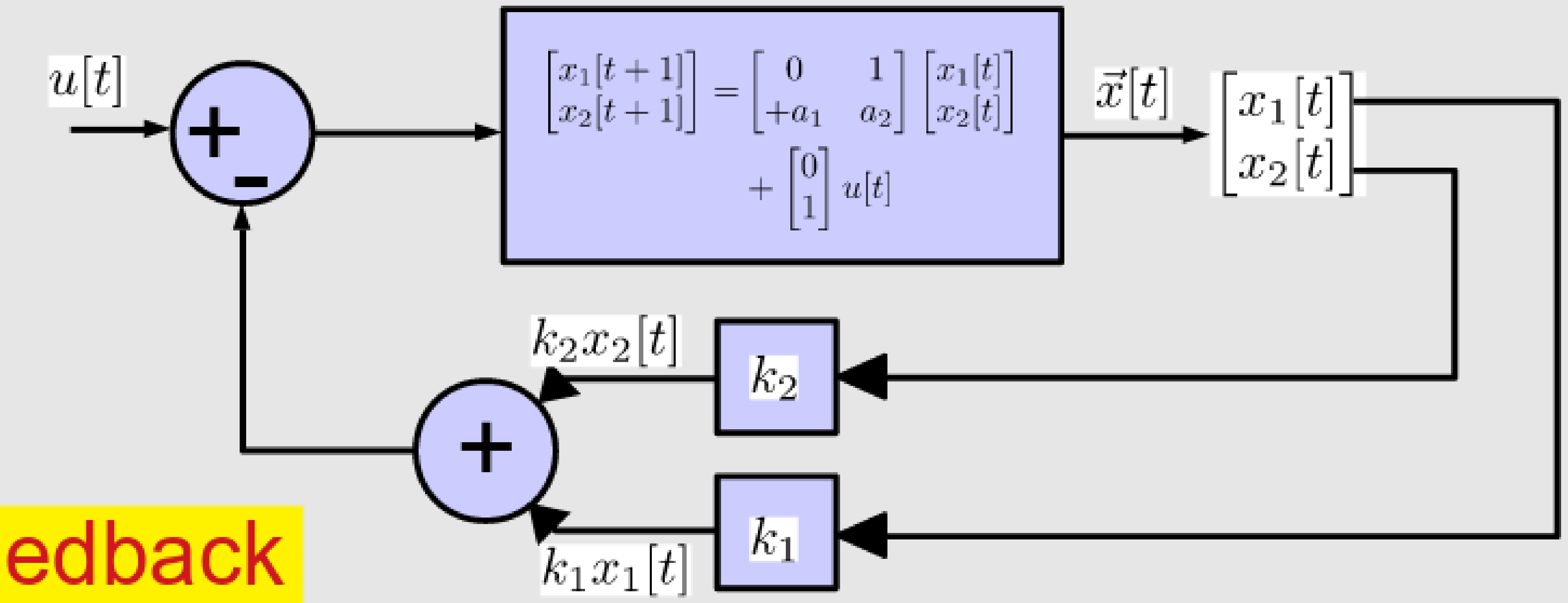
$$\boxed{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}}$$

w feedback



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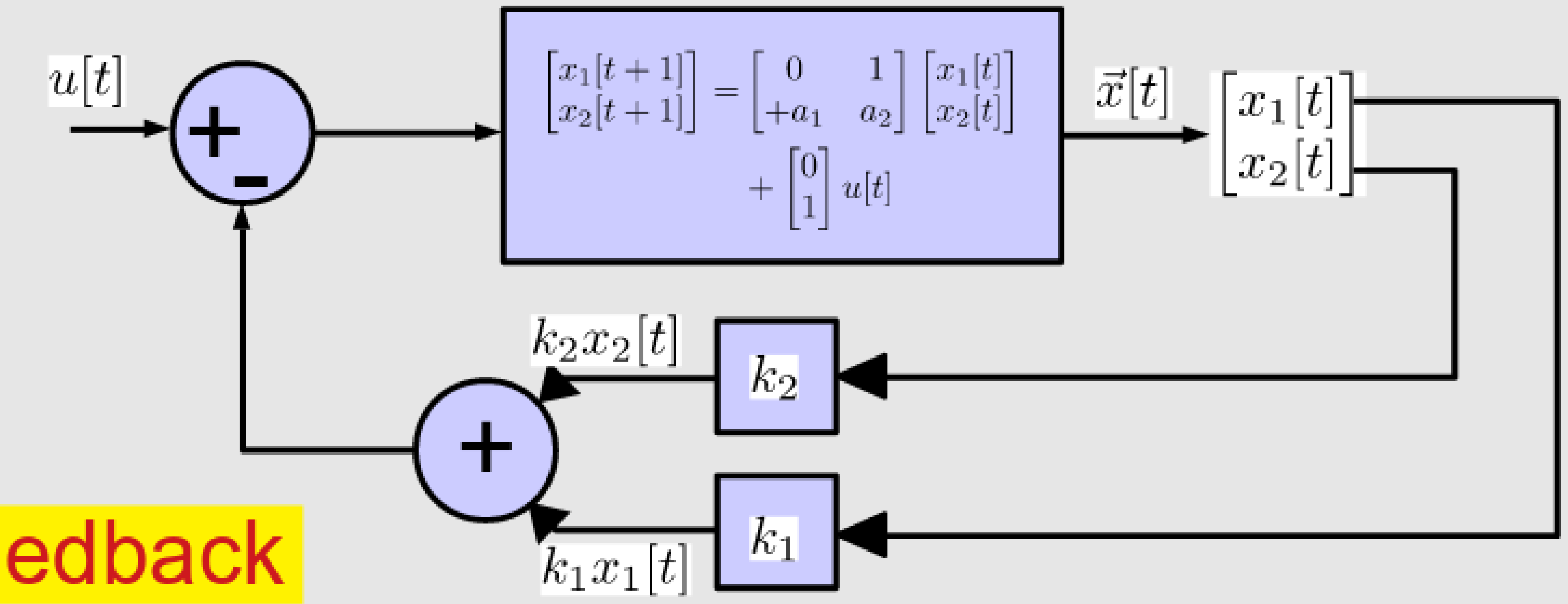
$$\boxed{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}} \longleftarrow \text{w feedback}$$



- char. poly.: $\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$

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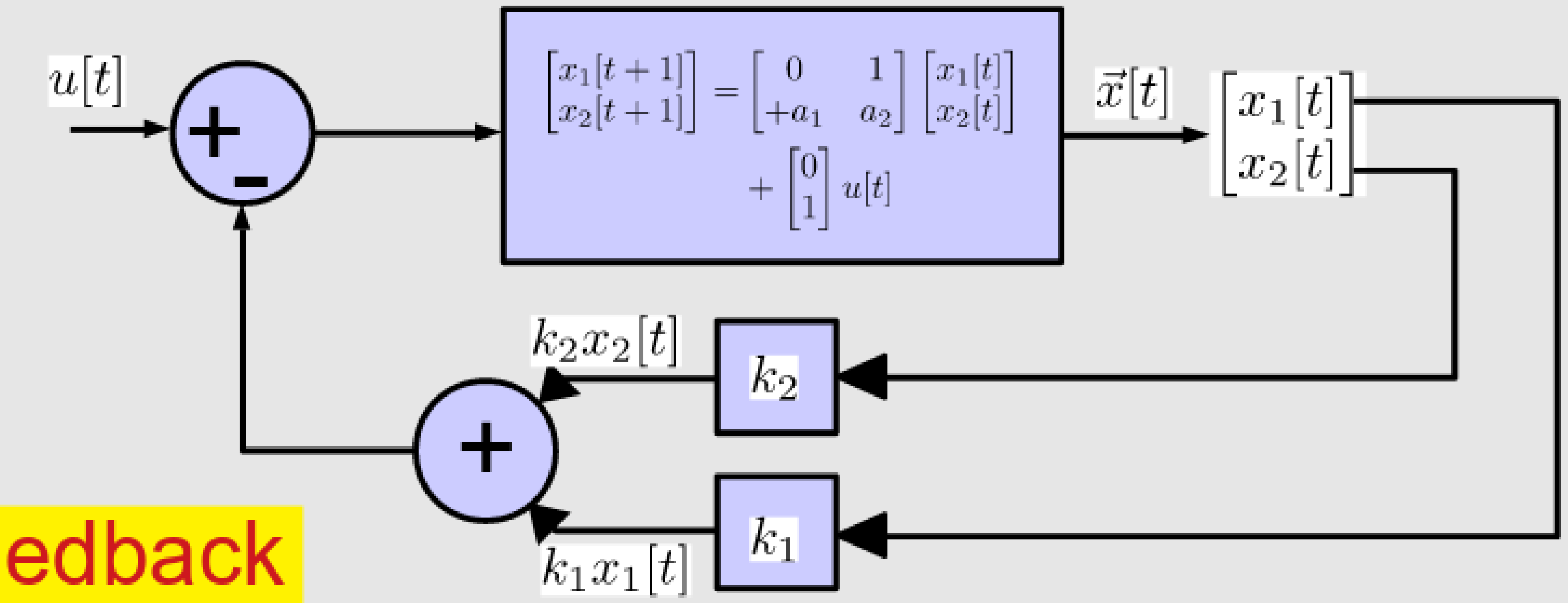
$$\boxed{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}}$$

w feedback



- char. poly.: $\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$
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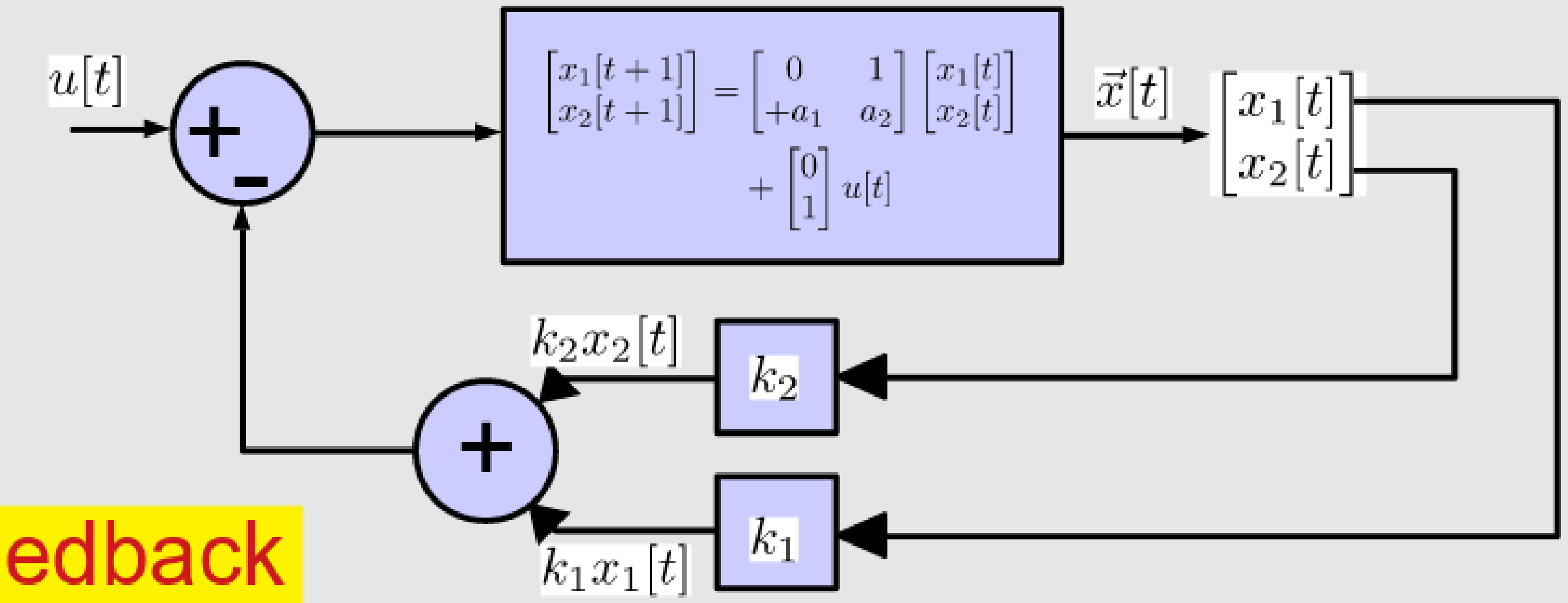


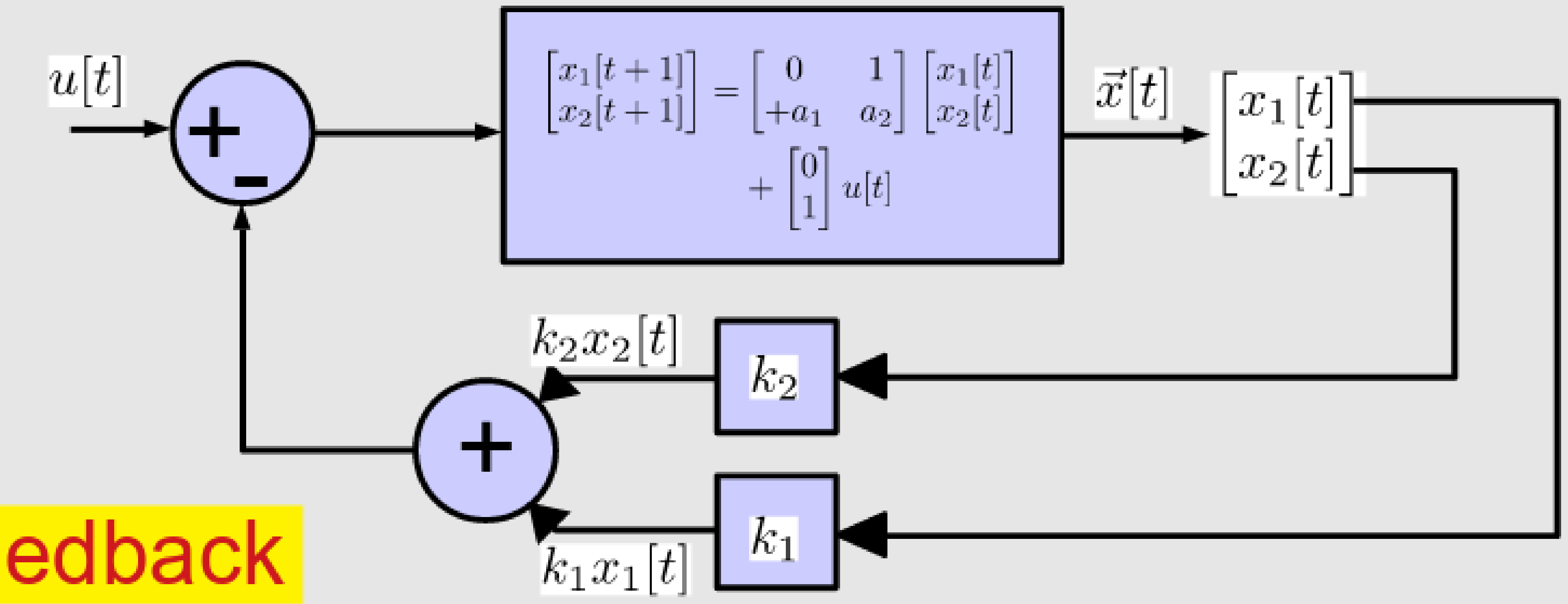
- char. poly.: $\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$
  - roots: $\lambda_{1,2} = \frac{a_2 - k_2}{2} \pm \frac{1}{2}\sqrt{(a_2 - k_2)^2 + 4(a_1 - k_1)}$
- easy to express k_1, k_2 in terms of λ_1, λ_2:
  - $k_1 = \lambda_1\lambda_2 - a_1$
    $k_2 = a_2 - \lambda_1 - \lambda_2$

Example: Discrete-Time Feedback

- $\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$

$$\boxed{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}}$$

w feedback



- char. poly.: $\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$
  - roots: $\lambda_{1,2} = \frac{a_2 - k_2}{2} \pm \frac{1}{2}\sqrt{(a_2 - k_2)^2 + 4(a_1 - k_1)}$
- easy to express k_1, k_2 in terms of λ_1, λ_2:
  - $k_1 = \lambda_1 \lambda_2 - a_1$
    $k_2 = a_2 - \lambda_1 - \lambda_2$

choose any λ_1 and λ_2 (eg, stable ones); set k_1 and k_2

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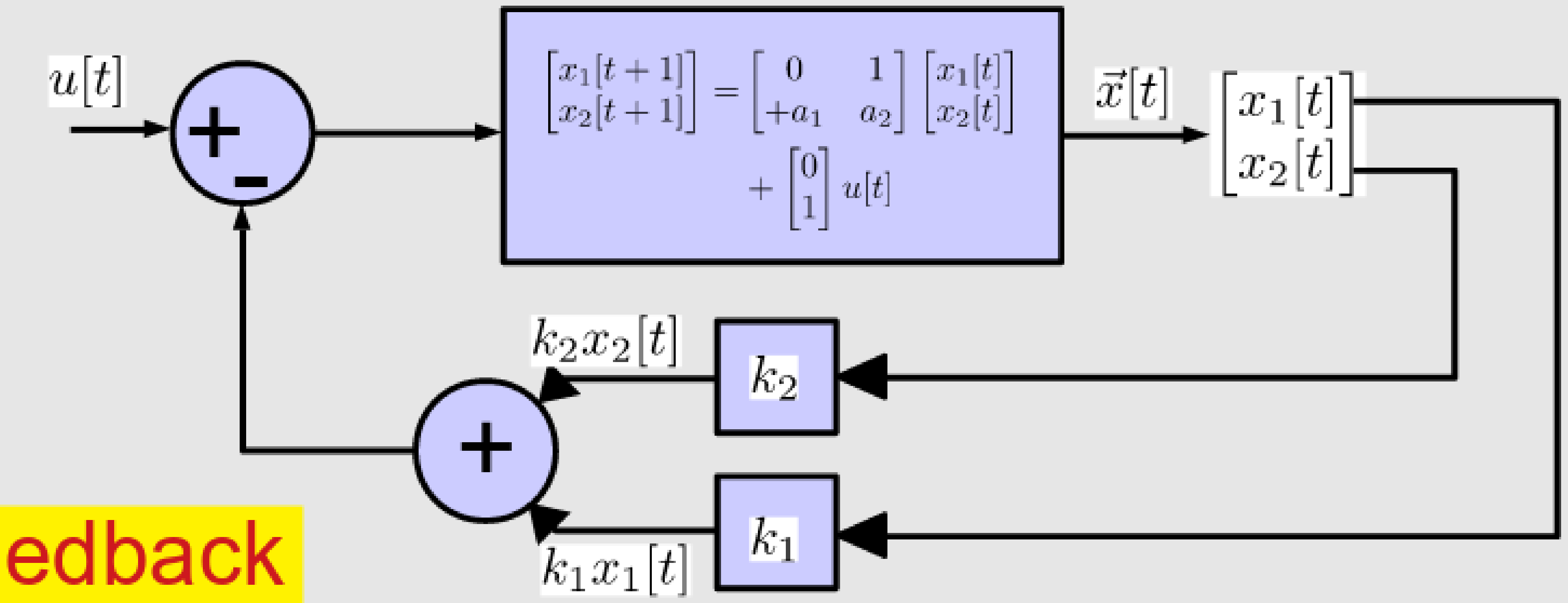

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 $k_2 = a_2 - \lambda_1 - \lambda_2$ ← choose any λ_1 and λ_2 (eg, stable ones); set k_1 and k_2
  - **<u>if</u>** λ_1 is complex: **<u>make sure</u>** λ_2 is the conjugate of λ_1!
    - ➔ otherwise, $k_1/k_2/x_1/x_2$ will have imaginary components
      - which would be physically meaningless

Another D-T. Feedback Example

- $\vec{x}[t+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$

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  - roots: $\lambda_1 = 1-k_1,\quad \boxed{\lambda_2 = 2}$ ← does not depend on k_1 or k_2; ie, cannot be altered via feedback
- suspicions (based on a few examples)
  - controllable → can place all eigenvalues via careful feedback
  - not controllable → might not be able to place all evs

Summary

- Controllability
  - controllability matrix must be full rank
    - C-H Theorem
  - examples: accelerating car (discrete), R-L1-L2 ckt
- Feedback
  - controllable + unstable = useless
    - **uncontrollable + unstable = REALLY useless?**
  - feedback (from state to input) can stabilize (evs moved)
    - inverted pendulum and other examples