# EE16B, Spring 2018
# UC Berkeley EECS

## Maharbiz and Roychowdhury

## Lectures 8A, 8B & 9A: Overview Slides

# Data Analysis

## Singular Value Decomposition
## and
## Principal Component Analysis

# The SVD
# (Singular Value Decomposition)

# Singular Value Decomposition

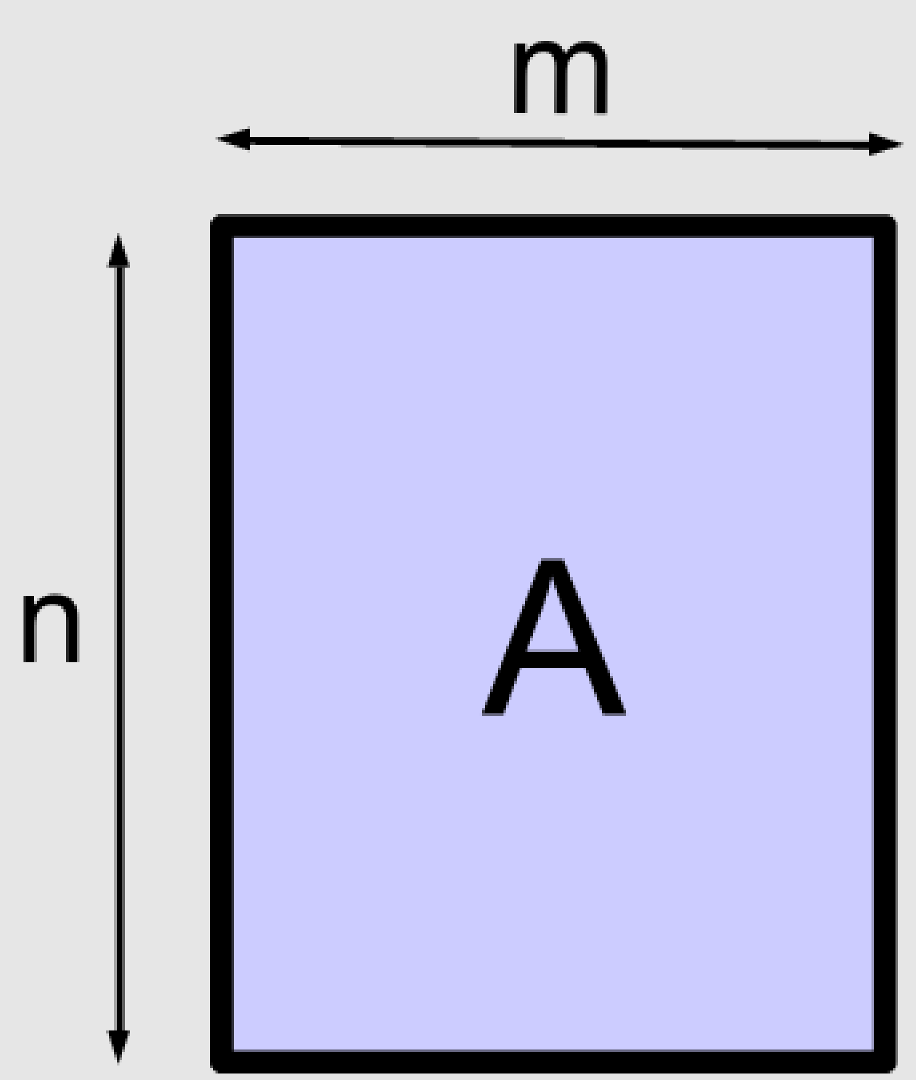
- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

$$A = U \Sigma V^T$$

# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

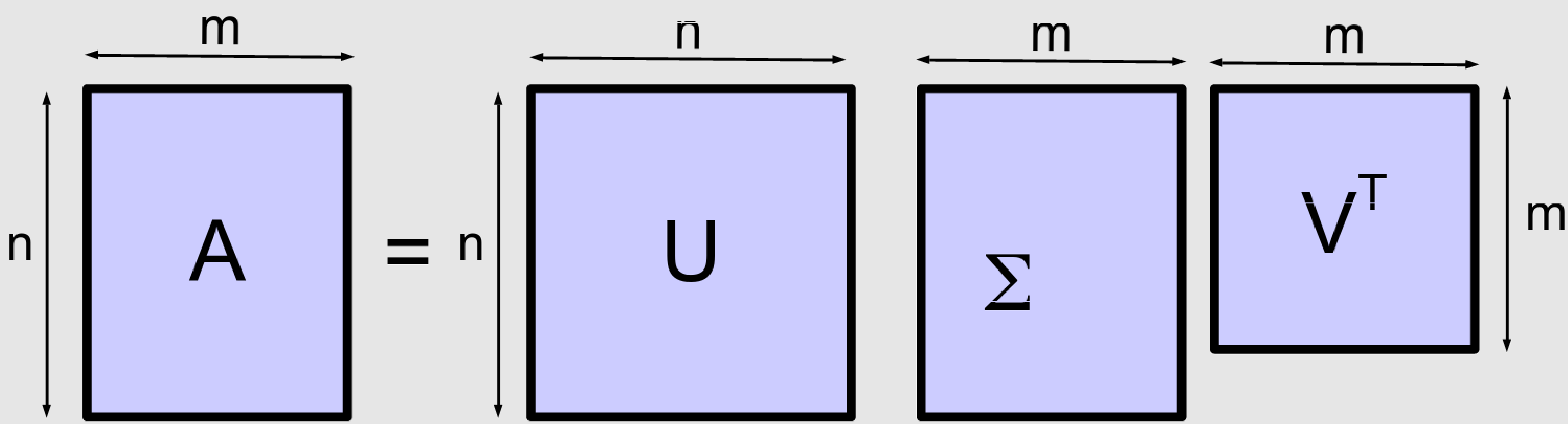
$$A = U \, \Sigma \, V^T$$



# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

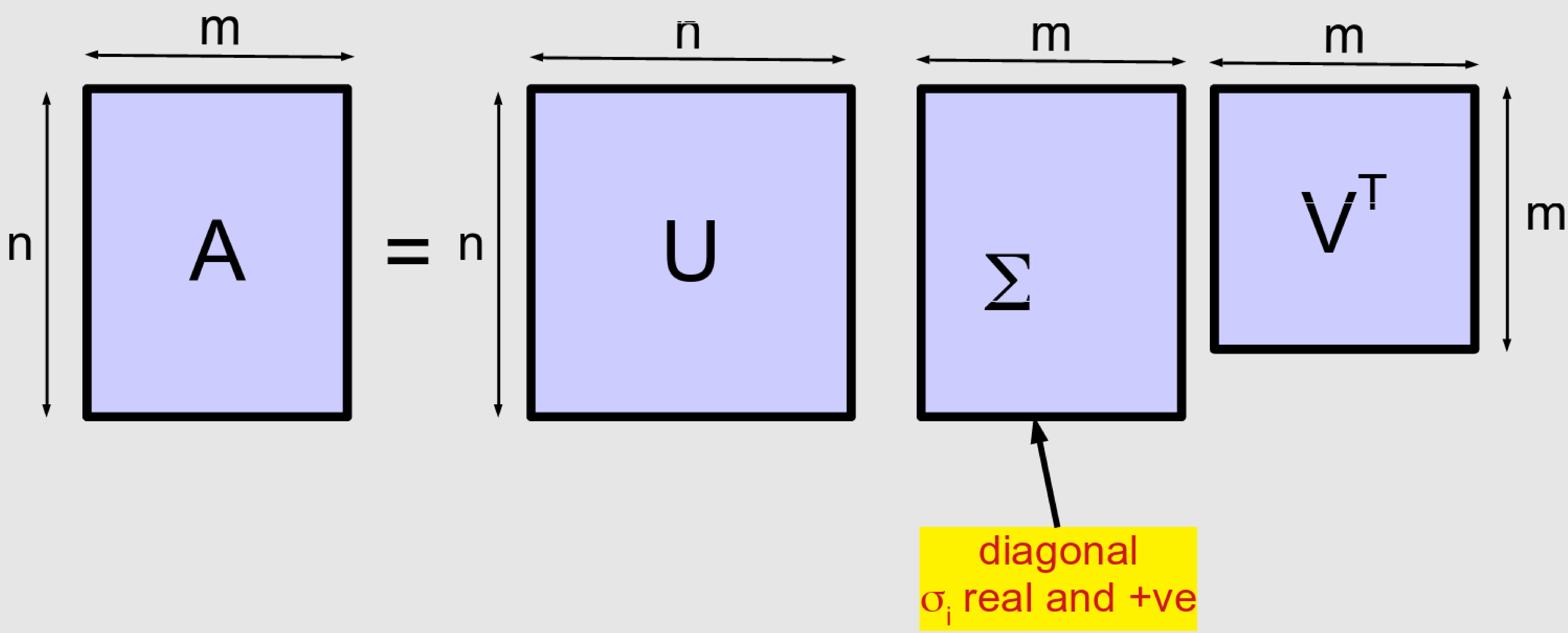
$$A = U\,\Sigma\,V^T$$



# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

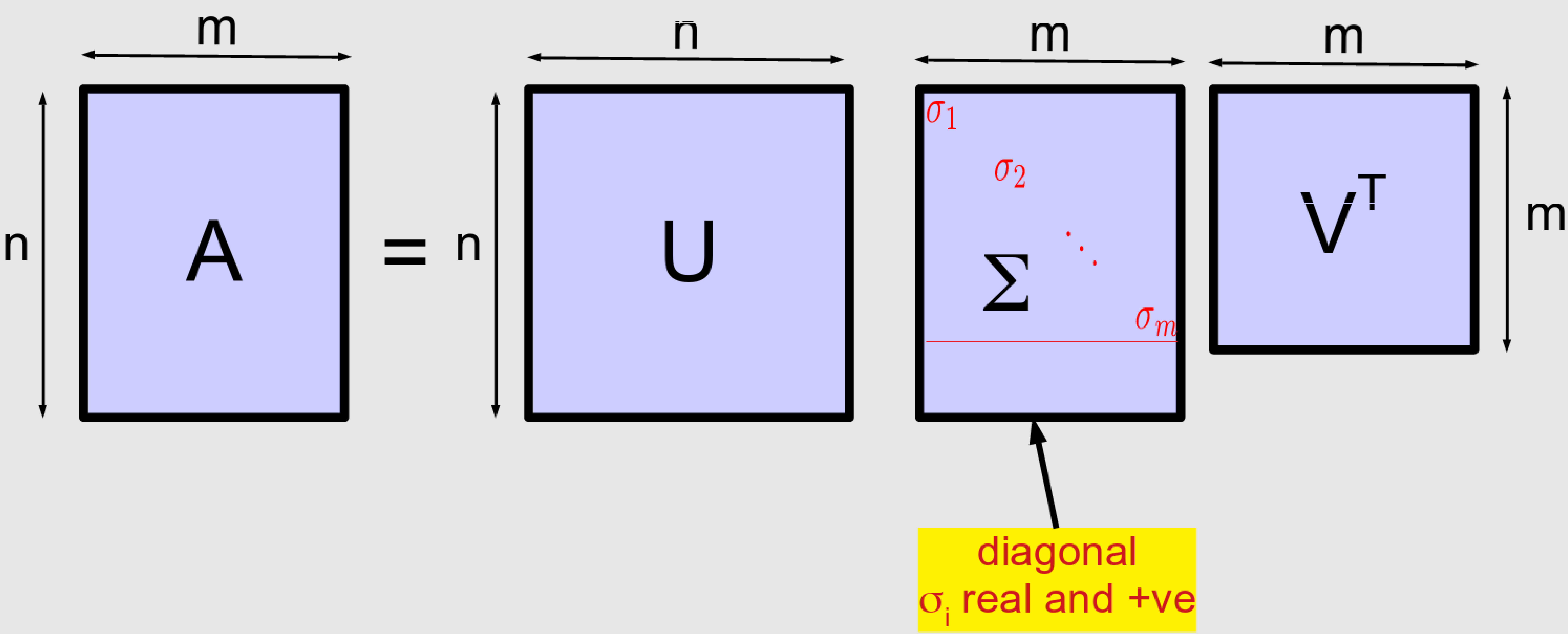
$$A = U\,\Sigma\,V^T$$



# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

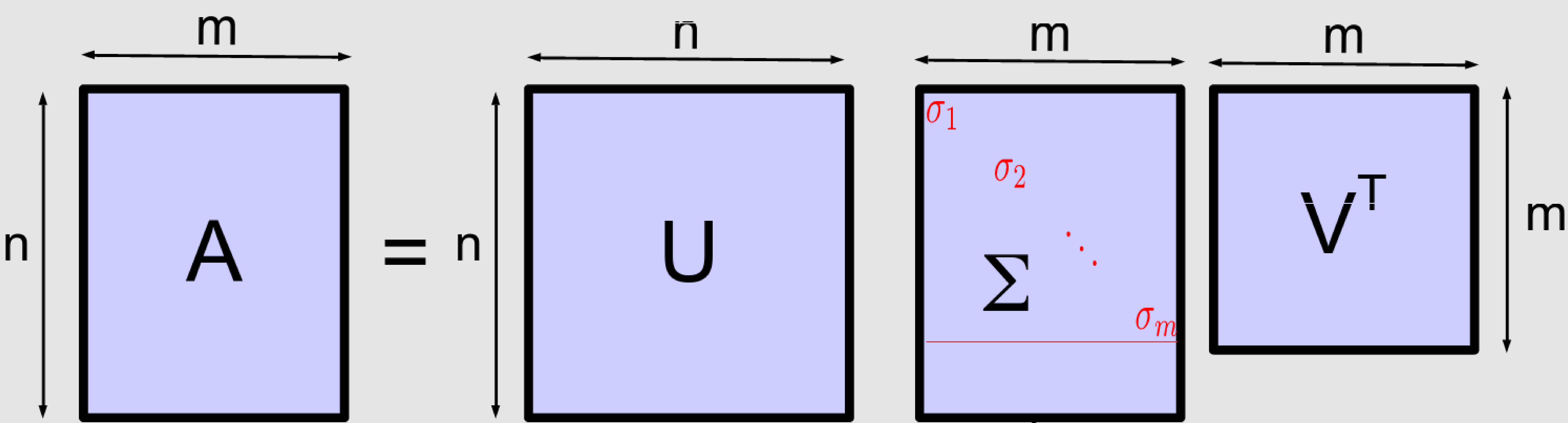
$$A = U \Sigma V^T$$



# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

$$A = U \Sigma V^T$$




**SINGULAR VALUES**

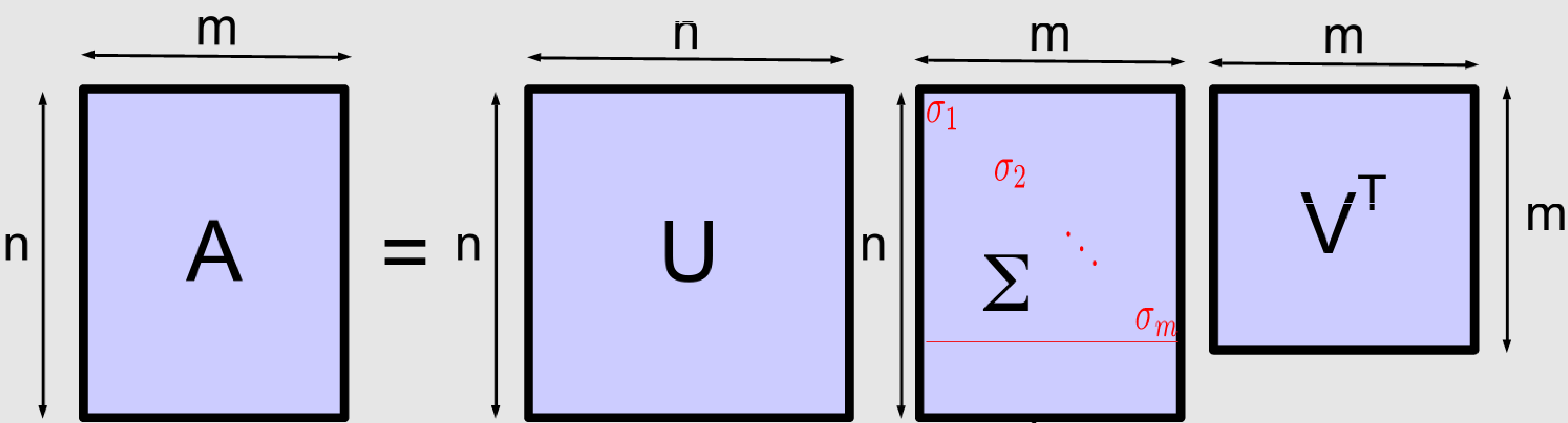
$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_m \geq 0$

diagonal
$\sigma_i$ real and +ve

# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

$$A = U\,\Sigma\,V^T$$




**SINGULAR VALUES**

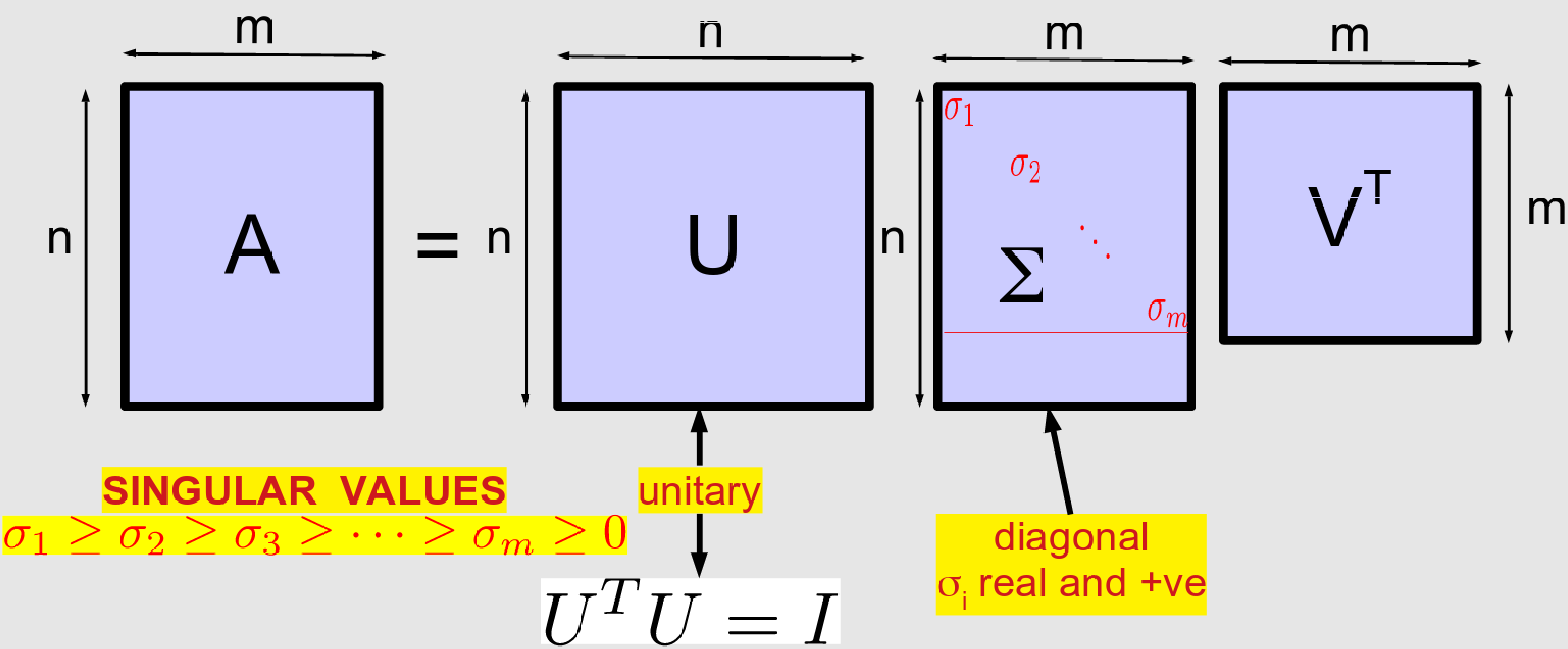
$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_m \geq 0$

diagonal
$\sigma_i$ real and +ve

# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

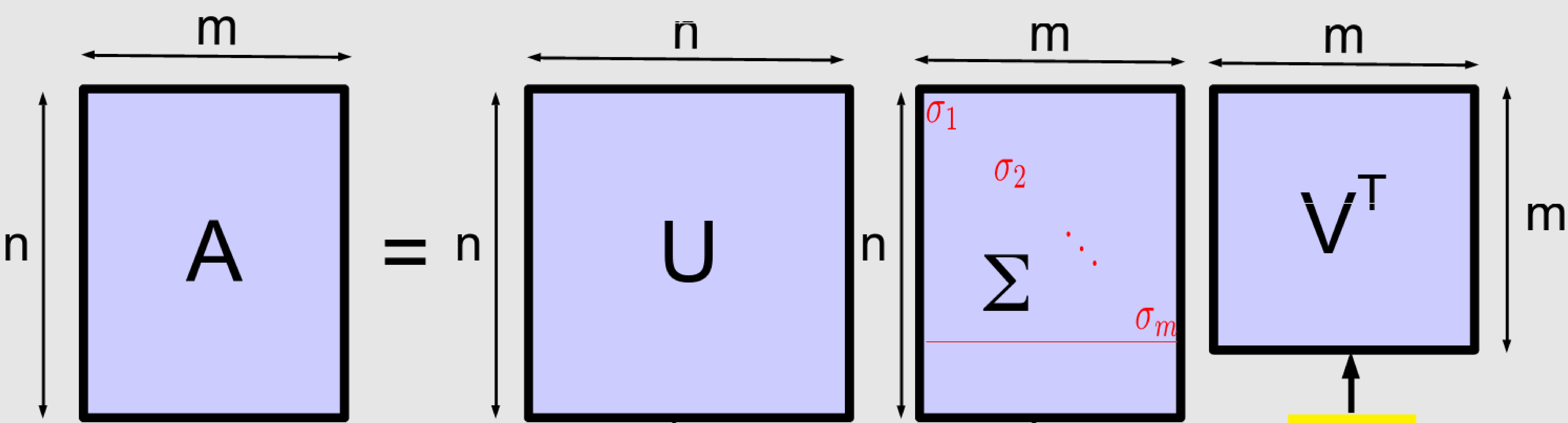
$$A = U \Sigma V^T$$



# Singular Value Decomposition

- Looks like eigendecomposition, **but is different**
  - **Any matrix A** (no exceptions) can be decomposed as

$$A = U \Sigma V^T$$




**SINGULAR VALUES**

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_m \geq 0$

unitary

$U^T U = I$

diagonal
$\sigma_i$ real and +ve

unitary

$V^T V = I$

# Unitary Matrices: Orthonormality

U

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

# Unitary Matrices: Orthonormality

$$\begin{array}{cc} U^T & U \\ \begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix} & \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} \end{array}$$

# Unitary Matrices: Orthonormality

$$\overset{U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

# Unitary Matrices: Orthonormality

$$\overset{U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

$\vec{u}_1^T \vec{u}_1 = 1$

# Unitary Matrices: Orthonormality

$$\overset{U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

$\vec{u}_1^T \vec{u}_1 = 1$ $\quad$ $\vec{u}_1^T \vec{u}_2 = 0$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{\mathrm{U^T}} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{\mathrm{U}} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{\mathrm{I}}$$

$\vec{u}_1^T \vec{u}_1 = 1$ $\quad \vec{u}_1^T \vec{u}_2 = 0$ $\quad \vec{u}_1^T \vec{u}_3 = 0$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{I}$$

$\vec{u}_1^T \vec{u}_1 = 1$ $\quad \vec{u}_1^T \vec{u}_2 = 0$ $\quad \vec{u}_1^T \vec{u}_3 = 0$ $\quad \cdots$ $\quad \vec{u}_1^T \vec{u}_n = 0$

# Unitary Matrices: Orthonormality

$$\overset{U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

$\vec{u}_1^T \vec{u}_1 = 1$ $\quad$ $\vec{u}_1^T \vec{u}_2 = 0$ $\quad$ $\vec{u}_1^T \vec{u}_3 = 0$ $\quad \cdots \quad$ $\vec{u}_1^T \vec{u}_n = 0$

$\vec{u}_2^T \vec{u}_1 = 0$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{I}$$

$\vec{u}_1^T \vec{u}_1 = 1$ $\quad \vec{u}_1^T \vec{u}_2 = 0$ $\quad \vec{u}_1^T \vec{u}_3 = 0$ $\quad \cdots$ $\quad \vec{u}_1^T \vec{u}_n = 0$

$\vec{u}_2^T \vec{u}_1 = 0$ $\quad \vec{u}_2^T \vec{u}_2 = 1$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{\color{blue}{U^T}} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{\color{blue}{U}} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{\color{blue}{I}}$$

$$\color{blue}{\vec{u}_1^T \vec{u}_1 = 1} \quad \vec{u}_1^T \vec{u}_2 = 0 \quad \vec{u}_1^T \vec{u}_3 = 0 \quad \cdots \quad \vec{u}_1^T \vec{u}_n = 0$$

$$\vec{u}_2^T \vec{u}_1 = 0 \quad \color{blue}{\vec{u}_2^T \vec{u}_2 = 1} \quad \vec{u}_3^T \vec{u}_3 = 0 \quad \cdots \quad \vec{u}_2^T \vec{u}_n = 0$$

# Unitary Matrices: Orthonormality

$$\underset{U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \underset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

| | | | | |
|---|---|---|---|---|
| $\vec{u}_1^T\vec{u}_1 = 1$ | $\vec{u}_1^T\vec{u}_2 = 0$ | $\vec{u}_1^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_1^T\vec{u}_n = 0$ |
| $\vec{u}_2^T\vec{u}_1 = 0$ | $\vec{u}_2^T\vec{u}_2 = 1$ | $\vec{u}_3^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_2^T\vec{u}_n = 0$ |
| | $\vdots$ | | $\vdots$ | |
| $\vec{u}_n^T\vec{u}_1 = 0$ | $\vec{u}_n^T\vec{u}_2 = 0$ | $\vec{u}_n^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_n^T\vec{u}_n = 1$ |

# Unitary Matrices: Orthonormality

$$\overset{\textstyle U^T}{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}} \overset{\textstyle U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} = \overset{\textstyle I}{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

$$\begin{array}{ccccc} \vec{u}_1^T\vec{u}_1 = 1 & \vec{u}_1^T\vec{u}_2 = 0 & \vec{u}_1^T\vec{u}_3 = 0 & \cdots & \vec{u}_1^T\vec{u}_n = 0 \\ \vec{u}_2^T\vec{u}_1 = 0 & \vec{u}_2^T\vec{u}_2 = 1 & \vec{u}_3^T\vec{u}_3 = 0 & \cdots & \vec{u}_2^T\vec{u}_n = 0 \\ & \vdots & & \vdots & \\ \vec{u}_n^T\vec{u}_1 = 0 & \vec{u}_n^T\vec{u}_2 = 0 & \vec{u}_n^T\vec{u}_3 = 0 & \cdots & \vec{u}_n^T\vec{u}_n = 1 \end{array}$$

$$\vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{I}$$

$$\begin{matrix} \vec{u}_1^T\vec{u}_1 = 1 & \vec{u}_1^T\vec{u}_2 = 0 & \vec{u}_1^T\vec{u}_3 = 0 & \cdots & \vec{u}_1^T\vec{u}_n = 0 \\ \vec{u}_2^T\vec{u}_1 = 0 & \vec{u}_2^T\vec{u}_2 = 1 & \vec{u}_3^T\vec{u}_3 = 0 & \cdots & \vec{u}_2^T\vec{u}_n = 0 \\ & \vdots & & \vdots & \\ \vec{u}_n^T\vec{u}_1 = 0 & \vec{u}_n^T\vec{u}_2 = 0 & \vec{u}_n^T\vec{u}_3 = 0 & \cdots & \vec{u}_n^T\vec{u}_n = 1 \end{matrix}$$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

$$\vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{I}$$

| | | | | |
|---|---|---|---|---|
| $\vec{u}_1^T\vec{u}_1 = 1$ | $\vec{u}_1^T\vec{u}_2 = 0$ | $\vec{u}_1^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_1^T\vec{u}_n = 0$ |
| $\vec{u}_2^T\vec{u}_1 = 0$ | $\vec{u}_2^T\vec{u}_2 = 1$ | $\vec{u}_3^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_2^T\vec{u}_n = 0$ |
| | $\vdots$ | | $\vdots$ | |
| $\vec{u}_n^T\vec{u}_1 = 0$ | $\vec{u}_n^T\vec{u}_2 = 0$ | $\vec{u}_n^T\vec{u}_3 = 0$ | $\cdots$ | $\vec{u}_n^T\vec{u}_n = 1$ |

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$
called **ORTHONORMAL**

$$\vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$\|\vec{u}_i\| = 1$

# Unitary Matrices: Orthonormality

$$\underbrace{\begin{bmatrix} \longleftarrow & \vec{u}_1^T & \longrightarrow \\ \longleftarrow & \vec{u}_2^T & \longrightarrow \\ \longleftarrow & \vec{u}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{u}_n^T & \longrightarrow \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}_{I}$$

$$\begin{array}{ccccc} \vec{u}_1^T\vec{u}_1 = 1 & \vec{u}_1^T\vec{u}_2 = 0 & \vec{u}_1^T\vec{u}_3 = 0 & \cdots & \vec{u}_1^T\vec{u}_n = 0 \\ \vec{u}_2^T\vec{u}_1 = 0 & \vec{u}_2^T\vec{u}_2 = 1 & \vec{u}_3^T\vec{u}_3 = 0 & \cdots & \vec{u}_2^T\vec{u}_n = 0 \\ & \vdots & & \vdots & \\ \vec{u}_n^T\vec{u}_1 = 0 & \vec{u}_n^T\vec{u}_2 = 0 & \vec{u}_n^T\vec{u}_3 = 0 & \cdots & \vec{u}_n^T\vec{u}_n = 1 \end{array}$$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$
called **ORTHONORMAL**

$$\vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$\|\vec{u}_i\| = 1$

Similarly,
$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$
are **ORTHONORMAL**

$\|\vec{v}_j\| = 1$

# Rank 1 Matrices and Outer Products

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$

# Rank 1 Matrices and Outer Products

rank=1

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$

# Rank 1 Matrices and Outer Products

- Consider $A = \overset{\text{rank=1}}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}} = \overset{\text{col}}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ row}$

# Rank 1 Matrices and Outer Products

- Consider $$\overset{\text{rank=1}}{A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}} = \overset{\text{col}}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \;\text{row}$$
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors
  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix}$

# Rank 1 Matrices and Outer Products

- Consider $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

  rank=1

  col

  row

  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors
  - $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$$

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

  rank=1

  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

  rank=1

  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

rank=1

  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very “simple” type of matrix
  - its “data” can be “compressed” very easily
    - → can be written as outer product: $A = \vec{x}\vec{y}^T$

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

  rank=1
  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily
    - ➔ can be written as outer product: $A = \vec{x}\vec{y}^T$

      nxm

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix}1&1&1&1&1\\3&3&3&3&3\\2&2&2&2&2\end{bmatrix} = \begin{bmatrix}1\\3\\2\end{bmatrix}\begin{bmatrix}1&1&1&1&1\end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

rank=1

- $\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}a&b&c&d&e\end{bmatrix} = \begin{bmatrix}xa&xb&xc&xd&xe\\ya&yb&yc&yd&ye\\za&zb&zc&zd&ze\end{bmatrix}$
- rank-1: a very “simple” type of matrix
  - its “data” can be “compressed” very easily
    - → can be written as outer product: $A = \vec{x}\vec{y}^T$

nxm nx1

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

  rank=1

  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily
    - → can be written as outer product: $A = \vec{x}\vec{y}^T$

      nxm nx1 1xm

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

rank=1

  - $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily
    - → can be written as outer product: $A = \vec{x}\vec{y}^T$

nm numbers nxm nx1 1xm

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

rank=1

- $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily
    - can be written as outer product: $A = \vec{x}\vec{y}^T$

nm numbers nxm nx1 1xm

n+m numbers

# Rank 1 Matrices and Outer Products

rank=1 col

- Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ row
  - rank-1 matrix can be written as $\vec{x}\vec{y}^T$: an **outer product**
- outer product: product of col and row vectors

rank=1

- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = \begin{bmatrix} xa & xb & xc & xd & xe \\ ya & yb & yc & yd & ye \\ za & zb & zc & zd & ze \end{bmatrix}$
- rank-1: a very "simple" type of matrix
  - its "data" can be "compressed" very easily
    - ➔ can be written as outer product: $A = \vec{x}\vec{y}^T$

nm numbers nxm nx1 1xm

n+m << nm: **data compression**

n+m numbers

# Matrix Multiplication using Outer Products

$$\overset{X}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \quad \overset{Y^T}{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}$$

# Matrix Multiplication using Outer Products

$$\begin{array}{cc} X & Y^T \\ \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} & \begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix} \end{array}$$

$$= \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

# Matrix Multiplication using Outer Products

each of these is a rank-1 OUTER PRODUCT

$$\overset{X}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{Y^T}{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}} = \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

# Matrix Multiplication using Outer Products

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{\mathrm{X}} \underbrace{\begin{bmatrix} \longleftarrow & \vec{y}_1^{\,T} & \longrightarrow \\ \longleftarrow & \vec{y}_2^{\,T} & \longrightarrow \\ \longleftarrow & \vec{y}_3^{\,T} & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^{\,T} & \longrightarrow \end{bmatrix}}_{\mathrm{Y}^\mathrm{T}}$$

**each of these is a rank-1 OUTER PRODUCT**

$$= \vec{x}_1\vec{y}_1^{\,T} + \vec{x}_2\vec{y}_2^{\,T} + \vec{x}_3\vec{y}_3^{\,T} + \cdots + \vec{x}_n\vec{y}_n^{\,T}$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\overset{\textstyle X}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\textstyle Y^{T}}{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}$$

each of these is a rank-1 OUTER PRODUCT

$$= \vec{x}_1 \vec{y}_1^T + \vec{x}_2 \vec{y}_2^T + \vec{x}_3 \vec{y}_3^T + \cdots + \vec{x}_n \vec{y}_n^T$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax + bp & ay + bq & az + br \\ cx + dp & cy + dq & cz + dr \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\overset{X}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{Y^T}{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}$$

**each of these is a rank-1 OUTER PRODUCT**

$$= \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax + bp & ay + bq & az + br \\ cx + dp & cy + dq & cz + dr \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}_{Y^T}$$

each of these is a rank-1 OUTER PRODUCT

$$= \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax+bp & ay+bq & az+br \\ cx+dp & cy+dq & cz+dr \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ cx & cy & cz \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}_{Y^T} = \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

**each of these is a rank-1 OUTER PRODUCT**

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax+bp & ay+bq & az+br \\ cx+dp & cy+dq & cz+dr \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ cx & cy & cz \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} \begin{bmatrix} p & q & r \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}_{Y^T}$$

**each of these is a rank-1 OUTER PRODUCT**

$$= \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax + bp & ay + bq & az + br \\ cx + dp & cy + dq & cz + dr \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ cx & cy & cz \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} \begin{bmatrix} p & q & r \end{bmatrix} = \begin{bmatrix} bp & bq & br \\ dp & dq & dr \end{bmatrix}$$

# Matrix Multiplication using Outer Products

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \longleftarrow & \vec{y}_1^T & \longrightarrow \\ \longleftarrow & \vec{y}_2^T & \longrightarrow \\ \longleftarrow & \vec{y}_3^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{y}_n^T & \longrightarrow \end{bmatrix}}_{Y^T}$$

each of these is a rank-1 OUTER PRODUCT

$$= \vec{x}_1\vec{y}_1^T + \vec{x}_2\vec{y}_2^T + \vec{x}_3\vec{y}_3^T + \cdots + \vec{x}_n\vec{y}_n^T$$

- Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} ax+bp & ay+bq & az+br \\ cx+dp & cy+dq & cz+dr \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ cx & cy & cz \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} \begin{bmatrix} p & q & r \end{bmatrix} = \begin{bmatrix} bp & bq & br \\ dp & dq & dr \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} \begin{bmatrix} p & q & r \end{bmatrix}}$$

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

# SVD: Sum of Outer Products Form

$$
A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}
\overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}}
\overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}
$$

$$
= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}
$$

# SVD: Sum of Outer Products Form

$$
A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}
\overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}}
\overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}
$$

$$
= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}
$$

outer product

n x m rank-1

matrix

$\vec{u}_1 \vec{v}_1^T$

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix}$$

outer product

nxm rank-1 matrix

$\vec{u}_1 \vec{v}_1^T$

# SVD: Sum of Outer Products Form

$$
A = \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U}
\underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}}_{\Sigma}
\underbrace{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}_{V^T}
$$

$$
= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix}
$$

outer product
nxm rank-1
matrix
$\vec{u}_1 \vec{v}_1^T$

outer product
nxm rank-1
matrix
$\vec{u}_2 \vec{v}_2^T$

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix} + \ldots + \sigma_m \begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}$$

outer product
nxm rank-1
matrix
$\vec{u}_1\vec{v}_1^T$

outer product
nxm rank-1
matrix
$\vec{u}_2\vec{v}_2^T$

outer product
nxm rank-1
matrix
$\vec{u}_m\vec{v}_m^T$

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

$$= \sigma_1 \underbrace{\begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}}_{\substack{\text{outer product} \\ \text{nxm rank-1} \\ \text{matrix} \\ \vec{u}_1\vec{v}_1^T}} + \sigma_2 \underbrace{\begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix}}_{\substack{\text{outer product} \\ \text{nxm rank-1} \\ \text{matrix} \\ \vec{u}_2\vec{v}_2^T}} + \ldots + \sigma_m \underbrace{\begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}_{\substack{\text{outer product} \\ \text{nxm rank-1} \\ \text{matrix} \\ \vec{u}_m\vec{v}_m^T}}$$

**Frobenius norm** (sqrt(sum of squares) = 1)

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

**biggest weight**

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix} + \ldots + \sigma_m \begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}$$

outer product nxm rank-1 matrix $\vec{u}_1 \vec{v}_1^T$

outer product nxm rank-1 matrix $\vec{u}_2 \vec{v}_2^T$

outer product nxm rank-1 matrix $\vec{u}_m \vec{v}_m^T$

**Frobenius norm** (sqrt(sum of squares) = 1)

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

biggest weight

next biggest weight

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix} + \ldots + \sigma_m \begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}$$

outer product nxm rank-1 matrix $\vec{u}_1\vec{v}_1^T$

outer product nxm rank-1 matrix $\vec{u}_2\vec{v}_2^T$

outer product nxm rank-1 matrix $\vec{u}_m\vec{v}_m^T$

**Frobenius norm** (sqrt(sum of squares) = 1)

# SVD: Sum of Outer Products Form

$$A = \overset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}$$

biggest weight

next biggest weight

smallest weight

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix} + \ldots + \sigma_m \begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}$$

outer product n×m rank-1 matrix $\vec{u}_1\vec{v}_1^T$

outer product n×m rank-1 matrix $\vec{u}_2\vec{v}_2^T$

outer product n×m rank-1 matrix $\vec{u}_m\vec{v}_m^T$

**Frobenius norm** (sqrt(sum of squares) = 1)

# SVD: Sum of Outer Products Form

$$A = \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_m \\ & & & & \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \\ \longleftarrow & \vec{v}_2^T & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}}_{V^T}$$

**biggest weight** — **next biggest weight** — **smallest weight**

$$= \sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix} + \sigma_2 \begin{bmatrix} \uparrow \\ \vec{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_2^T & \longrightarrow \end{bmatrix} + \ldots + \sigma_m \begin{bmatrix} \uparrow \\ \vec{u}_m \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_m^T & \longrightarrow \end{bmatrix}$$

outer product n x m rank-1 matrix $\vec{u}_1 \vec{v}_1^T$

outer product n x m rank-1 matrix $\vec{u}_2 \vec{v}_2^T$

outer product n x m rank-1 matrix $\vec{u}_m \vec{v}_m^T$

**Frobenius norm** (sqrt(sum of squares) = 1)

SVD splits a matrix into a **weighted sum of rank-1 matrices of norm 1**

# Using the SVD for Image Analysis and Compression

# Example: B&W Polish Flag as a Matrix

- size: 281x450

$A =$ 

original: 3.2MB

# Example: B&W Polish Flag as a Matrix

- size: 281x450

$A =$

original: 3.2MB

rank=1: 58kB

# Example: B&W Polish Flag as a Matrix

- size: 281x450

A =

original: 3.2MB

rank=1: 58kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

# Example: B&W Polish Flag as a Matrix

- size: 281x450

A =

original: 3.2MB

rank=1: 58kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0.2 \\ \vdots \\ 0.2 \end{bmatrix}$$

(actual values are normalized)

# Example: B&W Polish Flag as a Matrix

- size: 281x450

A =  

original: 3.2MB    rank=1: 58kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0.2 \\ \vdots \\ 0.2 \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

(actual values are normalized)

(actual values are normalized)

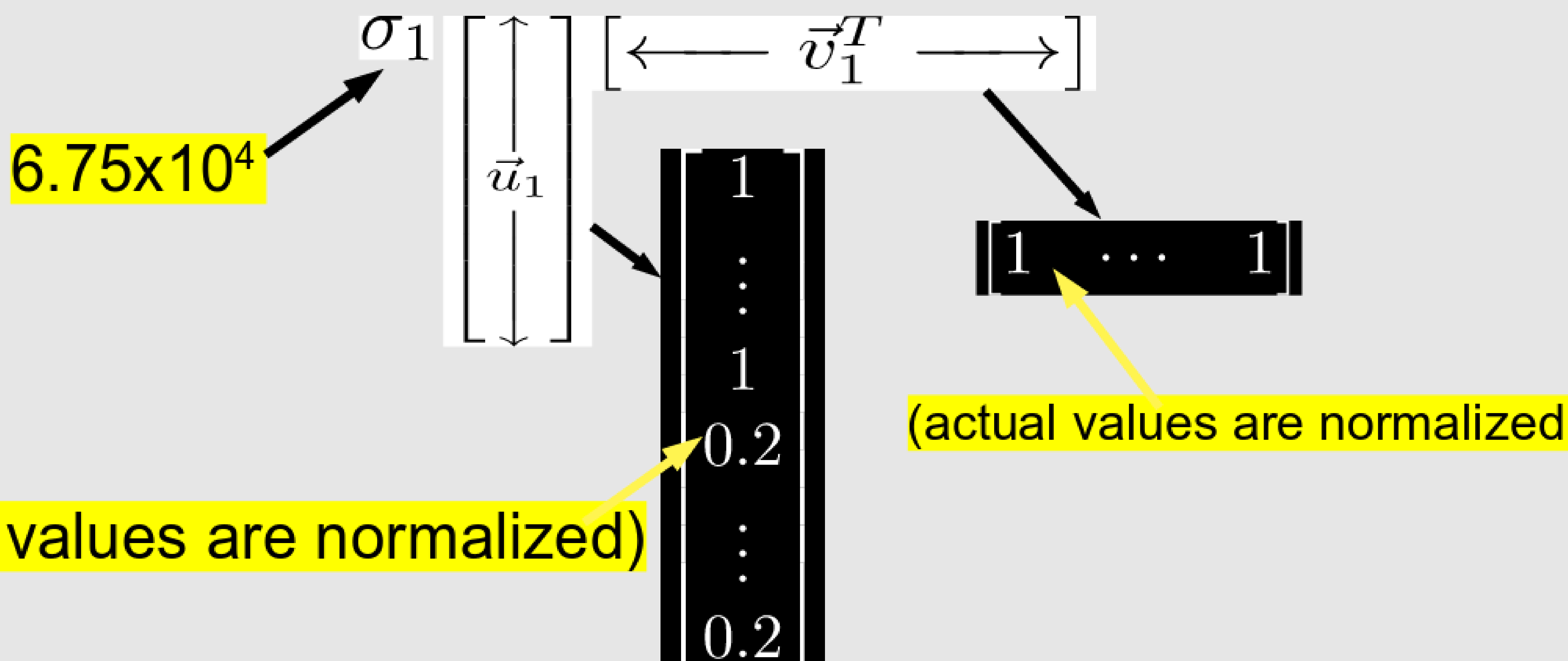
# Example: B&W Polish Flag as a Matrix

- size: 281x450

A =

original: 3.2MB

rank=1: 58kB



# Example: B&W Polish Flag as a Matrix

- size: 281x450

A = 

original: 3.2MB

rank=1: 58kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

6.75x10$^4$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0.2 \\ \vdots \\ 0.2 \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$\sigma_2 \simeq 10^{-10}$$

(actual values are normalized)

(actual values are normalized)

# Example: B&W Polish Flag as a Matrix

- size: 281x450

A =

original: 3.2MB

rank=1: 58kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

$6.75 \times 10^4$

$$\sigma_2 \simeq 10^{-10}$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0.2 \\ \vdots \\ 0.2 \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

(actual values are normalized)

(actual values are normalized)

**This is a RANK-1 FLAG**

# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A =

original: 10MB

# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A = 

original: 10MB

rank 1: 17.5kB

# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A = 

original: 10MB

rank 1: 17.5kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

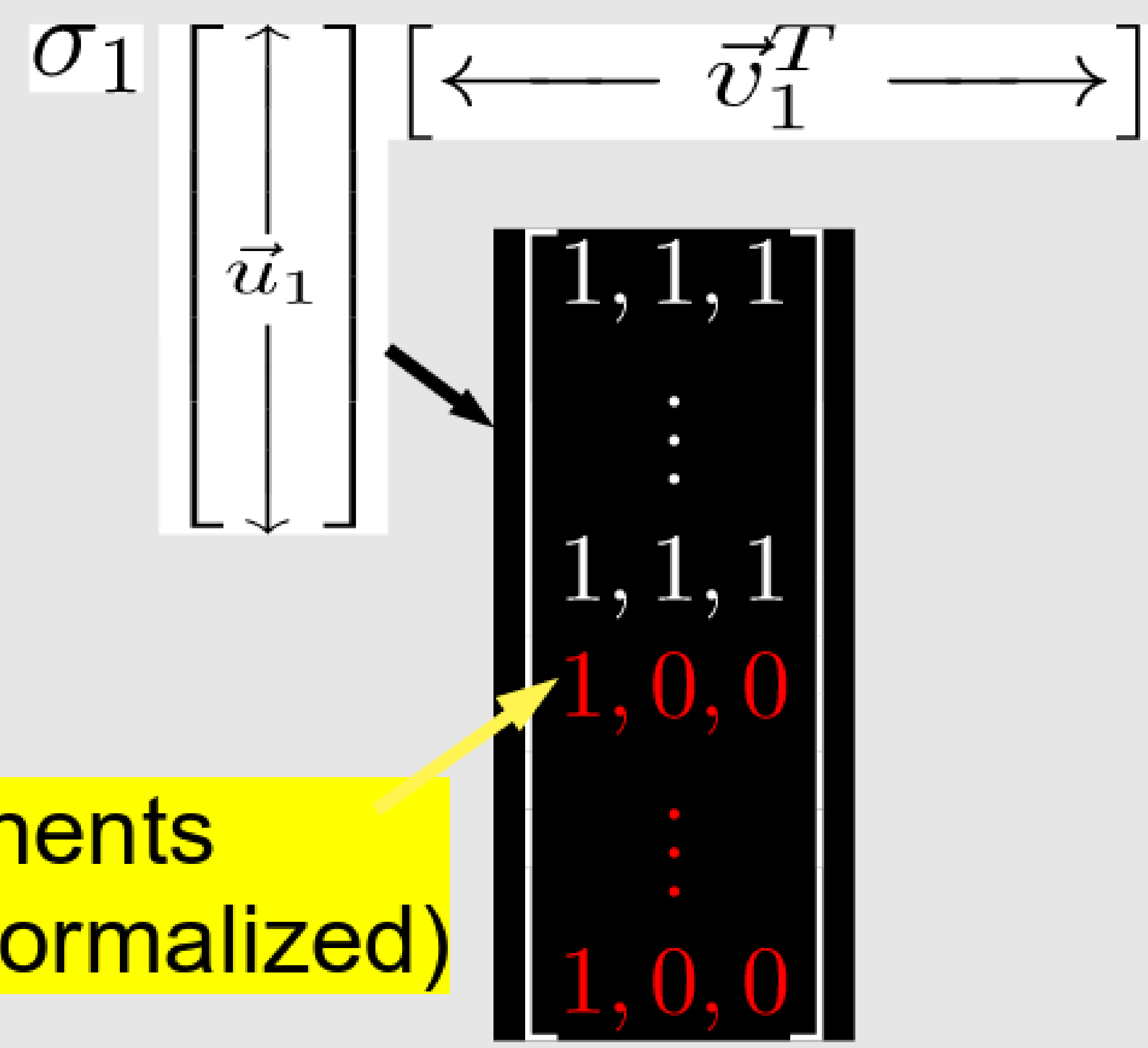
# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A =

original: 10MB

rank 1: 17.5kB




RGB components
(actual ones are normalized)

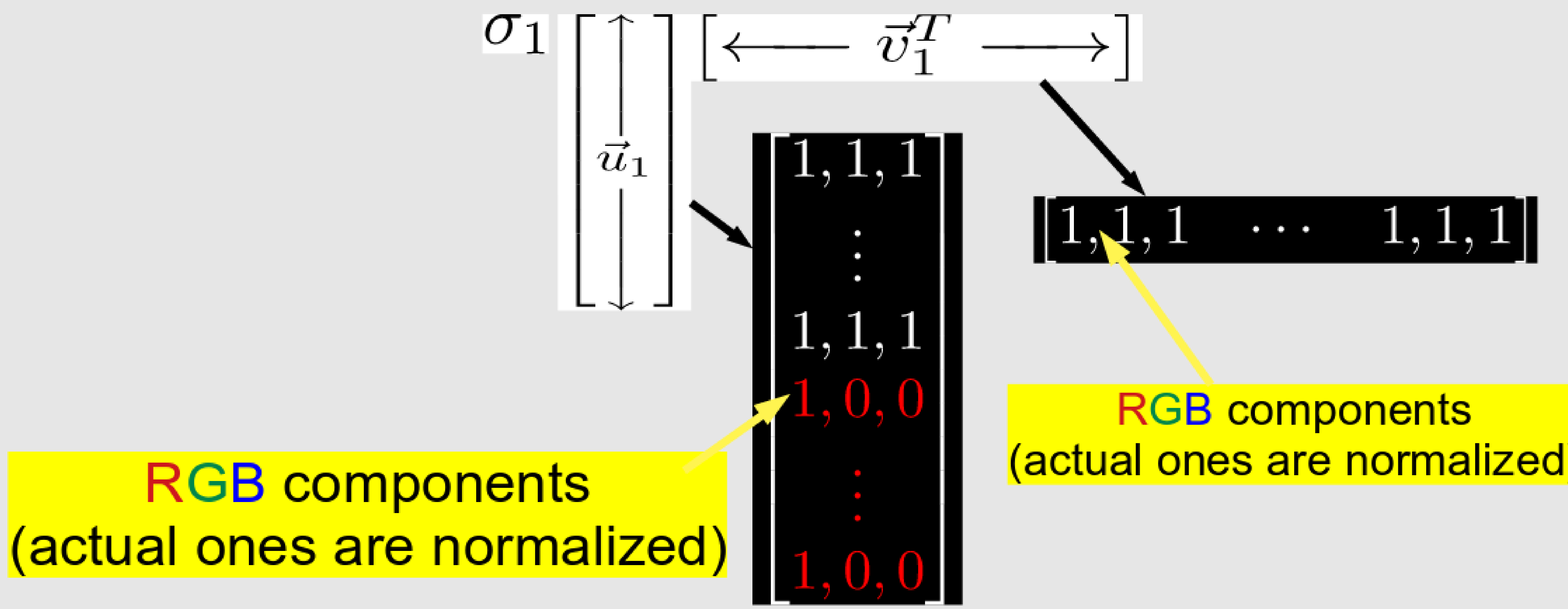
# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A =

original: 10MB

rank 1: 17.5kB



# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A =

original: 10MB

rank 1: 17.5kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

$(8.5, 6.4, 6.6) \times 10^4$

$$\begin{bmatrix} 1,1,1 \\ \vdots \\ 1,1,1 \\ 1,0,0 \\ \vdots \\ 1,0,0 \end{bmatrix}$$

RGB components
(actual ones are normalized)

$$\begin{bmatrix} 1,1,1 & \cdots & 1,1,1 \end{bmatrix}$$

RGB components
(actual ones are normalized)

# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A = 

original: 10MB

rank 1: 17.5kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow \; \vec{v}_1^T \; \longrightarrow \end{bmatrix}$$

$\sigma_1$: (8.5,6.4,6.6)x$10^4$

$$\vec{u}_1: \begin{bmatrix} 1,1,1 \\ \vdots \\ 1,1,1 \\ 1,0,0 \\ \vdots \\ 1,0,0 \end{bmatrix} \qquad \vec{v}_1^T: \begin{bmatrix} 1,1,1 & \cdots & 1,1,1 \end{bmatrix}$$

RGB components (actual ones are normalized)

RGB components (actual ones are normalized)

$$\sigma_2 \simeq 10^{-10} \quad \text{for R, G, B}$$

# Example: Polish Flag as a Matrix

- size: 281x450 (x 3 colours: R, G, B)

A = 

original: 10MB

rank 1: 17.5kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

$(8.5, 6.4, 6.6) \times 10^4$

$$\begin{bmatrix} 1,1,1 \\ \vdots \\ 1,1,1 \\ 1,0,0 \\ \vdots \\ 1,0,0 \end{bmatrix} \qquad \begin{bmatrix} 1,1,1 & \cdots & 1,1,1 \end{bmatrix}$$

RGB components
(actual ones are normalized)

RGB components
(actual ones are normalized)

$\sigma_2 \simeq 10^{-10}$ for R, G, B

**This is a RANK-1 FLAG**

# Example: SVD of the Austrian Flag

- size: 281x450 (x 3 colours: R, G, B)

A = 

original: 73MB

rank 1: 48.5kB

$$\sigma_1 \begin{bmatrix} \uparrow \\ \vec{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \longleftarrow & \vec{v}_1^T & \longrightarrow \end{bmatrix}$$

2.3x10$^5$

$$\begin{bmatrix} \color{red}{1,0,0} \\ \color{red}{\vdots} \\ \color{red}{1,0,0} \\ 1,1,1 \\ \vdots \\ 1,1,1 \\ \color{red}{1,0,0} \\ \color{red}{\vdots} \\ \color{red}{1,0,0} \end{bmatrix}$$

RGB components
(actual ones are normalized)

$$\begin{bmatrix} 1,1,1 & \cdots & 1,1,1 \end{bmatrix}$$

RGB components
(actual ones are normalized)

**This is ALSO a RANK-1 FLAG**

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

$\sigma_2 \vec{u}_2 \vec{v}_2^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

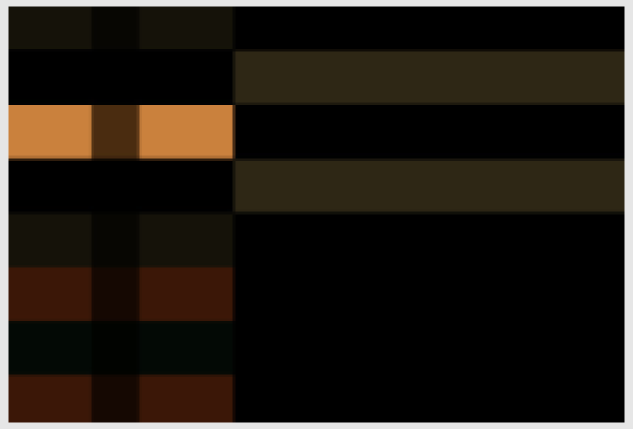
# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$



$\sigma_2 \vec{u}_2 \vec{v}_2^T$

rank 2: 36kB

$$\sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

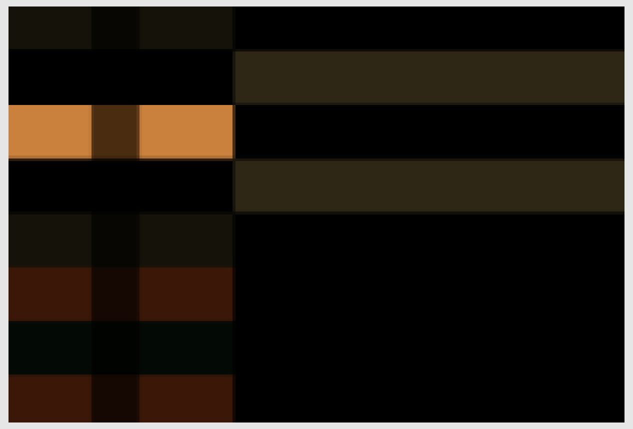
# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

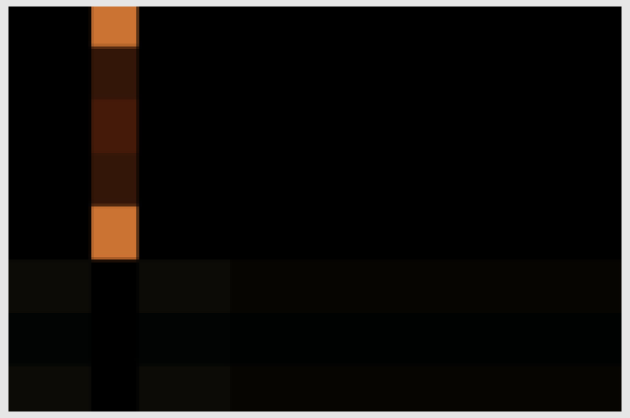
original: 10.1MB

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$



$\sigma_2 \vec{u}_2 \vec{v}_2^T$



$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 2: 36kB

$$\sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

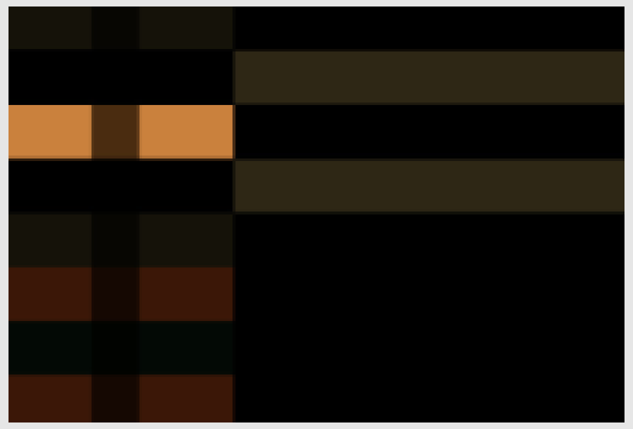
# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

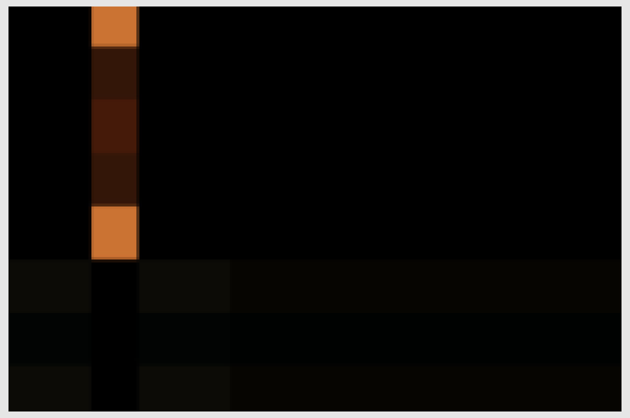
original: 10.1MB

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$



$\sigma_2 \vec{u}_2 \vec{v}_2^T$



$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 3: 54kb

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 2: 36kB

$\sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

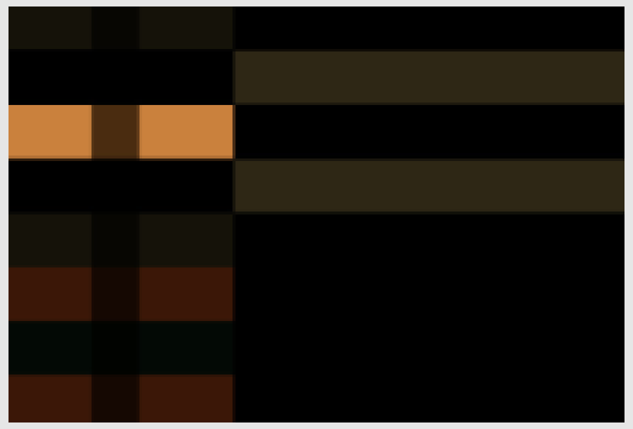
- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

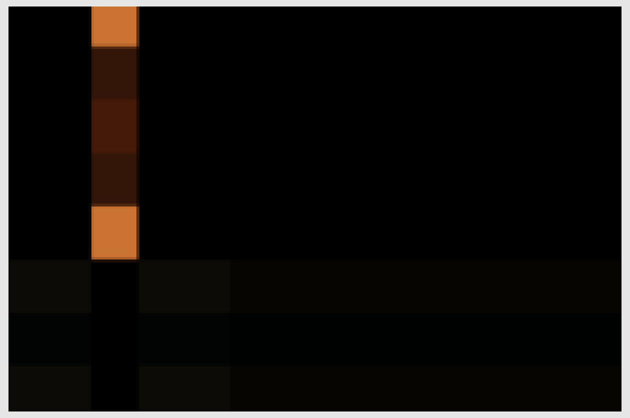
strongest **“feature”**

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$



$\sigma_2 \vec{u}_2 \vec{v}_2^T$



$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 3: 54kb

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 2: 36kB

$\sigma_1 \vec{u}_1 \vec{v}_1^T +$
$\sigma_2 \vec{u}_2 \vec{v}_2^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

original: 10.1MB

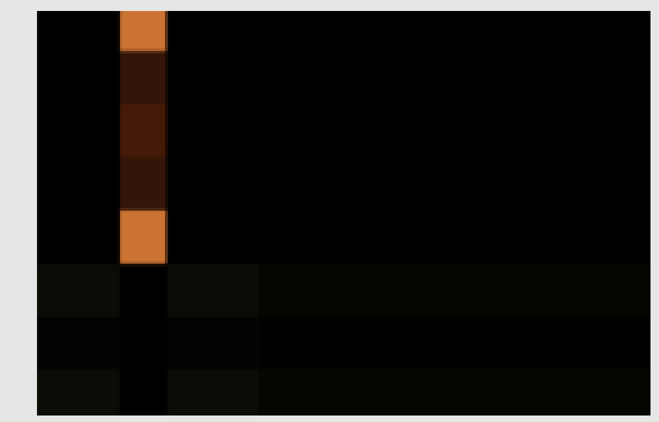
strongest **"feature"**

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

2nd strongest **"feature"**

$\sigma_2 \vec{u}_2 \vec{v}_2^T$



$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 3: 54kb

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 2: 36kB

$\sigma_1 \vec{u}_1 \vec{v}_1^T +$

$\sigma_2 \vec{u}_2 \vec{v}_2^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

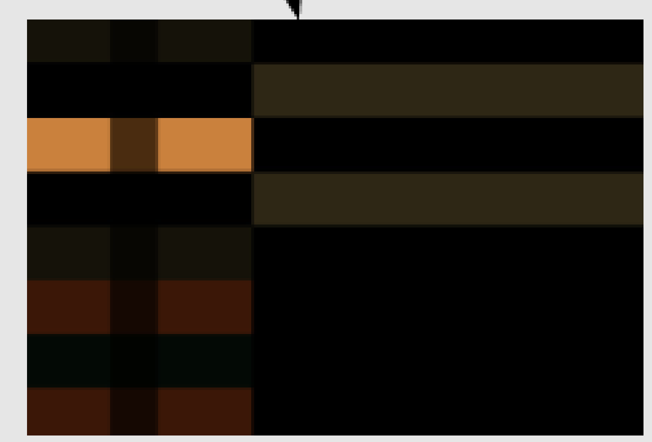
original: 10.1MB

strongest "feature"

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

2[nd] strongest "feature"



$\sigma_2 \vec{u}_2 \vec{v}_2^T$

3[rd] strongest "feature"

$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 3: 54kb

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 2: 36kB

$\sigma_1 \vec{u}_1 \vec{v}_1^T +$
$\sigma_2 \vec{u}_2 \vec{v}_2^T$

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of the Greek Flag

- size: 295x450 (x 3 colours: R, G, B)

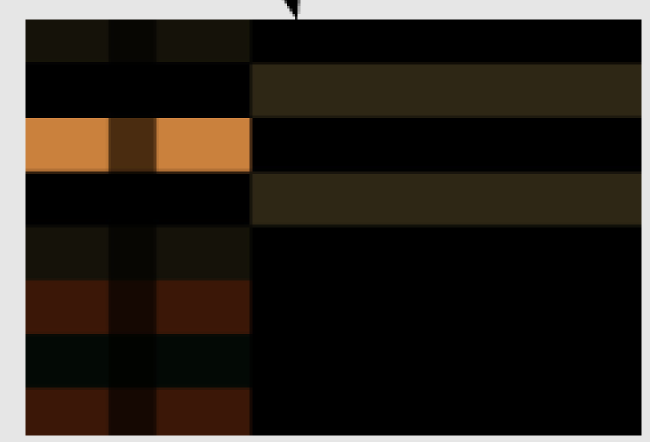
original: 10.1MB

strongest "feature"

rank 1: 18kb

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

2nd strongest "feature"



$\sigma_2 \vec{u}_2 \vec{v}_2^T$

3rd strongest "feature"

$\sigma_3 \vec{u}_3 \vec{v}_3^T$

rank 3: 54kb

$\sum_{i=1}^{3} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 2: 36kB

$$\sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

**This is a RANK-3 FLAG**

# Example: SVD of the US Flag

- size: 450x237 (x 3 colours: R, G, B)

original: 8.8MB

# Example: SVD of the US Flag

- size: 450x237 (x 3 colours: R, G, B)

original: 8.8MB

strongest **"feature"**

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

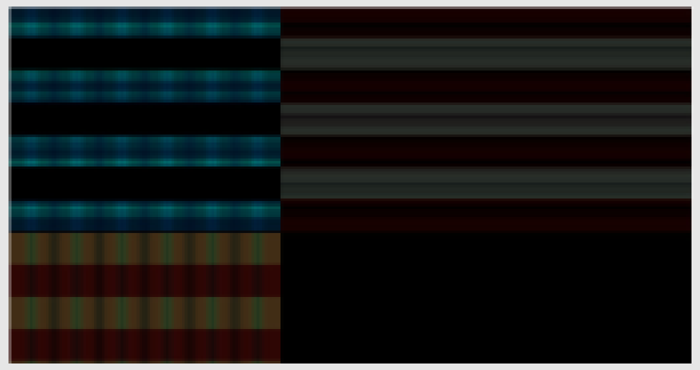
# Example: SVD of the US Flag

- size: 450x237 (x 3 colours: R, G, B)

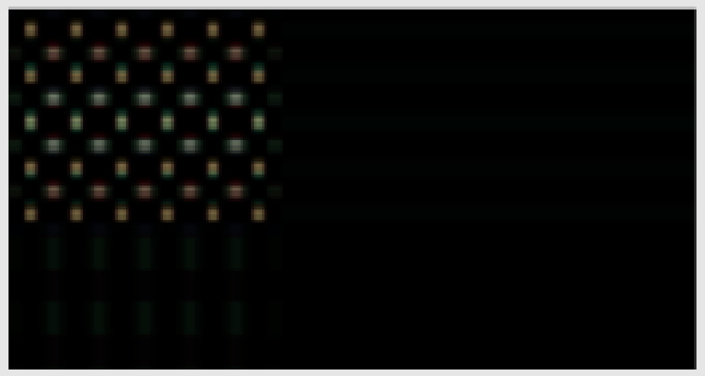
original: 8.8MB

strongest **"feature"**

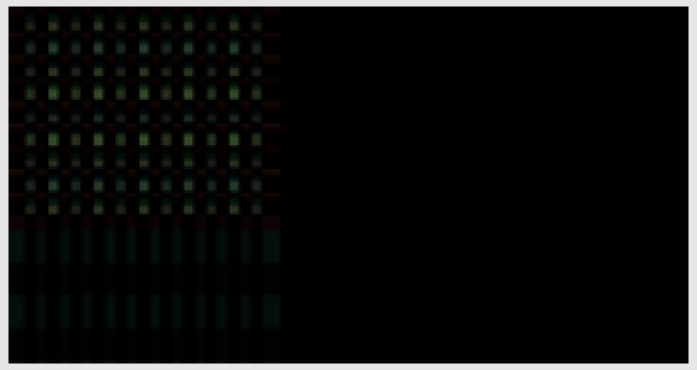
$\sigma_1 \vec{u}_1 \vec{v}_1^T$



$\sigma_2 \vec{u}_2 \vec{v}_2^T$ 16.5kb



$\sigma_3 \vec{u}_3 \vec{v}_3^T$ 16.5kB



$\sigma_4 \vec{u}_4 \vec{v}_4^T$ 16.5kB

$\sigma_5 \vec{u}_5 \vec{v}_5^T$ 16.5kb

$\sigma_6 \vec{u}_6 \vec{v}_6^T$ 16.5kb

# Example: SVD of the US Flag

- size: 450x237 (x 3 colours: R, G, B)

original: 8.8MB

strongest "feature"

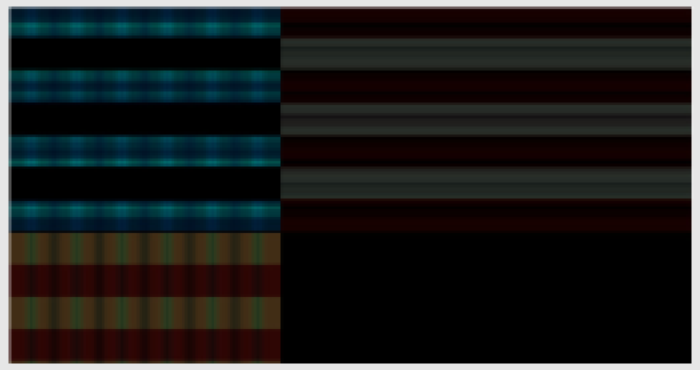
$\sigma_1 \vec{u}_1 \vec{v}_1^T$

$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$

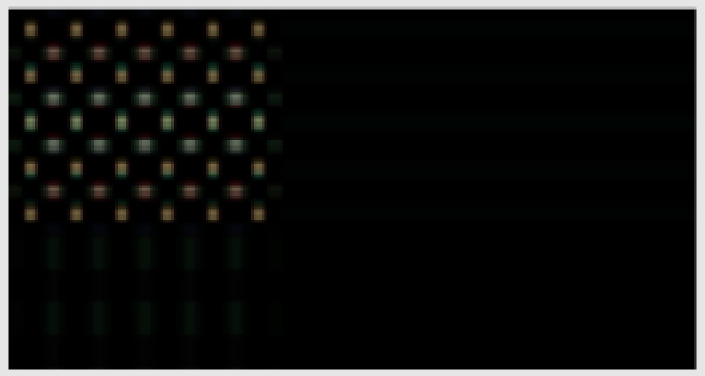
rank 5: 83kB

$\sum_{i=1}^{10} \sigma_i \vec{u}_i \vec{v}_i^T$

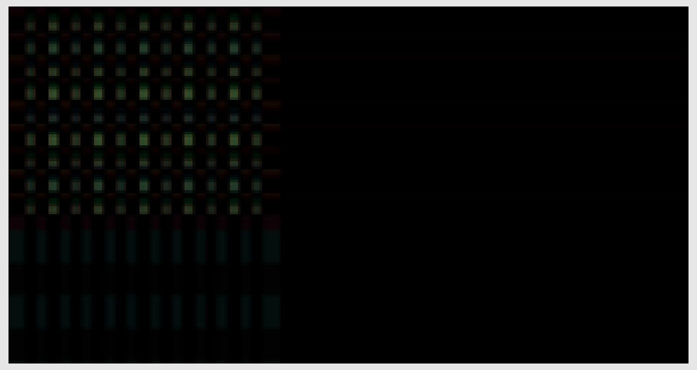
rank 10: 167kB



$\sigma_2 \vec{u}_2 \vec{v}_2^T$ 16.5kb



$\sigma_3 \vec{u}_3 \vec{v}_3^T$ 16.5kB



$\sigma_4 \vec{u}_4 \vec{v}_4^T$ 16.5kB

$\sigma_5 \vec{u}_5 \vec{v}_5^T$ 16.5kb

$\sigma_6 \vec{u}_6 \vec{v}_6^T$ 16.5kb

# Example: SVD of the US Flag

- size: 450x237 (x 3 colours: R, G, B)

original: 8.8MB

strongest **"feature"**

$\sigma_1 \vec{u}_1 \vec{v}_1^T$

$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$

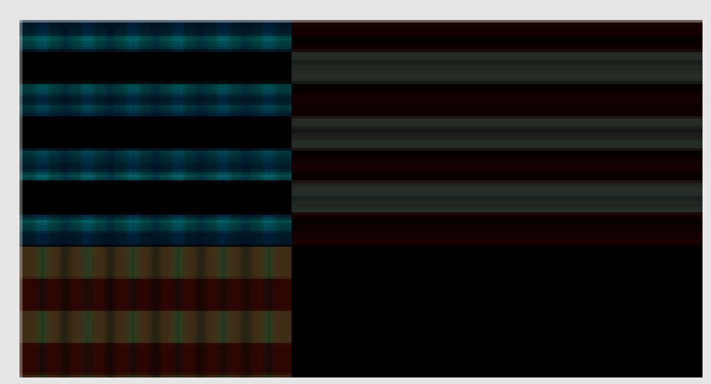
rank 5: 83kB

$\sum_{i=1}^{10} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 10: 167kB

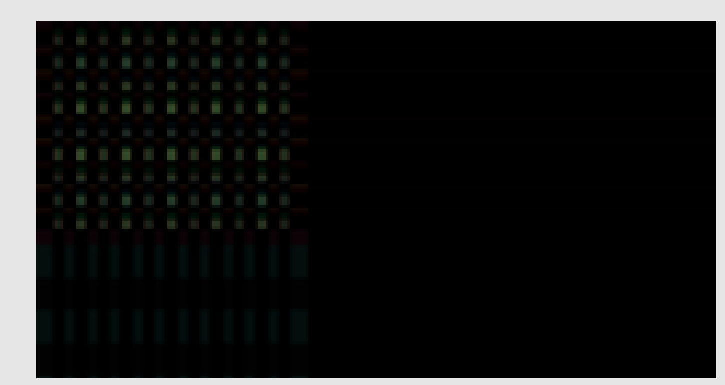
$\sum_{i=1}^{15} \sigma_i \vec{u}_i \vec{v}_i^T$

rank 15: 253kB

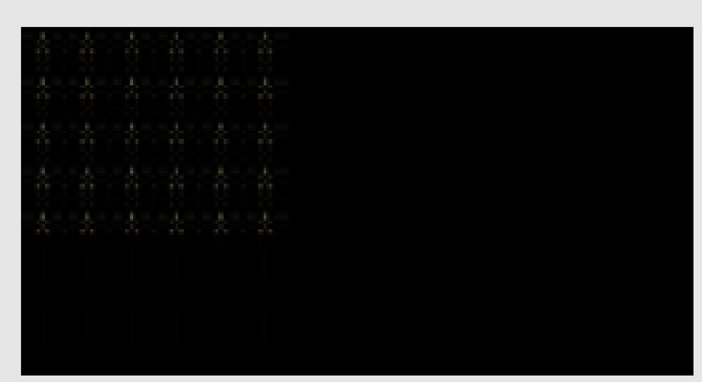


$\sigma_2 \vec{u}_2 \vec{v}_2^T$ 16.5kb

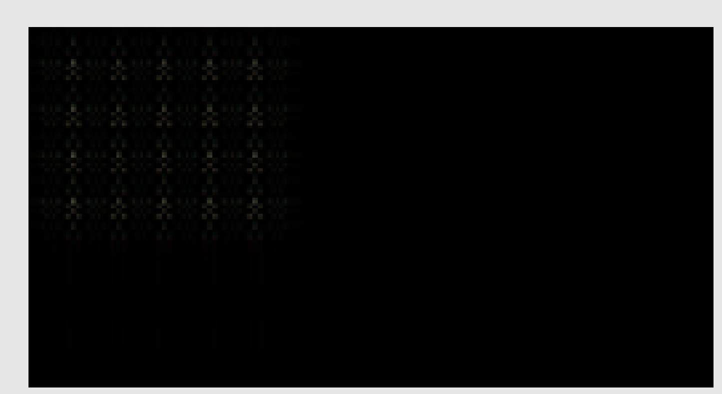
$\sigma_3 \vec{u}_3 \vec{v}_3^T$ 16.5kB



$\sigma_4 \vec{u}_4 \vec{v}_4^T$ 16.5kB



$\sigma_5 \vec{u}_5 \vec{v}_5^T$ 16.5kb



$\sigma_6 \vec{u}_6 \vec{v}_6^T$ 16.5kb

# Example: SVD of Michel Maharbiz

- size: 1100x757 (grayscale)

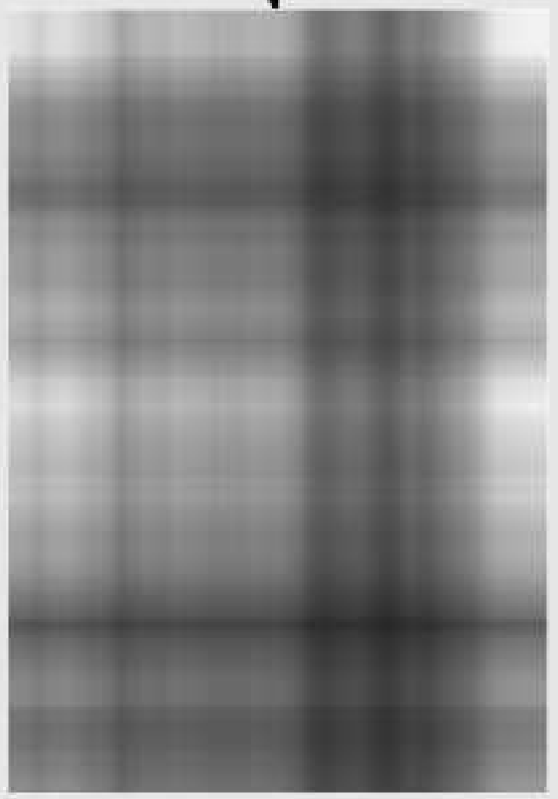
original

# Example: SVD of Michel Maharbiz

- size: 1100x757 (grayscale)

original

strongest **"feature"**
**?!**



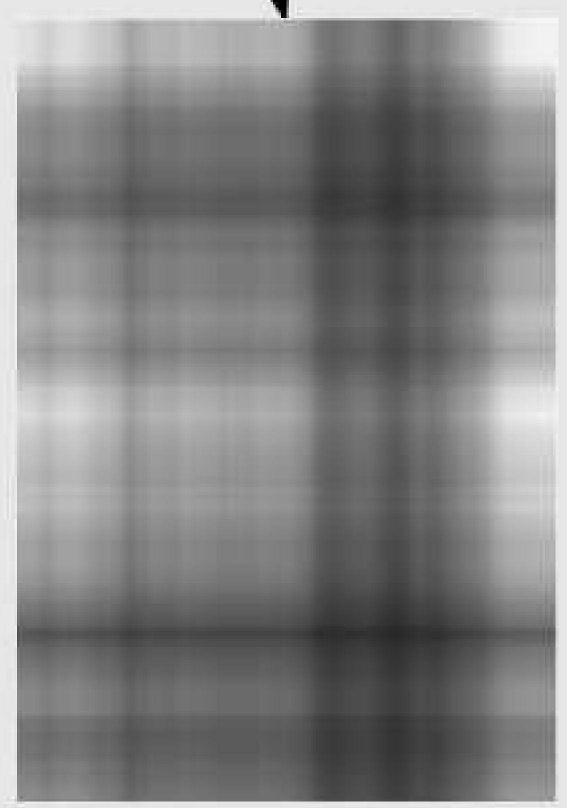
$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB

# Example: SVD of Michel Maharbiz

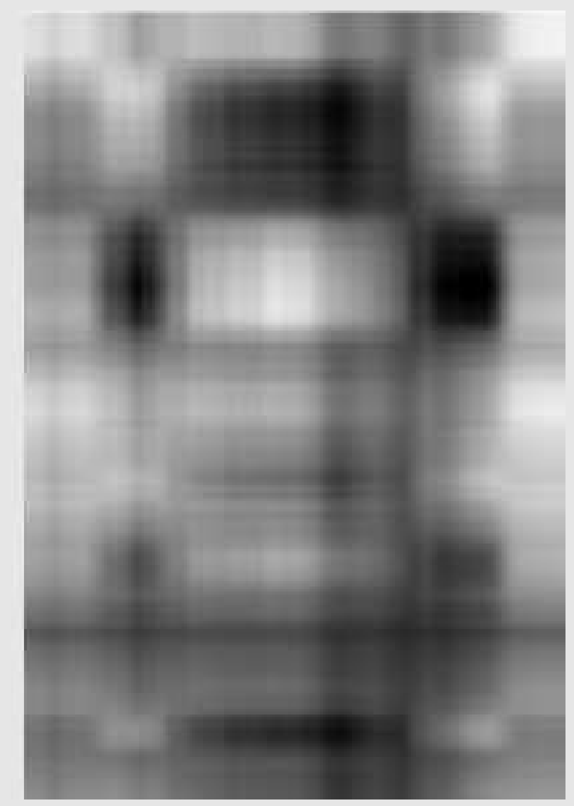
- size: 1100x757 (grayscale)

original

strongest **“feature” ?!**



$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB



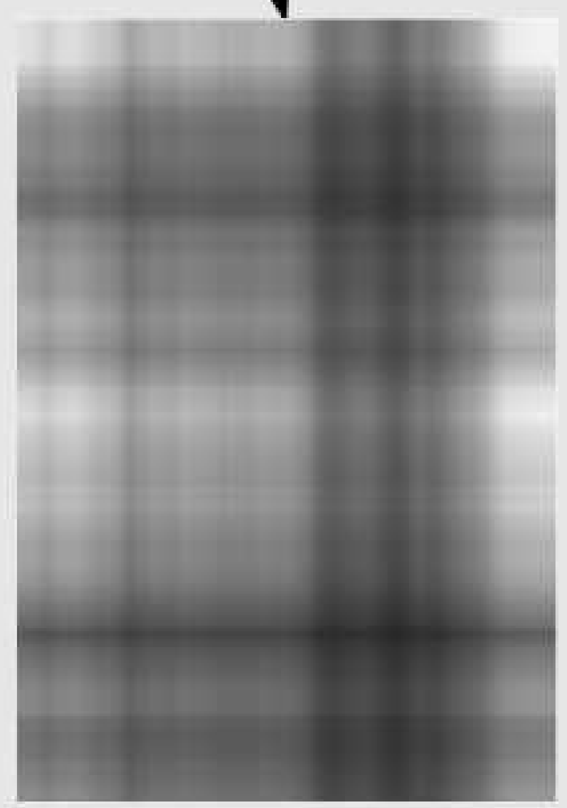
$\sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of Michel Maharbiz

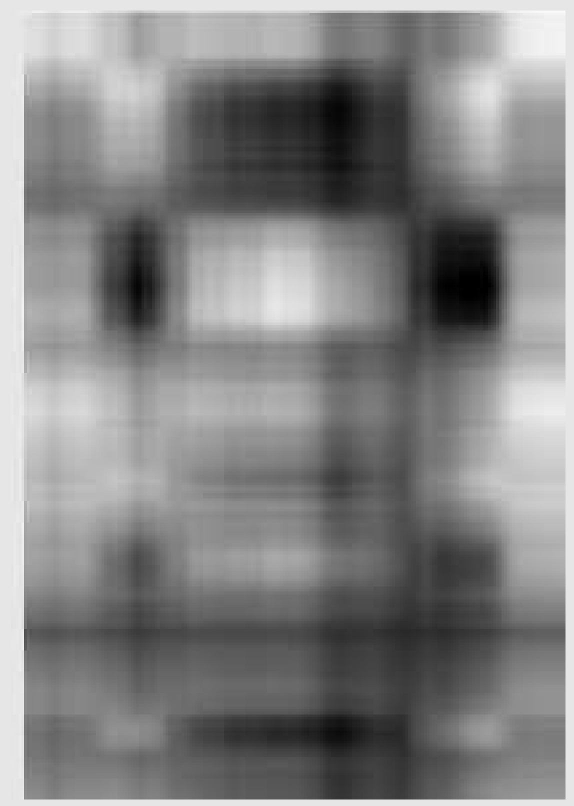
- size: 1100x757 (grayscale)

original

strongest **"feature"**
**?!**



$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB



$\sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^T$

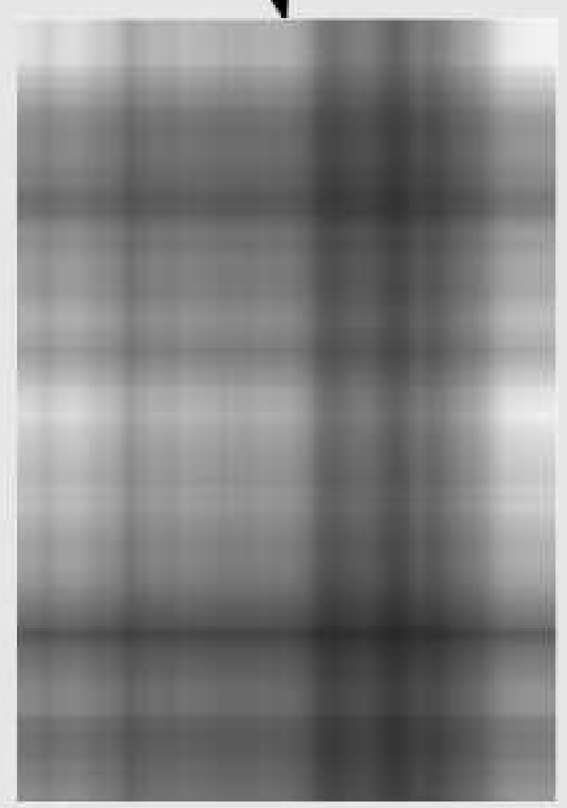
$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of Michel Maharbiz

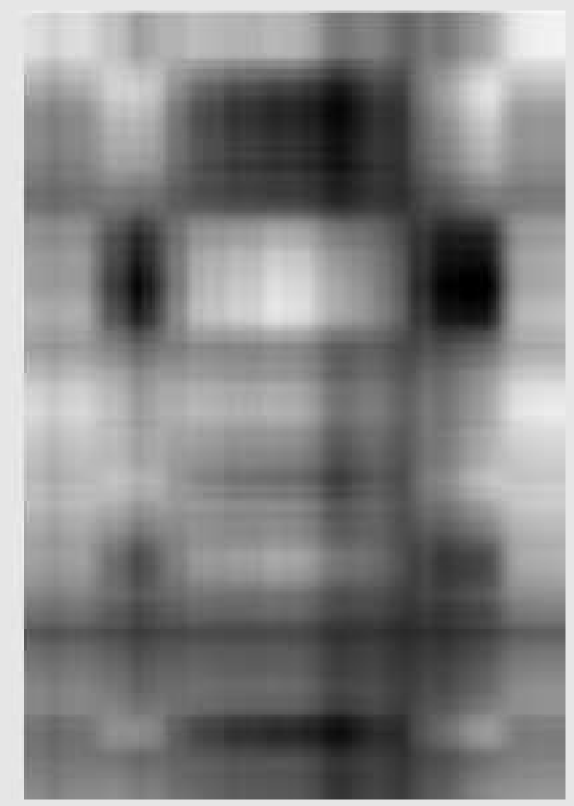
- size: 1100x757 (grayscale)

original

strongest **“feature”**
**?!**

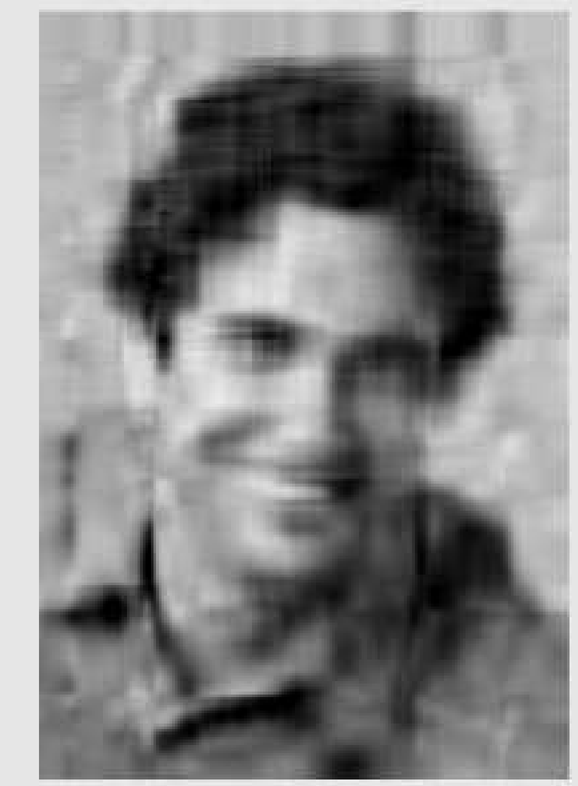


$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB



$\sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$



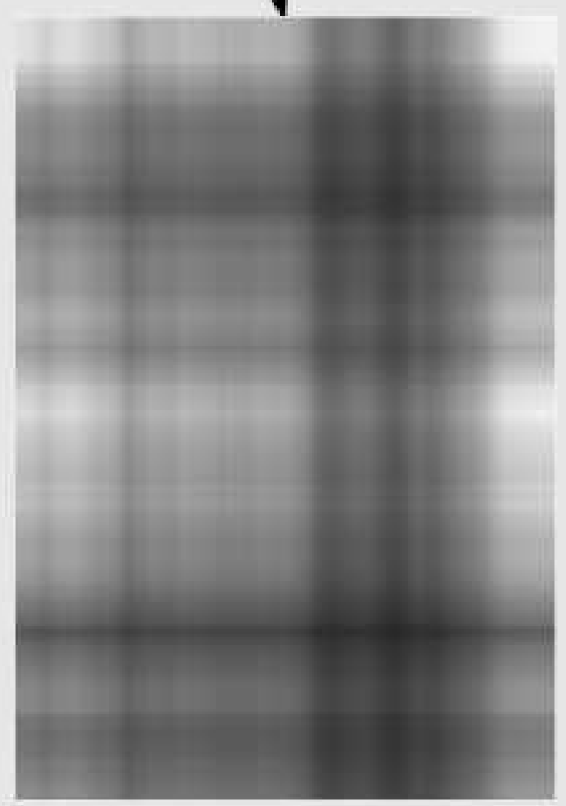
$\sum_{i=1}^{10} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of Michel Maharbiz

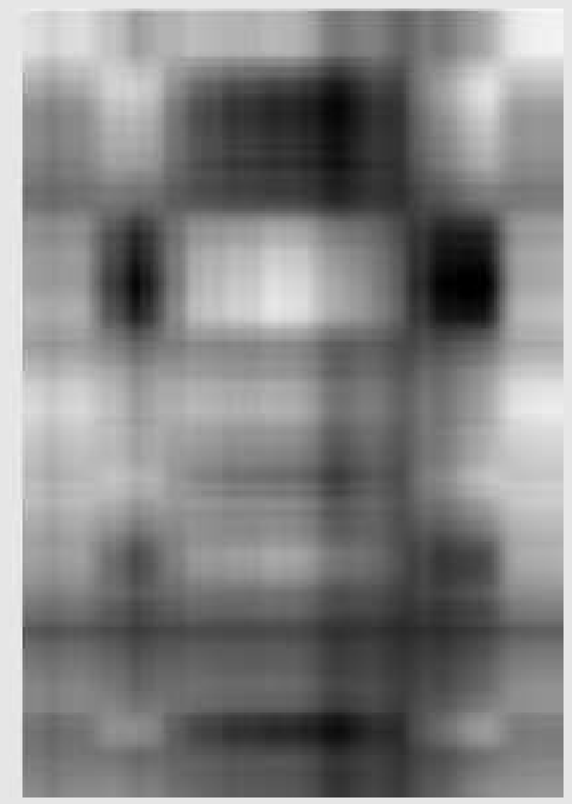
- size: 1100x757 (grayscale)

original

strongest **"feature"** **?!**



$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB



$\sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{10} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{20} \sigma_i \vec{u}_i \vec{v}_i^T$

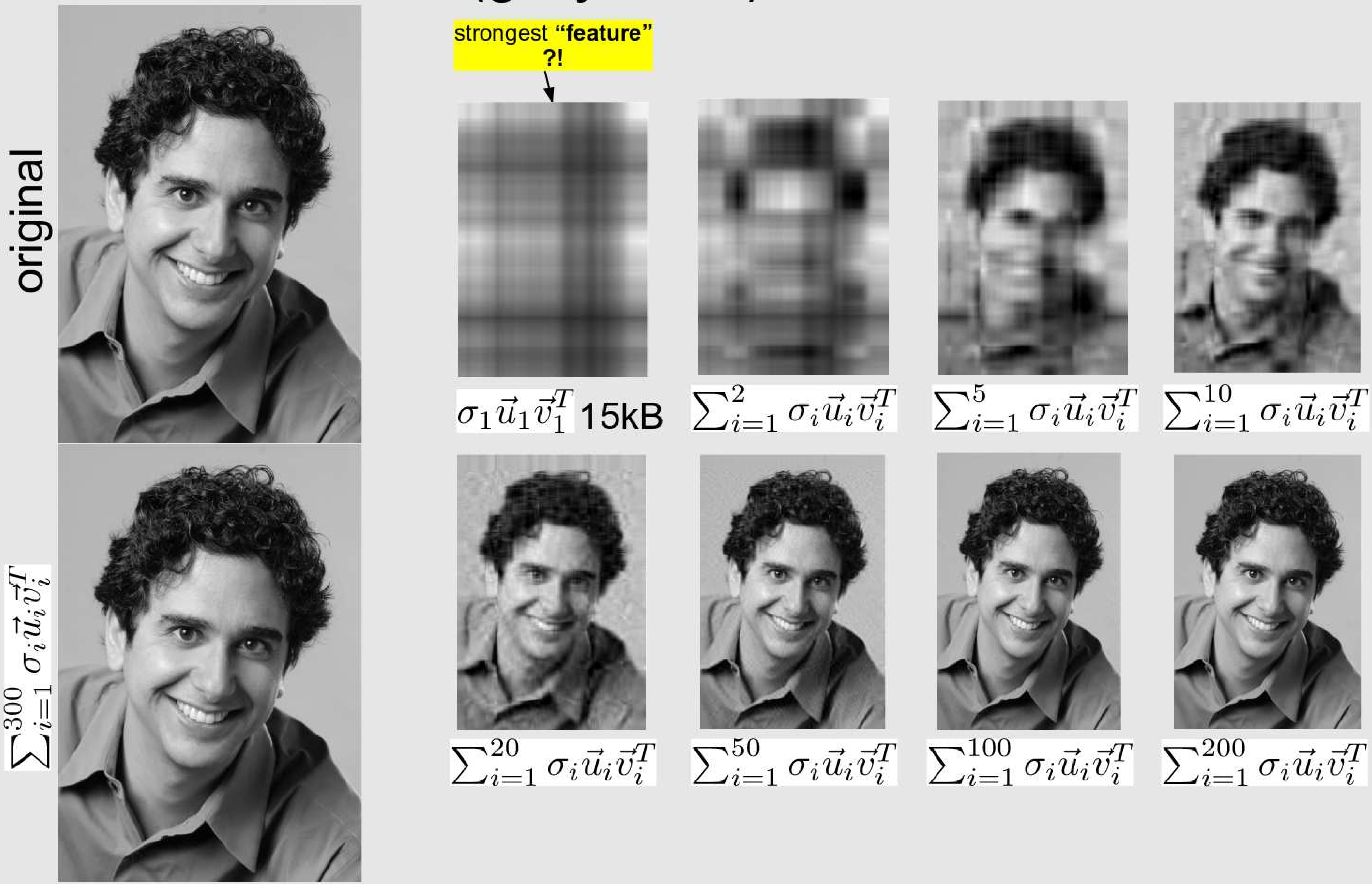
$\sum_{i=1}^{50} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{100} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{200} \sigma_i \vec{u}_i \vec{v}_i^T$

# Example: SVD of Michel Maharbiz

- size: 1100x757 (grayscale)



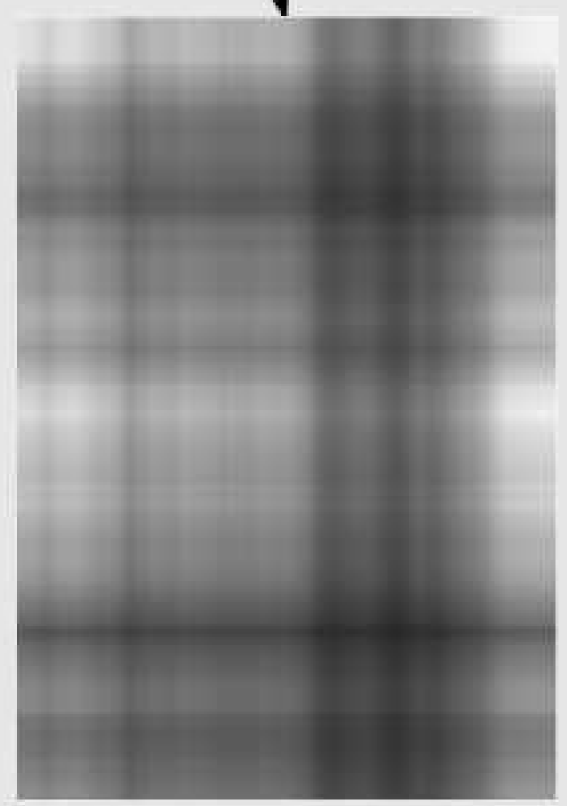
# Example: SVD of Michel Maharbiz

- size: 1100x757 (grayscale)

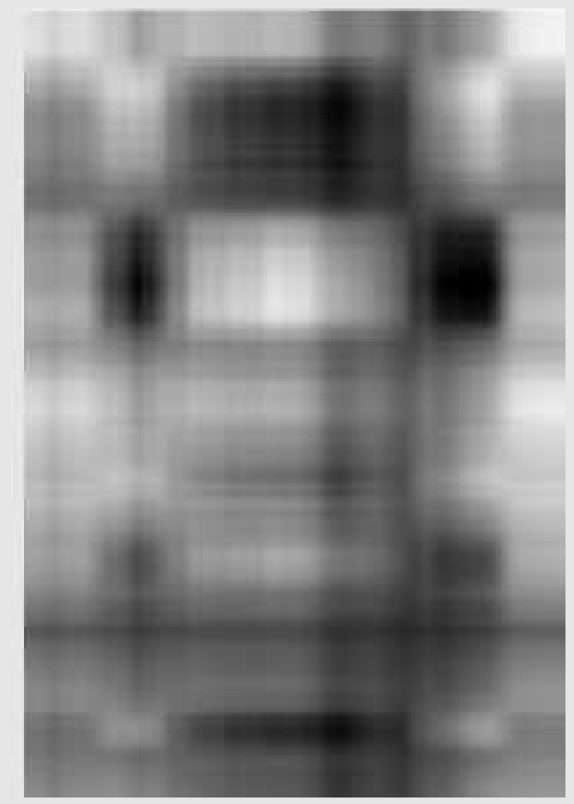
original

$\sum_{i=1}^{300} \sigma_i \vec{u}_i \vec{v}_i^T$

strongest **"feature"**
**?!**



$\sigma_1 \vec{u}_1 \vec{v}_1^T$ 15kB



$\sum_{i=1}^{2} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{10} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{20} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{50} \sigma_i \vec{u}_i \vec{v}_i^T$

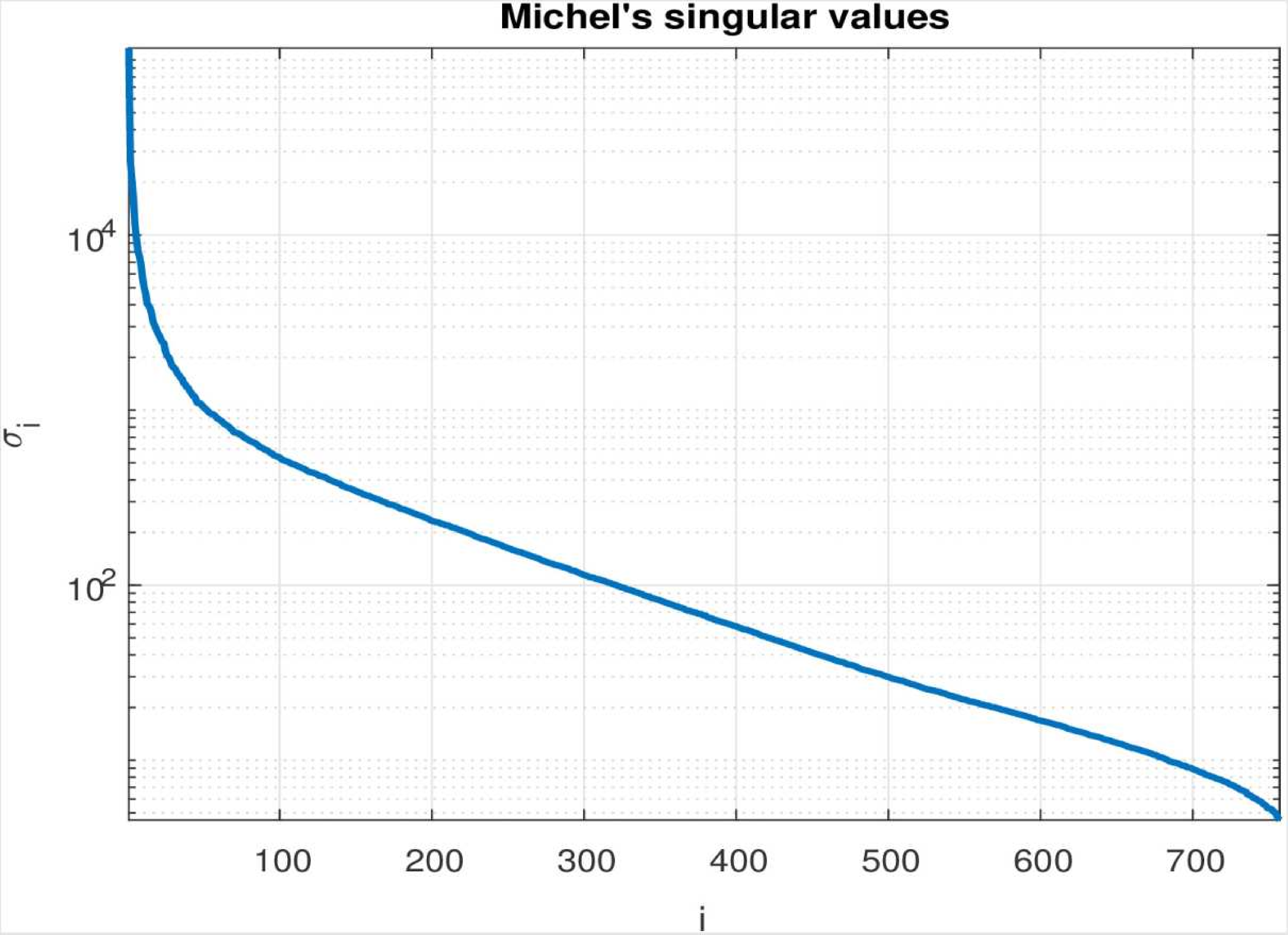
$\sum_{i=1}^{100} \sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=1}^{200} \sigma_i \vec{u}_i \vec{v}_i^T$

**Features not always intuitive**

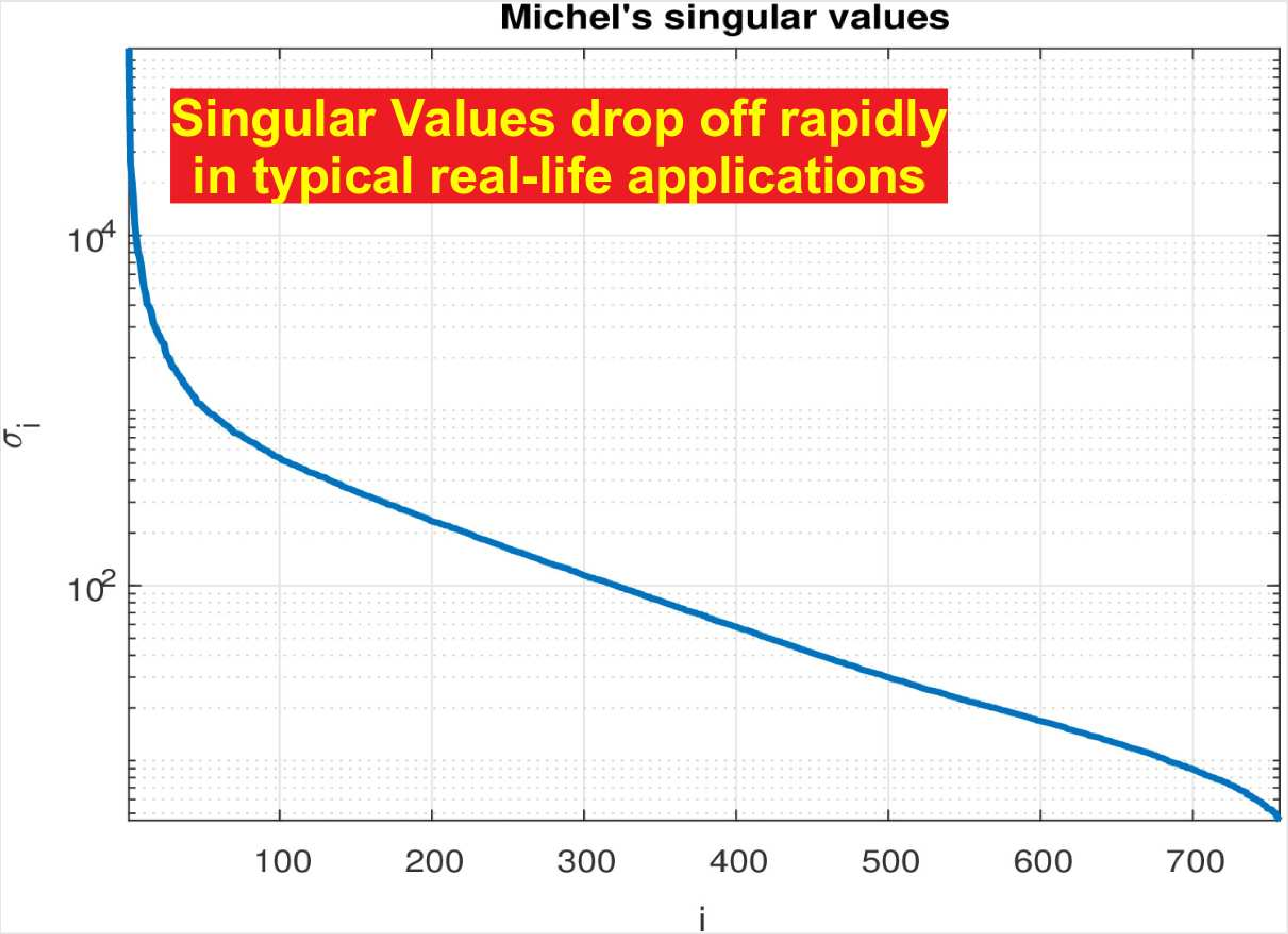
# Michel's Singular Values

- How Michel's singular values drop off



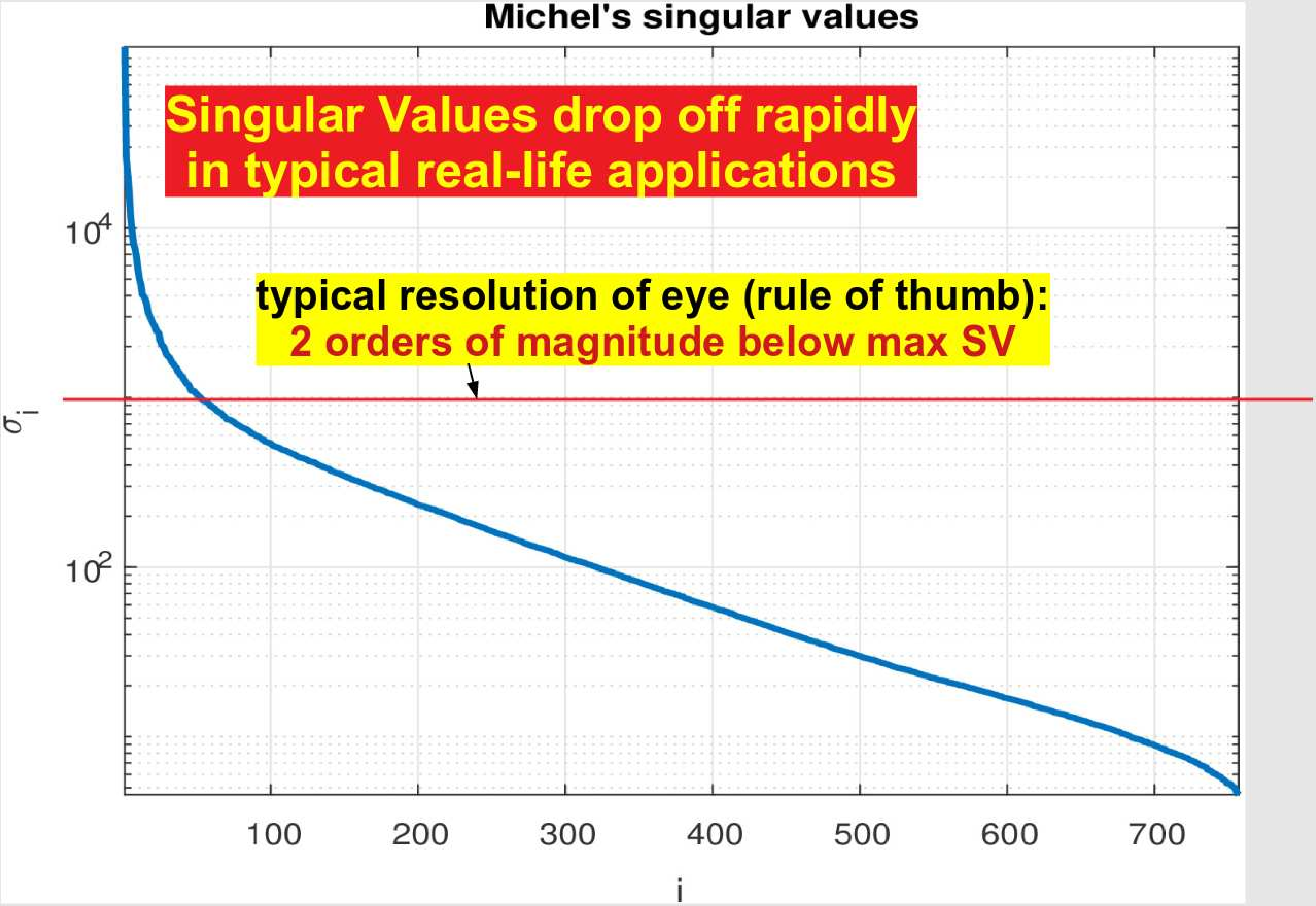
# Michel's Singular Values

- How Michel's singular values drop off



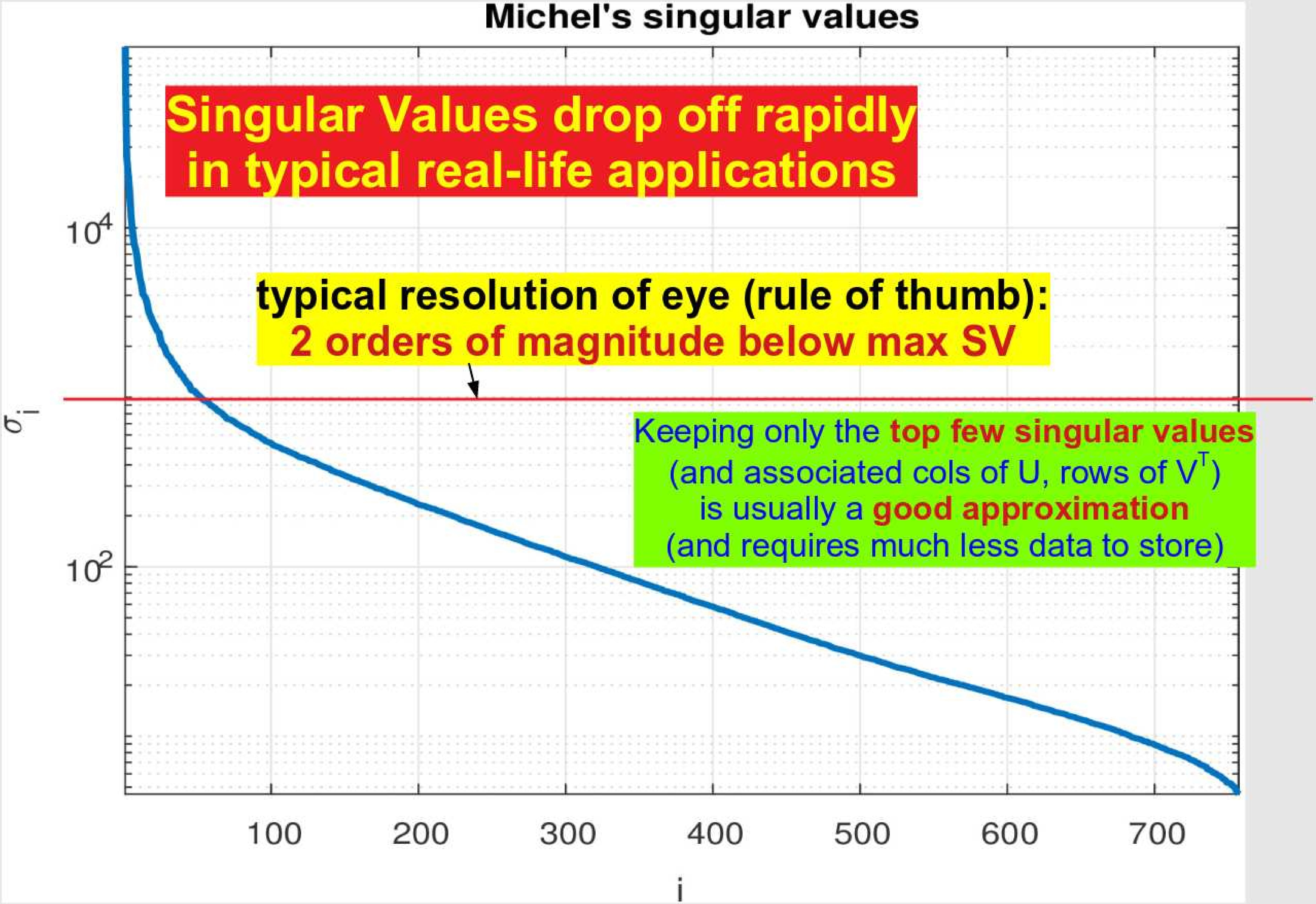
# Michel's Singular Values

- How Michel's singular values drop off



# Michel's Singular Values

- How Michel's singular values drop off



## Geometric View of Orthogonality

## Projection onto Orthonormal Bases

## Geometric View of Unitary Operations

# Geometric View of Orthogonality

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ ← $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

# Geometric View of Orthogonality

if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$
called **ORTHONORMAL**

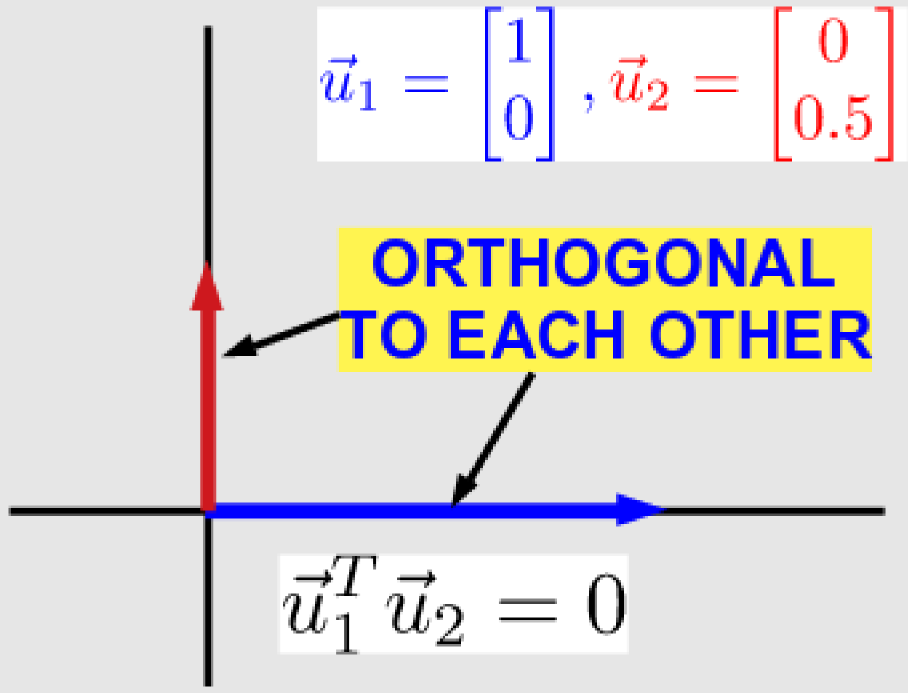
# Geometric View of Orthogonality

if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

- In 2D:



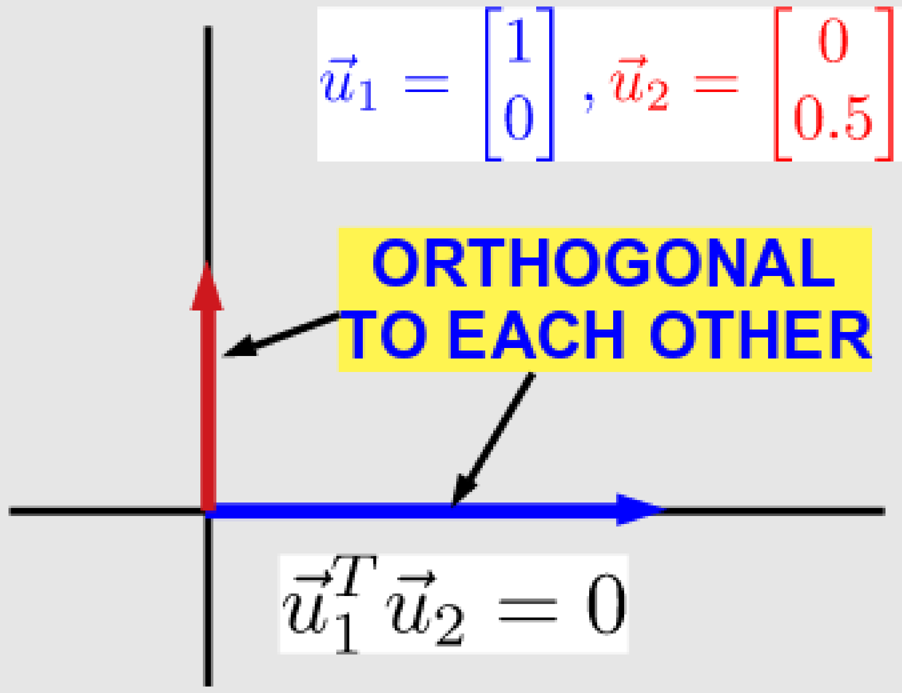
# Geometric View of Orthogonality

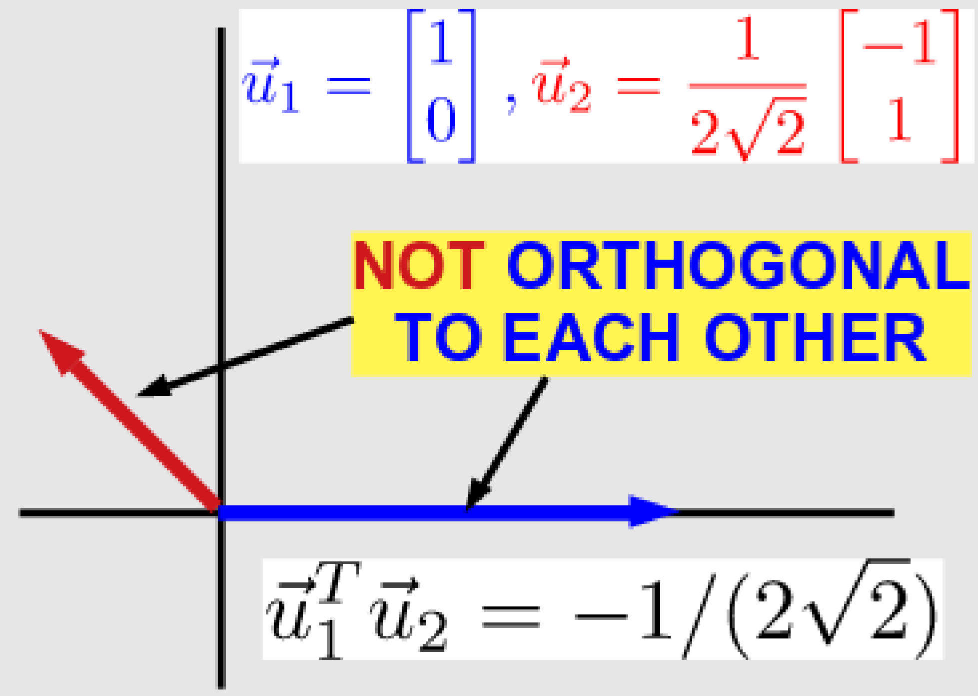
if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

- In 2D:



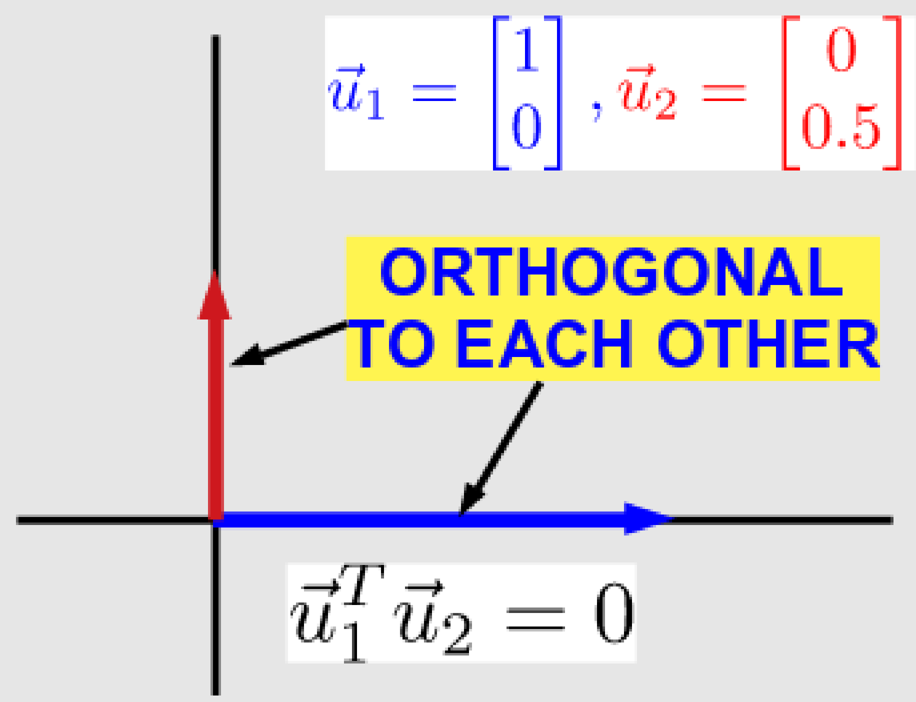
# Geometric View of Orthogonality

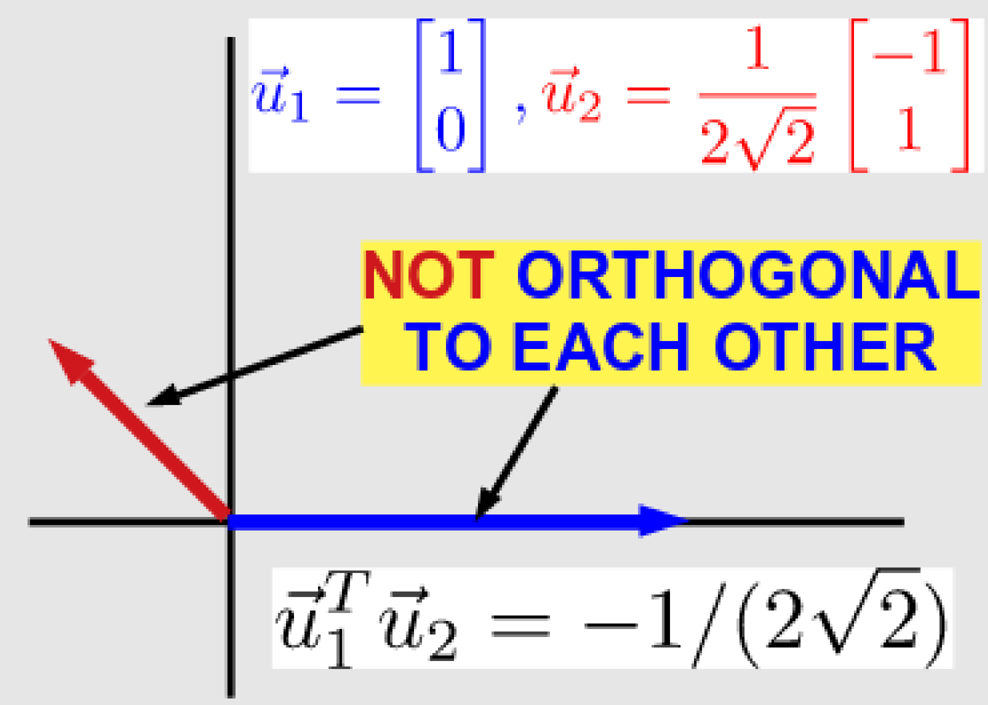
if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

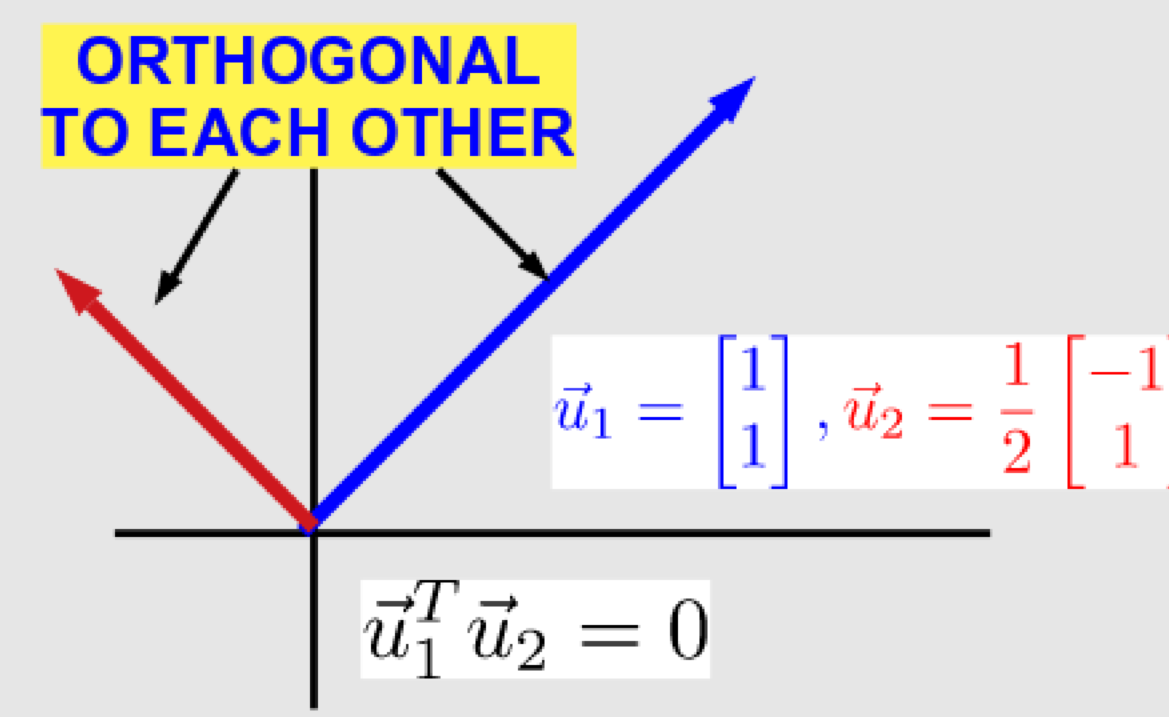
- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

- In 2D:



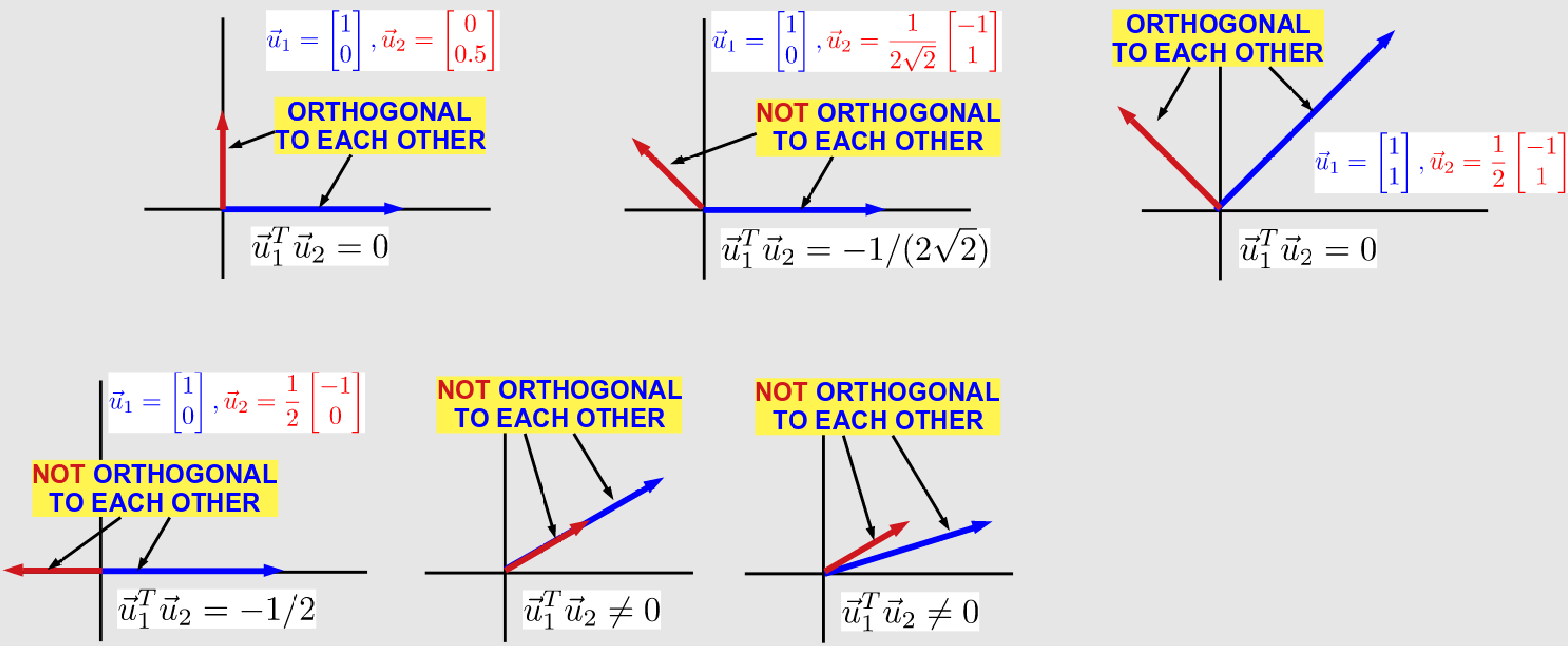
# Geometric View of Orthogonality

if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$
called **ORTHONORMAL**

- In 2D:



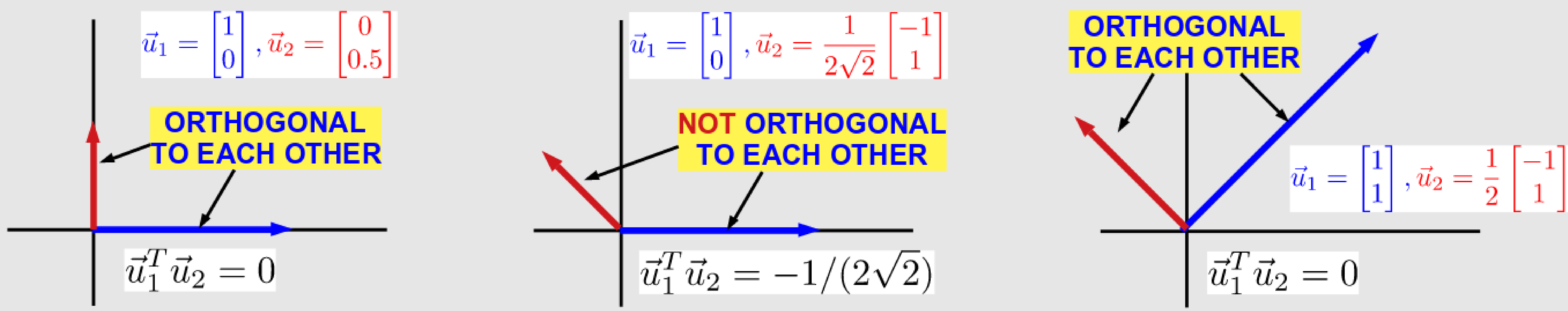
# Geometric View of Orthogonality

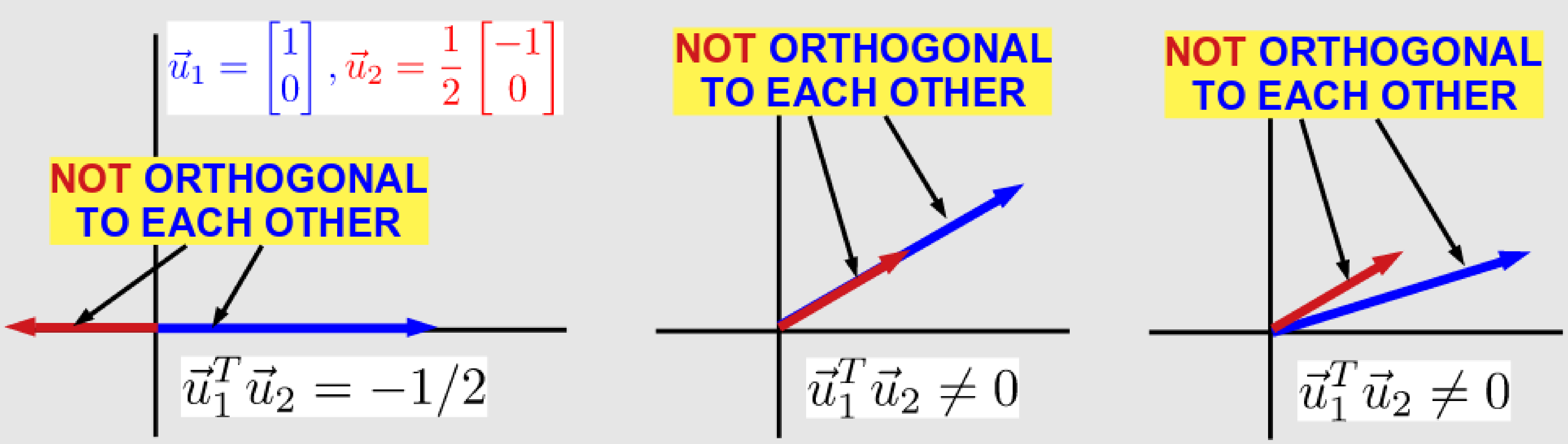
if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$
called **ORTHONORMAL**

- In 2D:




3D: orthogonality also means **at right angles**

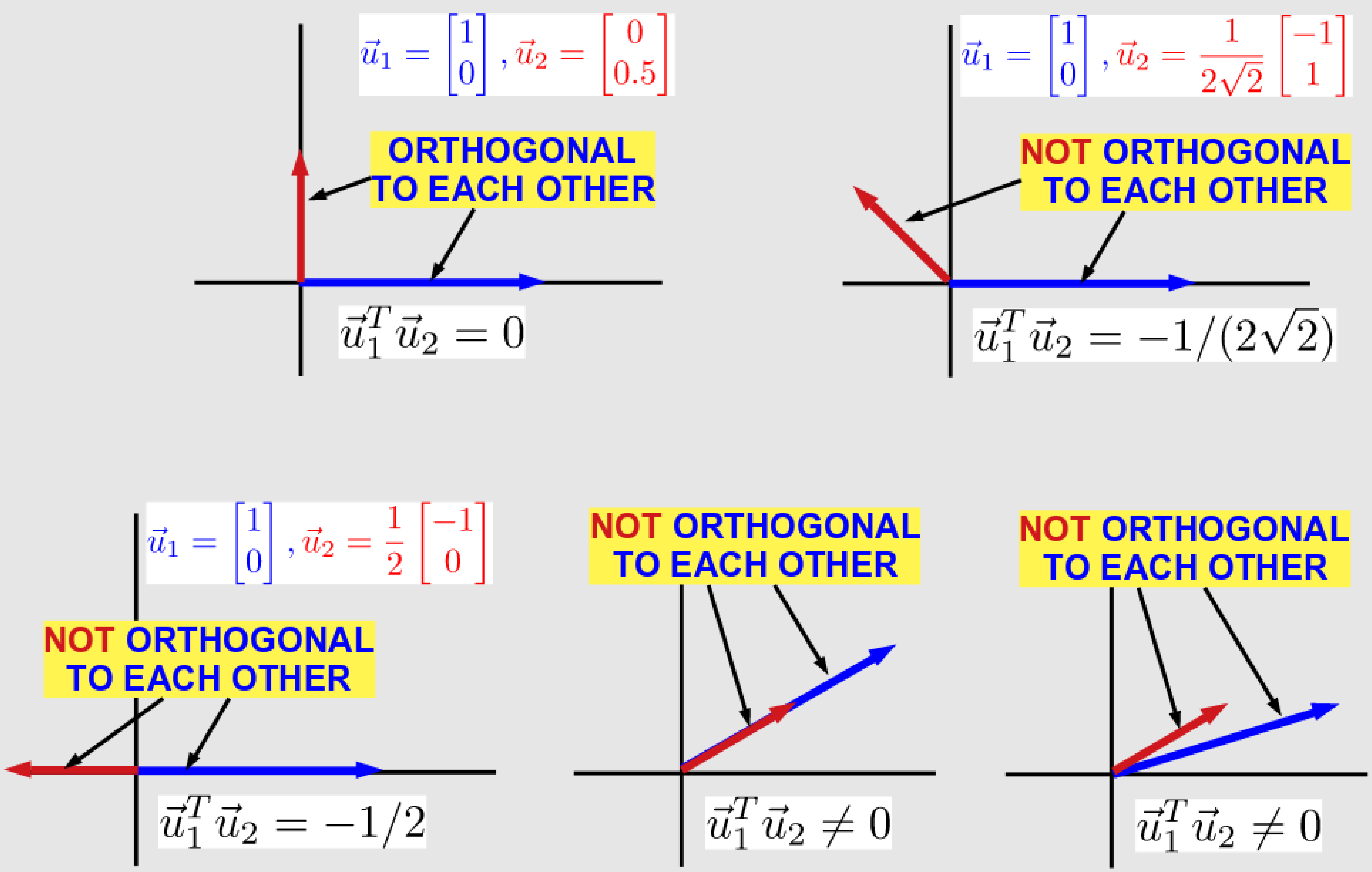
# Geometric View of Orthogonality

if not necessarily = 1 (but ≠ 0): then called **ORTHOGONAL**

- recall: $\vec{u}_i^T \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ called **ORTHONORMAL**

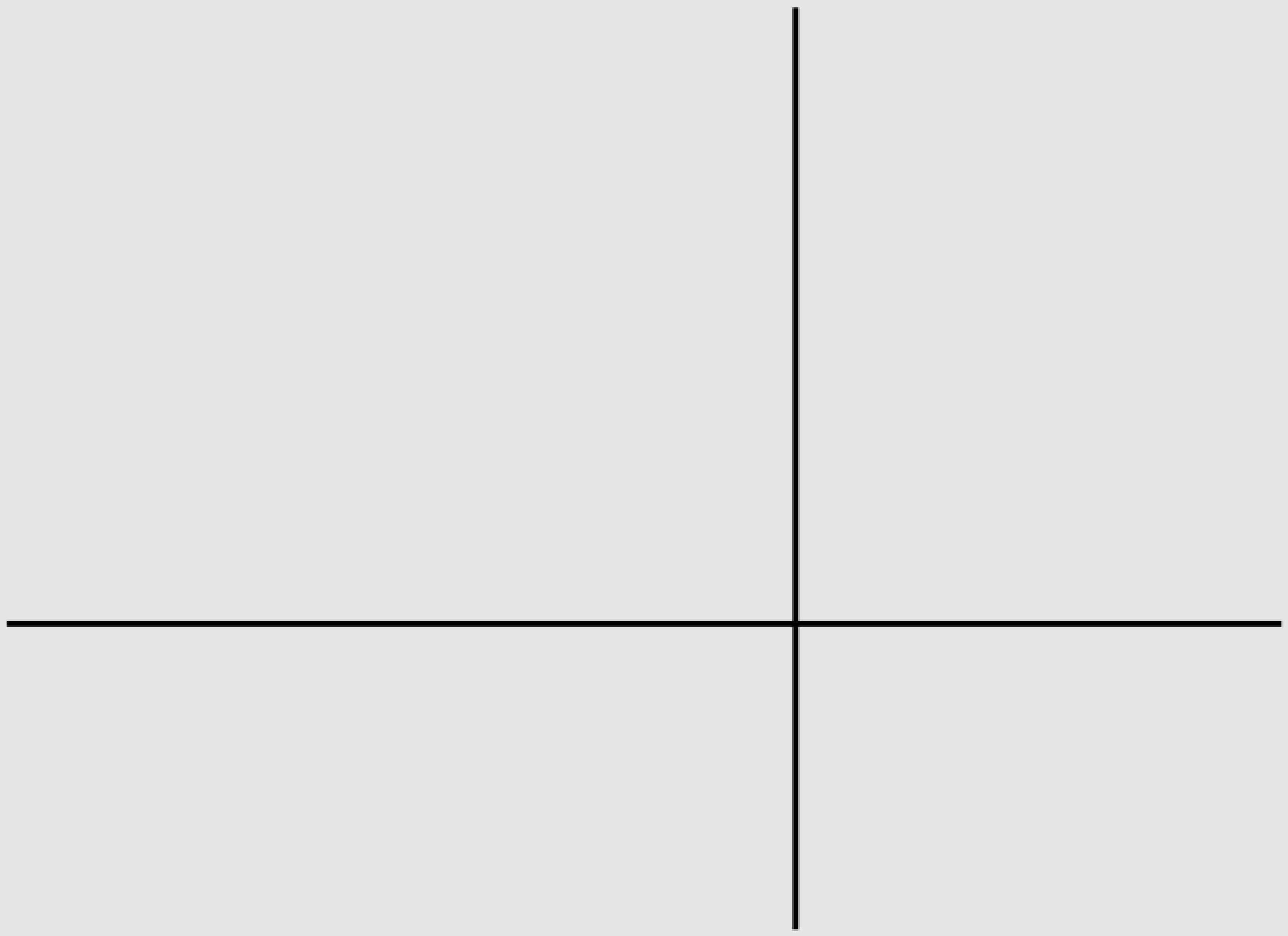
- In 2D:



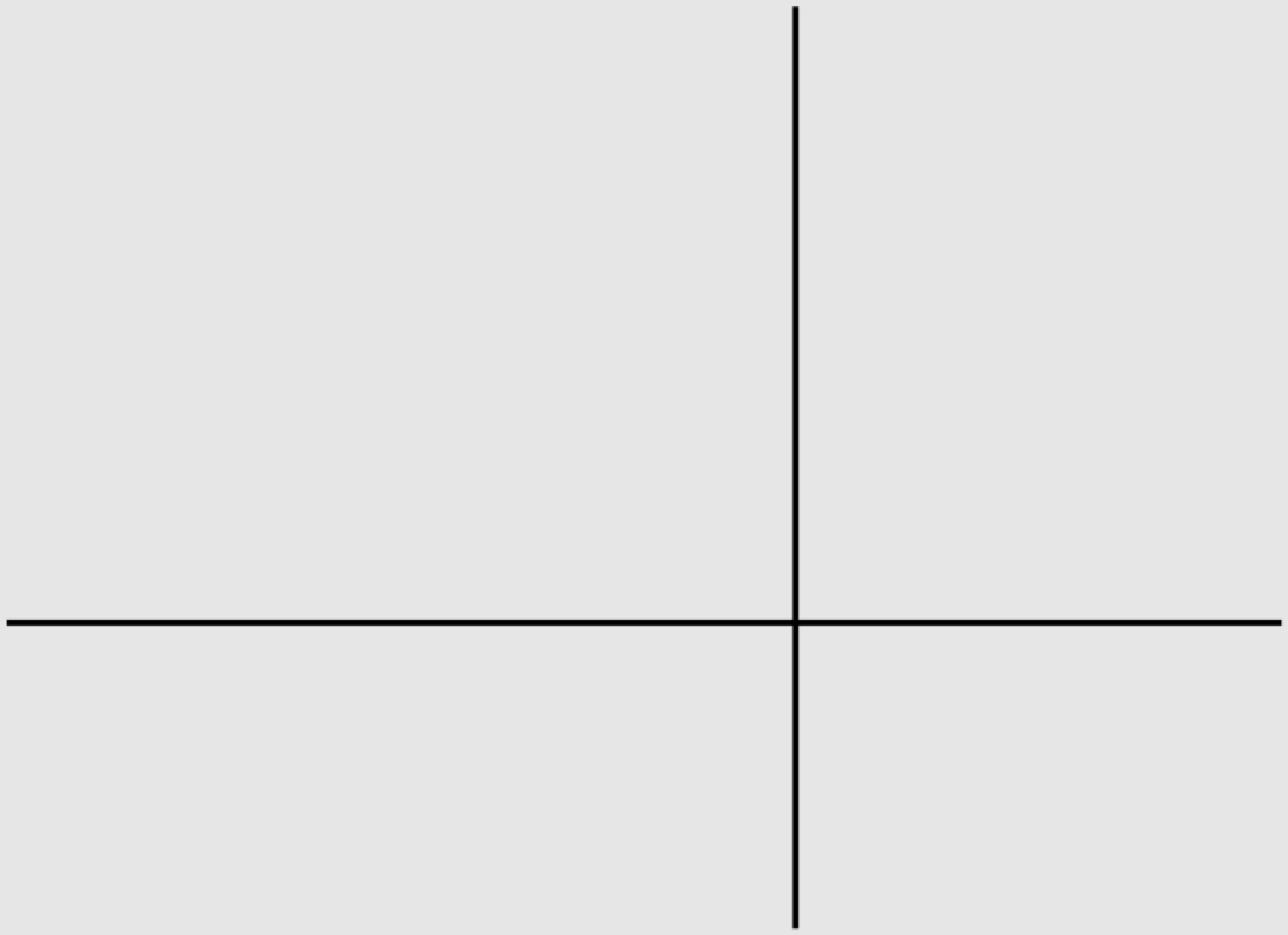

3D: orthogonality also means **at right angles**

4D and higher: "**right angles**" **means orthogonality!**

# Projection onto Orthonormal Bases



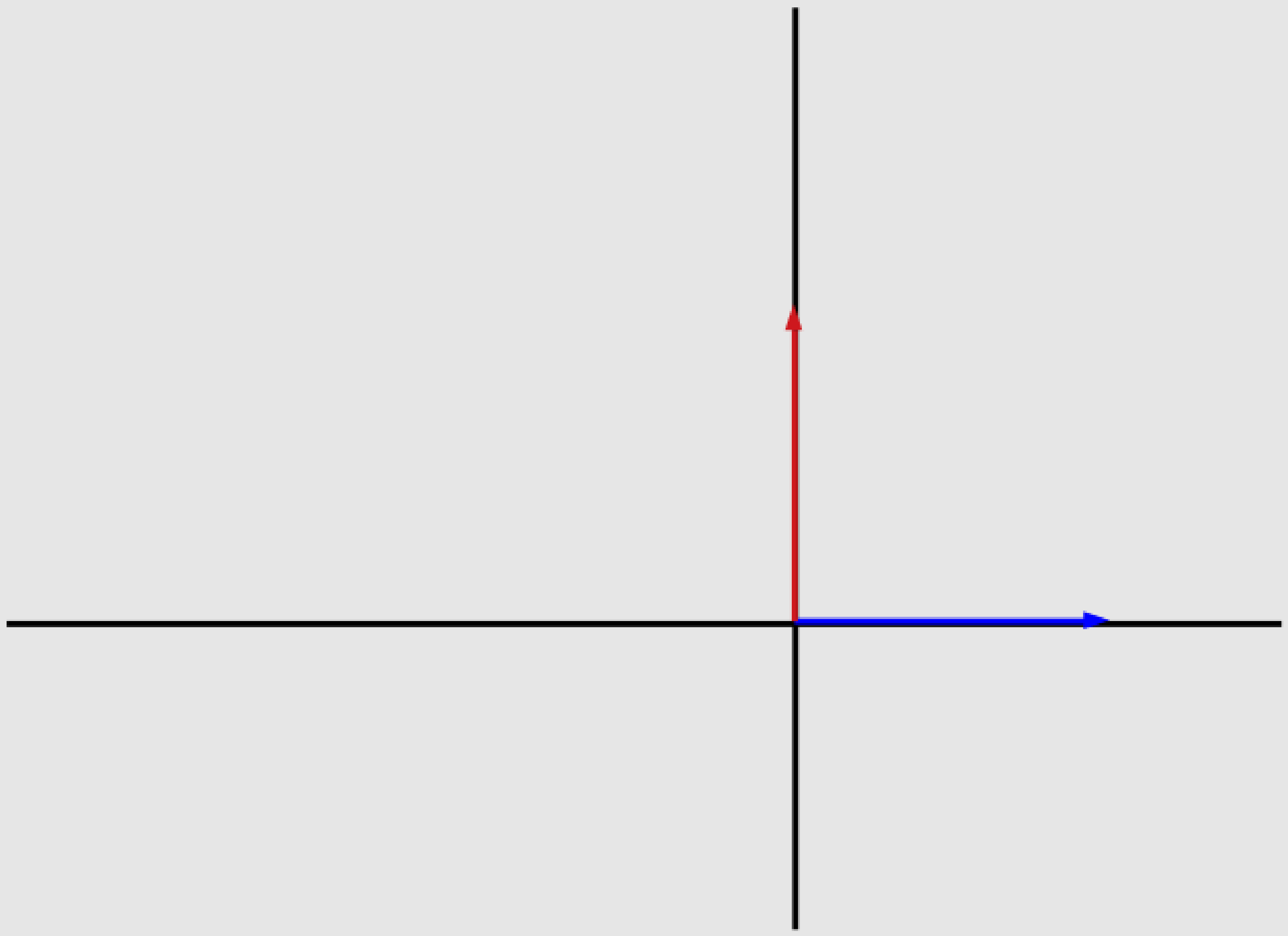
# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

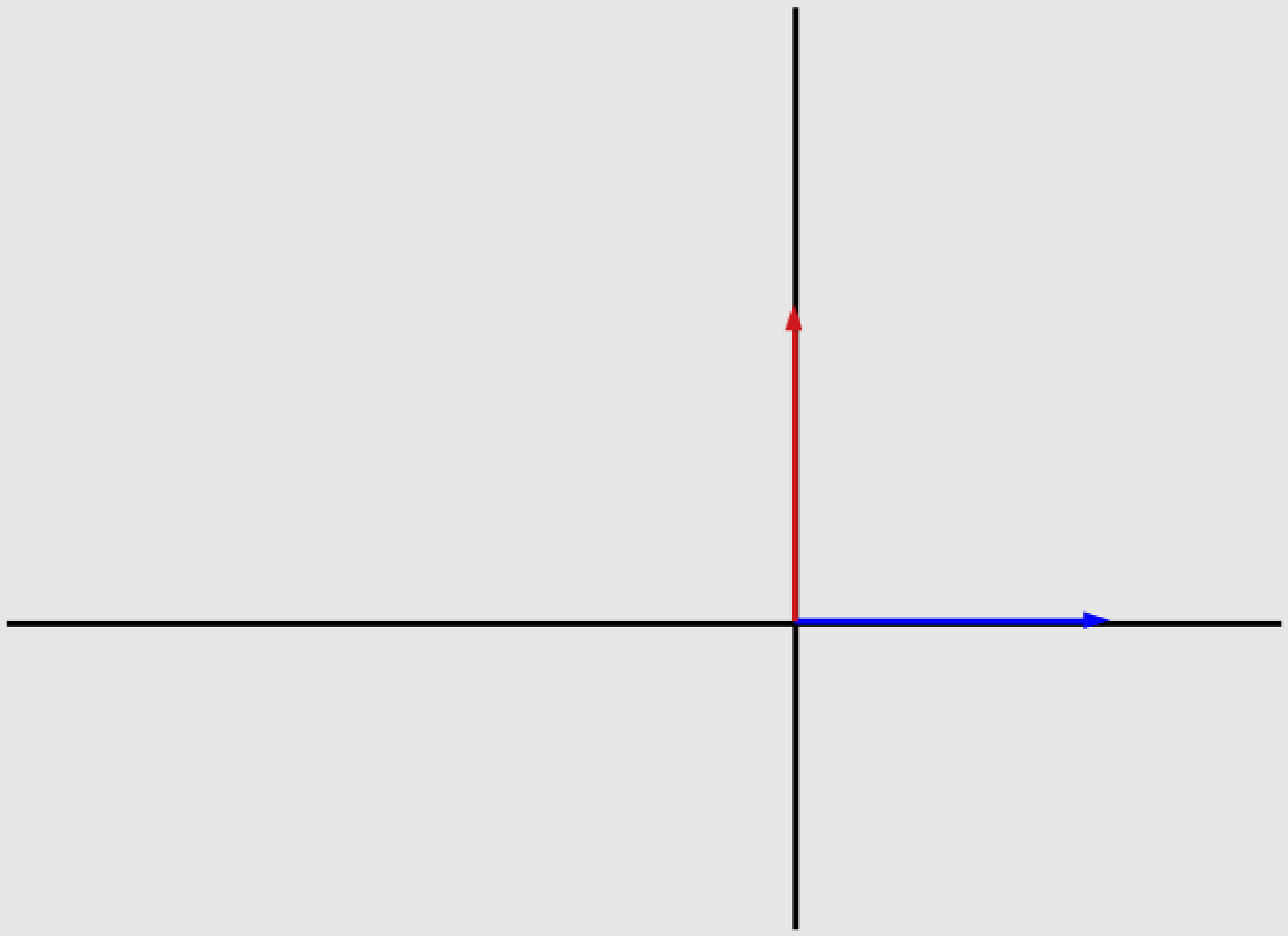
# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Projection onto Orthonormal Bases



orthonormal basis

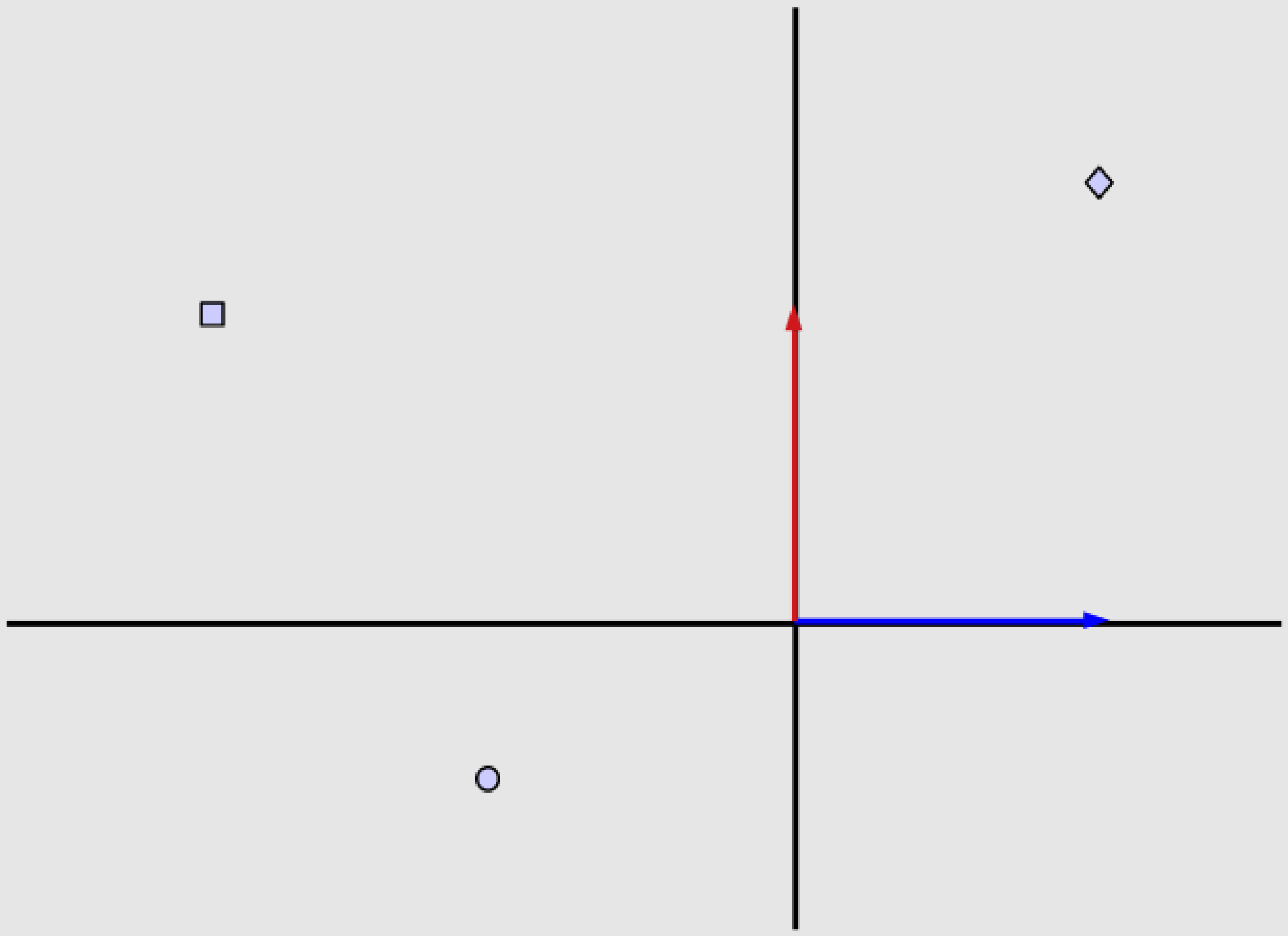
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} \mathbf{x} & \mathbf{y} \end{array}$$

$$D = \begin{bmatrix} 1 & 1.5 \\ -2 & 1 \\ -1 & -0.5 \end{bmatrix}$$

# Projection onto Orthonormal Bases



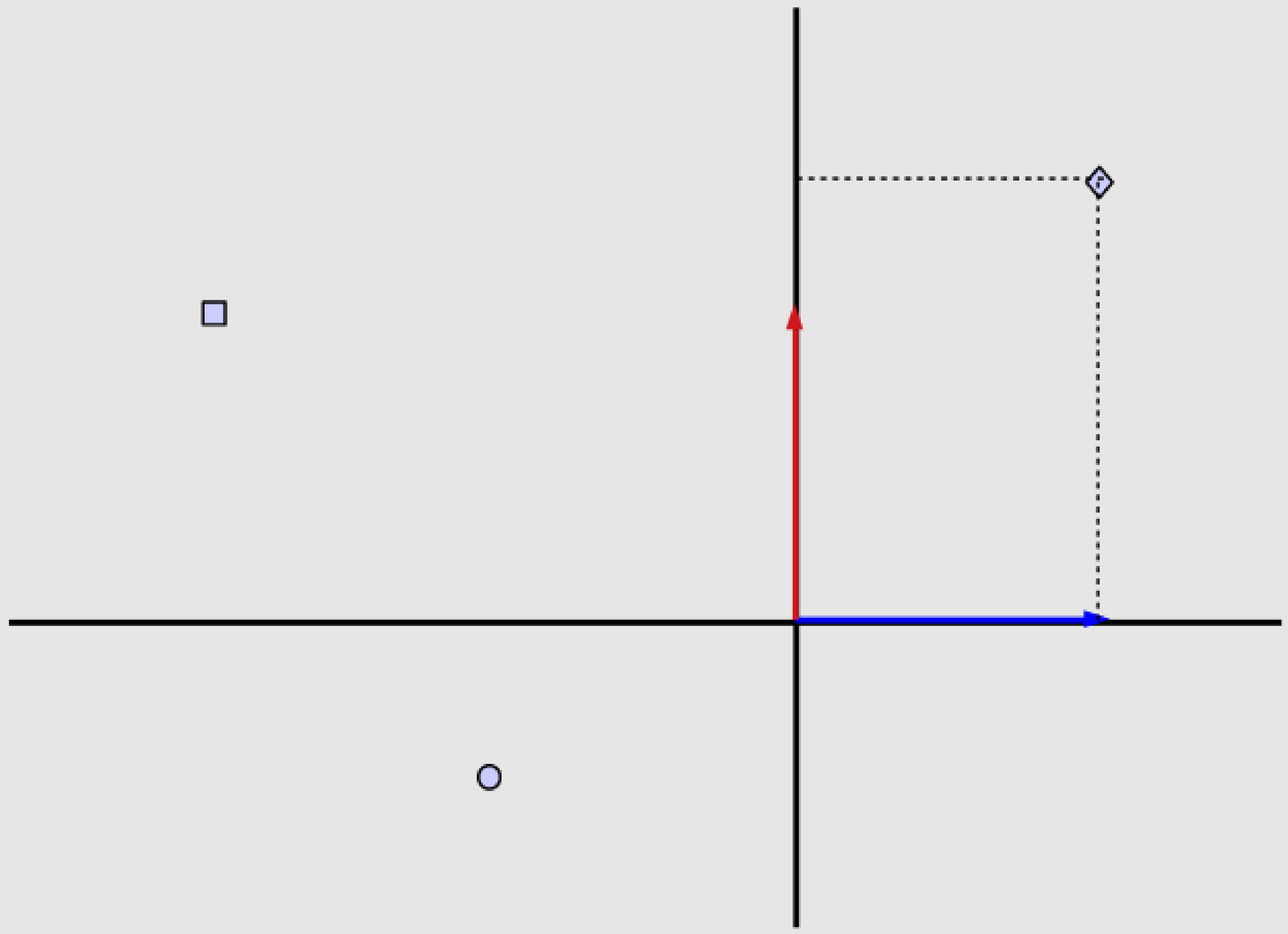
orthonormal basis

$$B_1 = \begin{bmatrix} \color{blue}{1} & \color{red}{0} \\ \color{blue}{0} & \color{red}{1} \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

# Projection onto Orthonormal Bases



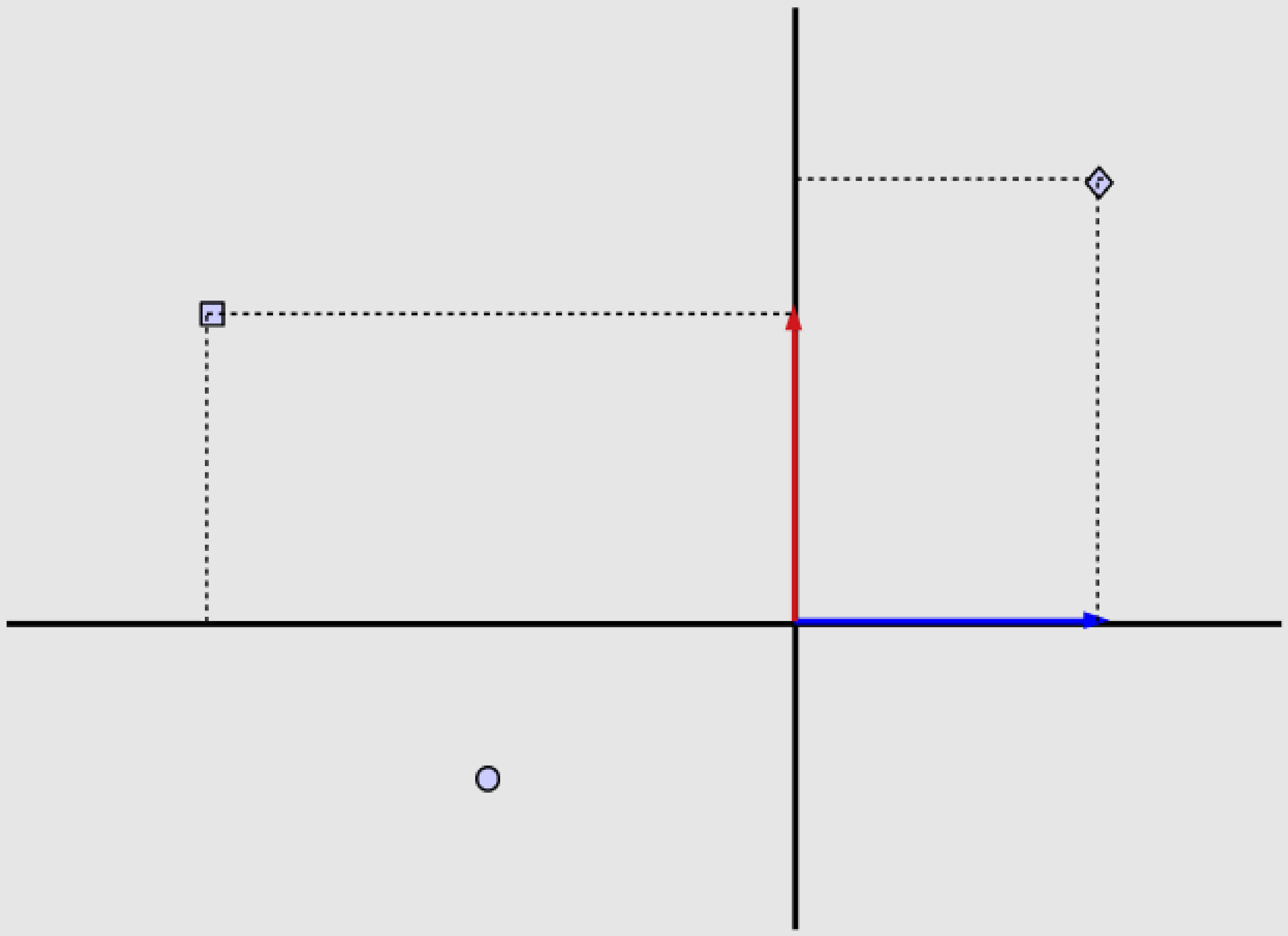
orthonormal basis

$$B_1 = \begin{bmatrix} \color{blue}{1} & \color{red}{0} \\ \color{blue}{0} & \color{red}{1} \end{bmatrix}$$

**SAMPLES**

$$D = \begin{array}{c} \begin{matrix} \mathbf{x} & \quad & \mathbf{y} \end{matrix} \\ \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

# Projection onto Orthonormal Bases



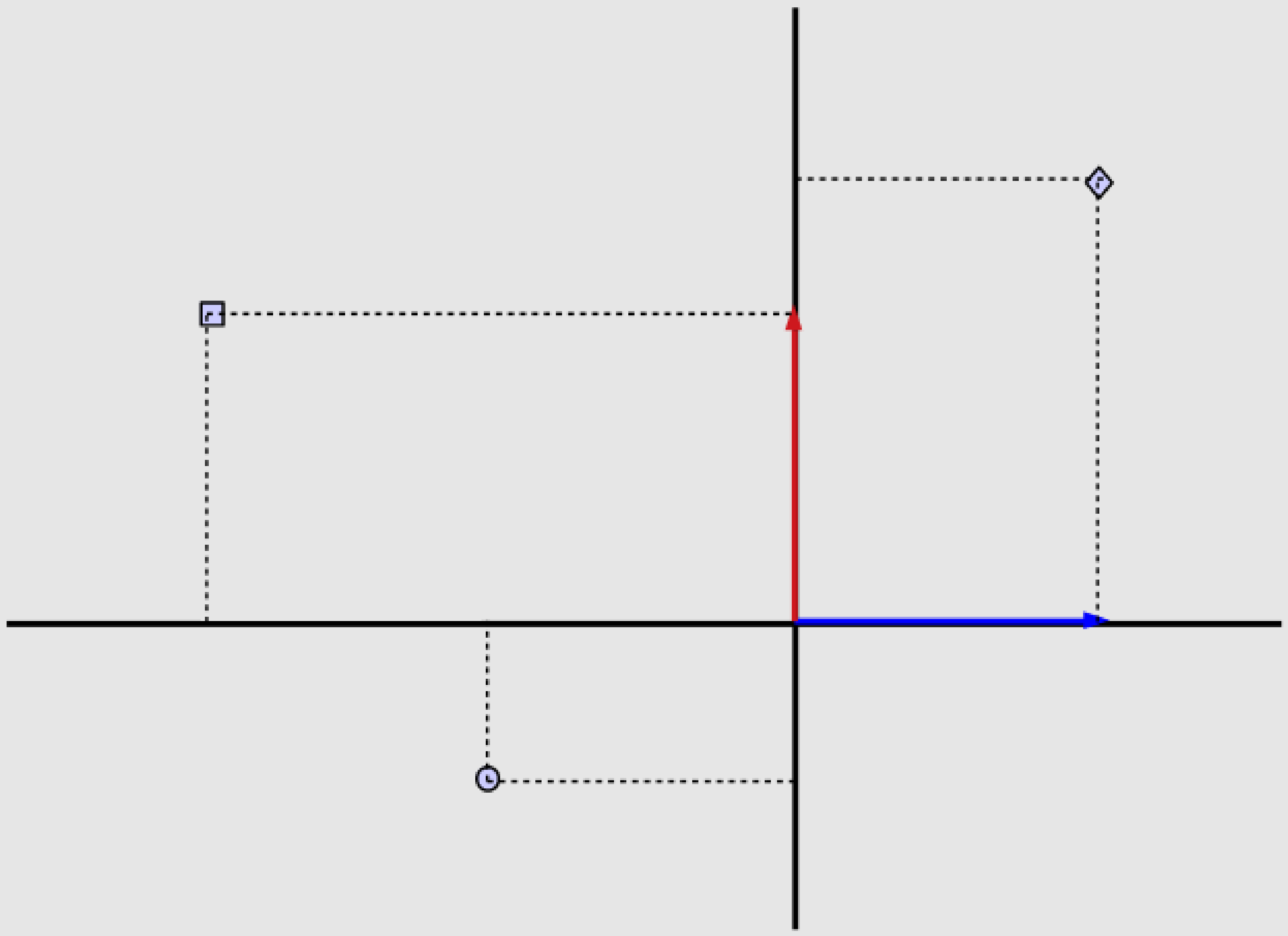
orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} x & \quad & y \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

# Projection onto Orthonormal Bases



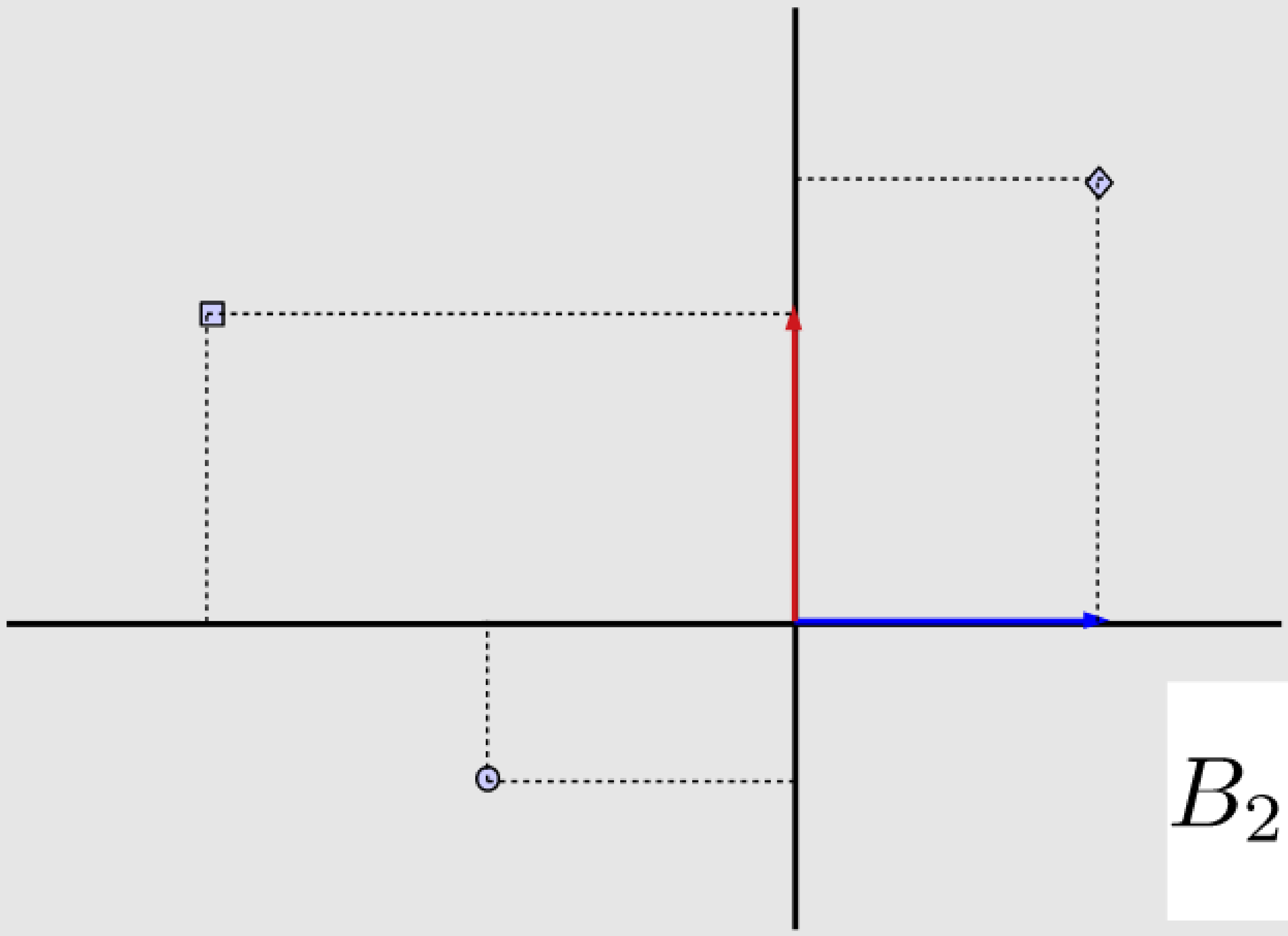
orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

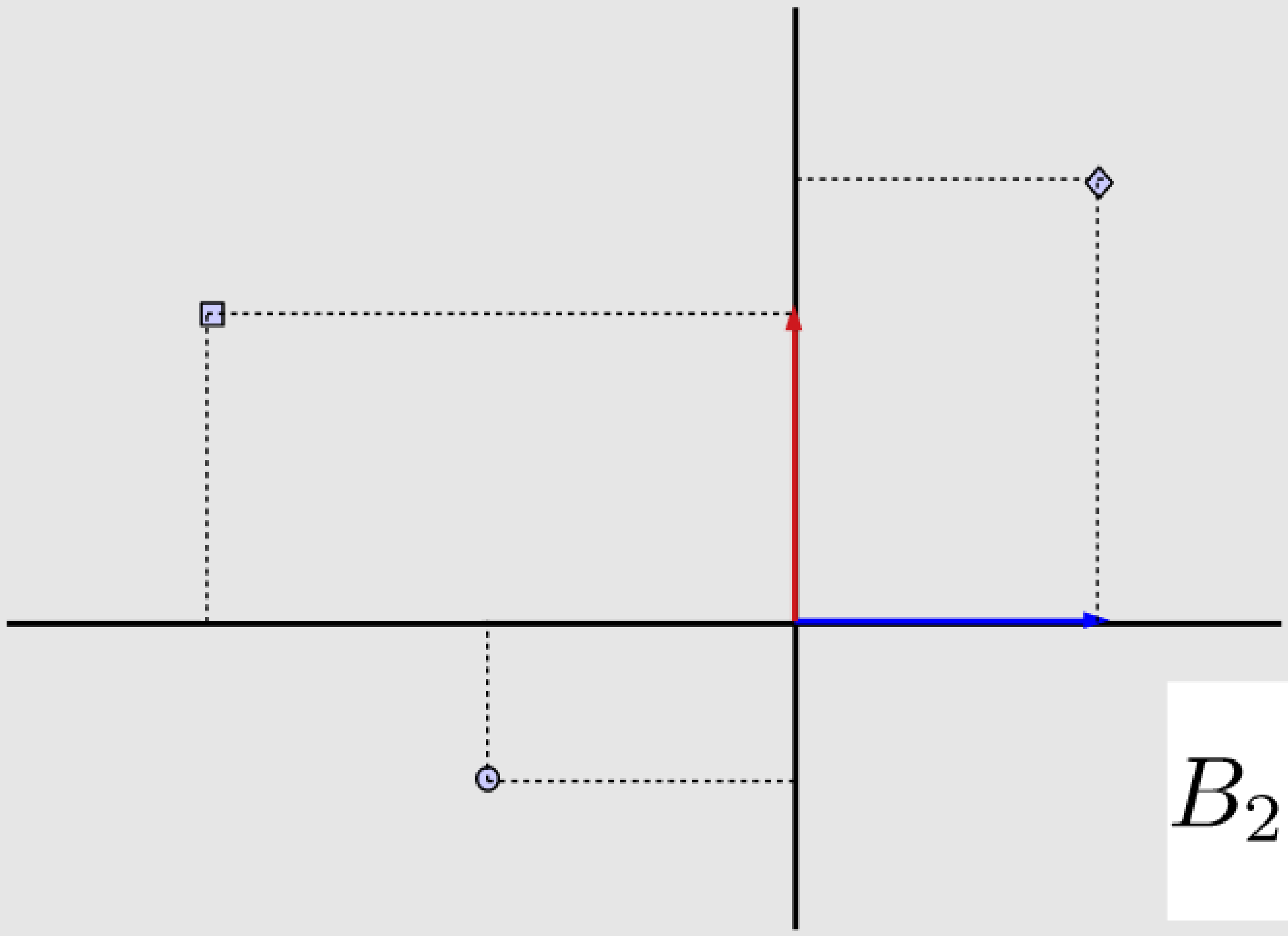
SAMPLES

$$D = \begin{array}{c} \begin{array}{ccc} \mathbf{x} & \quad\quad & \mathbf{y} \end{array} \\ \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$B_2 = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{c} \quad\; \mathbf{x} \qquad\quad \mathbf{y} \\ D = \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

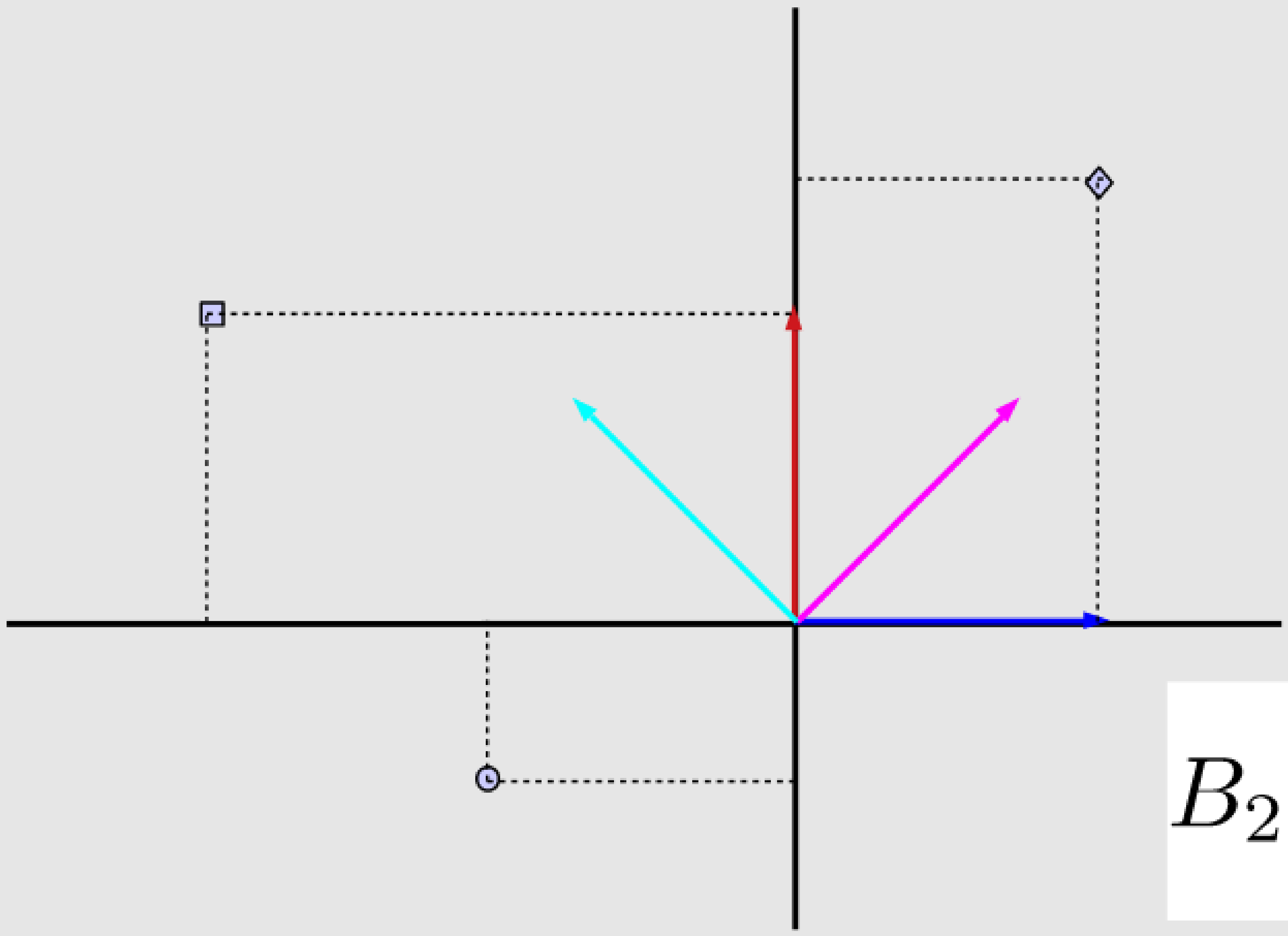
$$\begin{array}{c} \qquad\qquad\quad\;\; \vec{p}_1 \quad\;\; \vec{p}_2 \\ B_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\|\vec{p}_i\| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} \color{blue}{1} & \color{red}{0} \\ \color{blue}{0} & \color{red}{1} \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

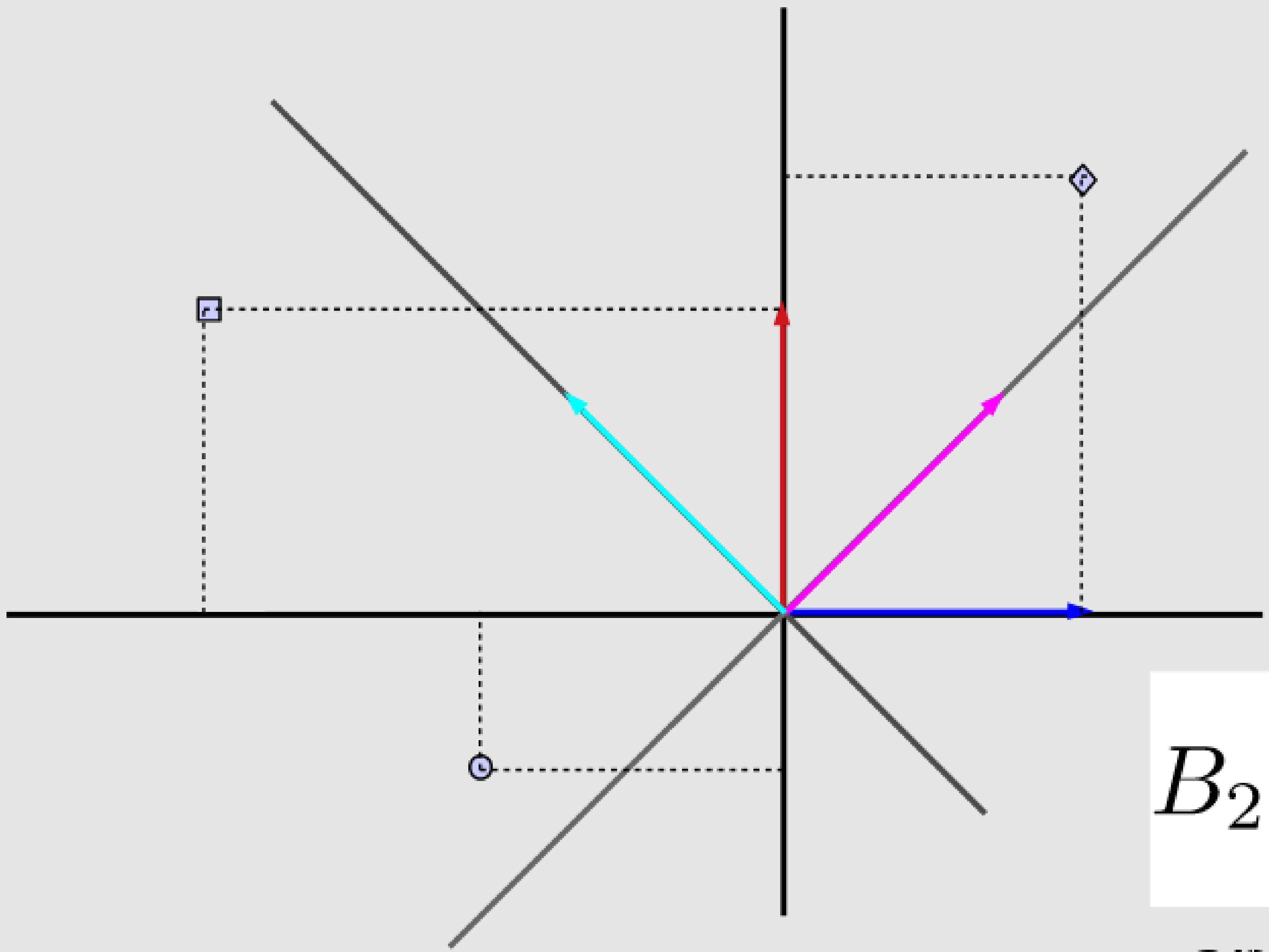
$$\begin{array}{cc} & \begin{array}{cc} \color{cyan}{\vec{p}_1} & \color{magenta}{\vec{p}_2} \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} \color{cyan}{1} & \color{magenta}{1} \\ \color{cyan}{-1} & \color{magenta}{1} \end{bmatrix} \end{array}$$

$$\|\vec{p}_i\| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \textbf{SAMPLES} \\ \begin{array}{cc} \mathbf{x} & \mathbf{y} \end{array} \end{array}$$

$$D = \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix}$$

$$\begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array}$$

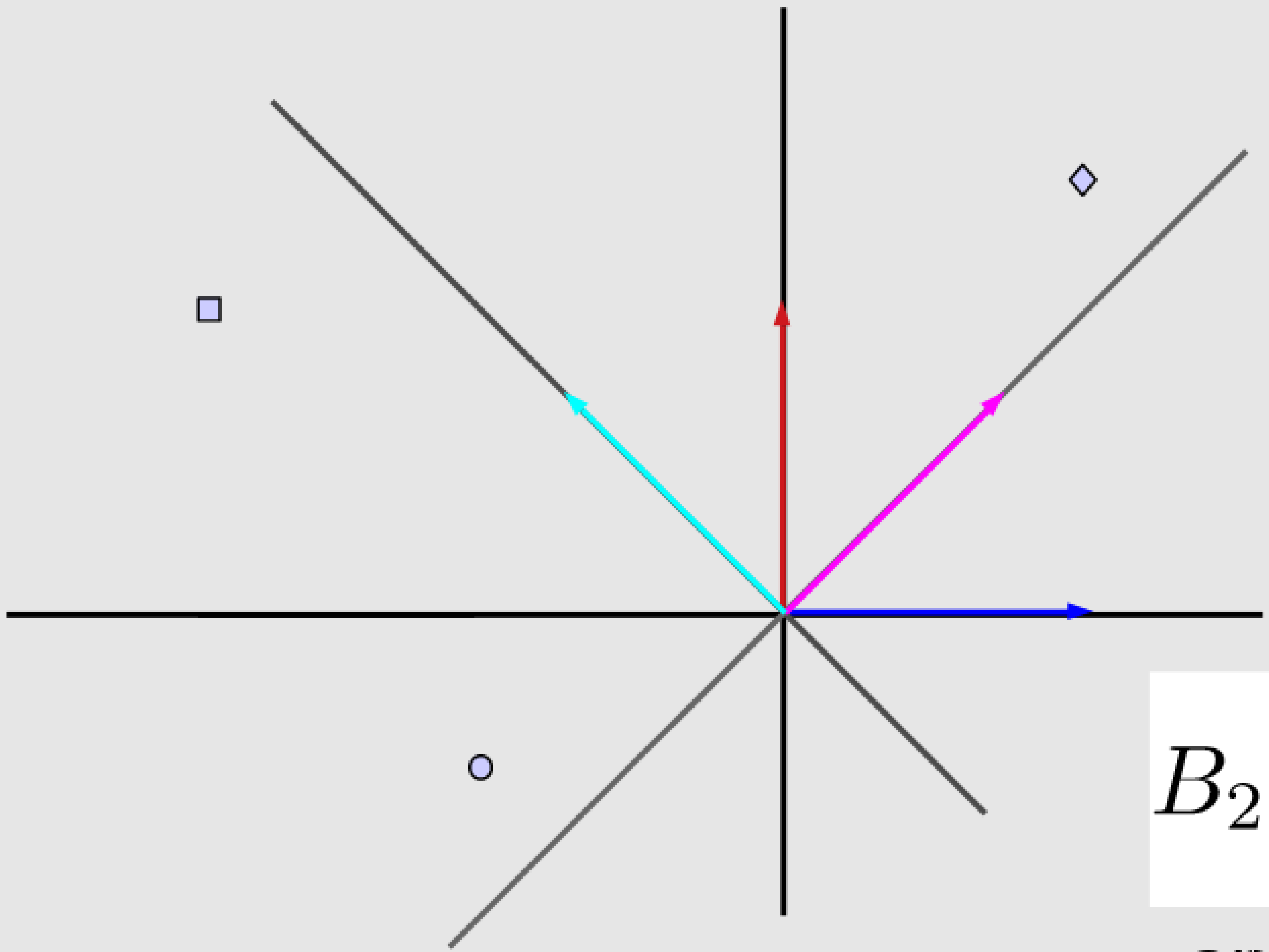
$$B_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\|\vec{p}_i\| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

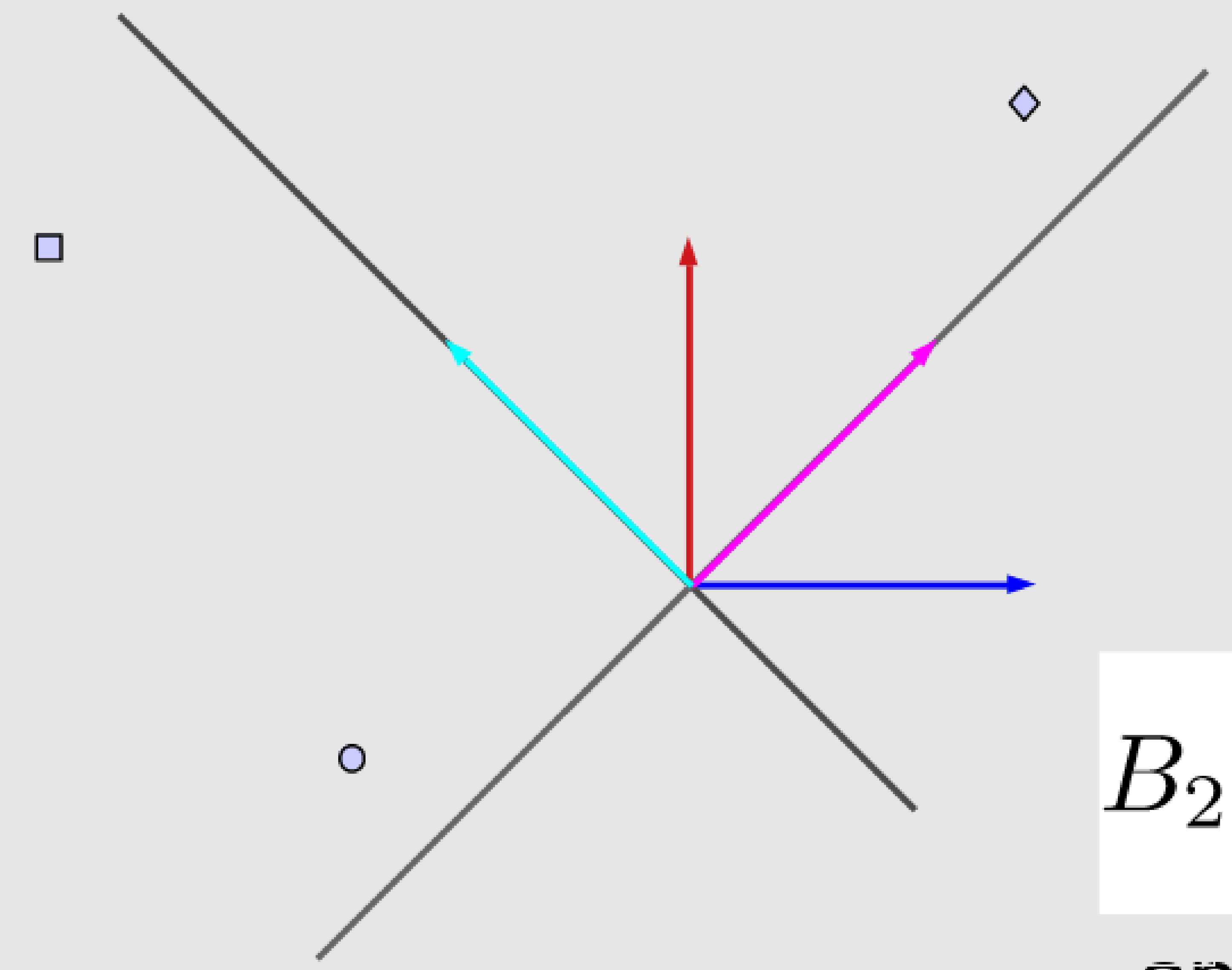
$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$||\vec{p}_i|| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & \quad & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

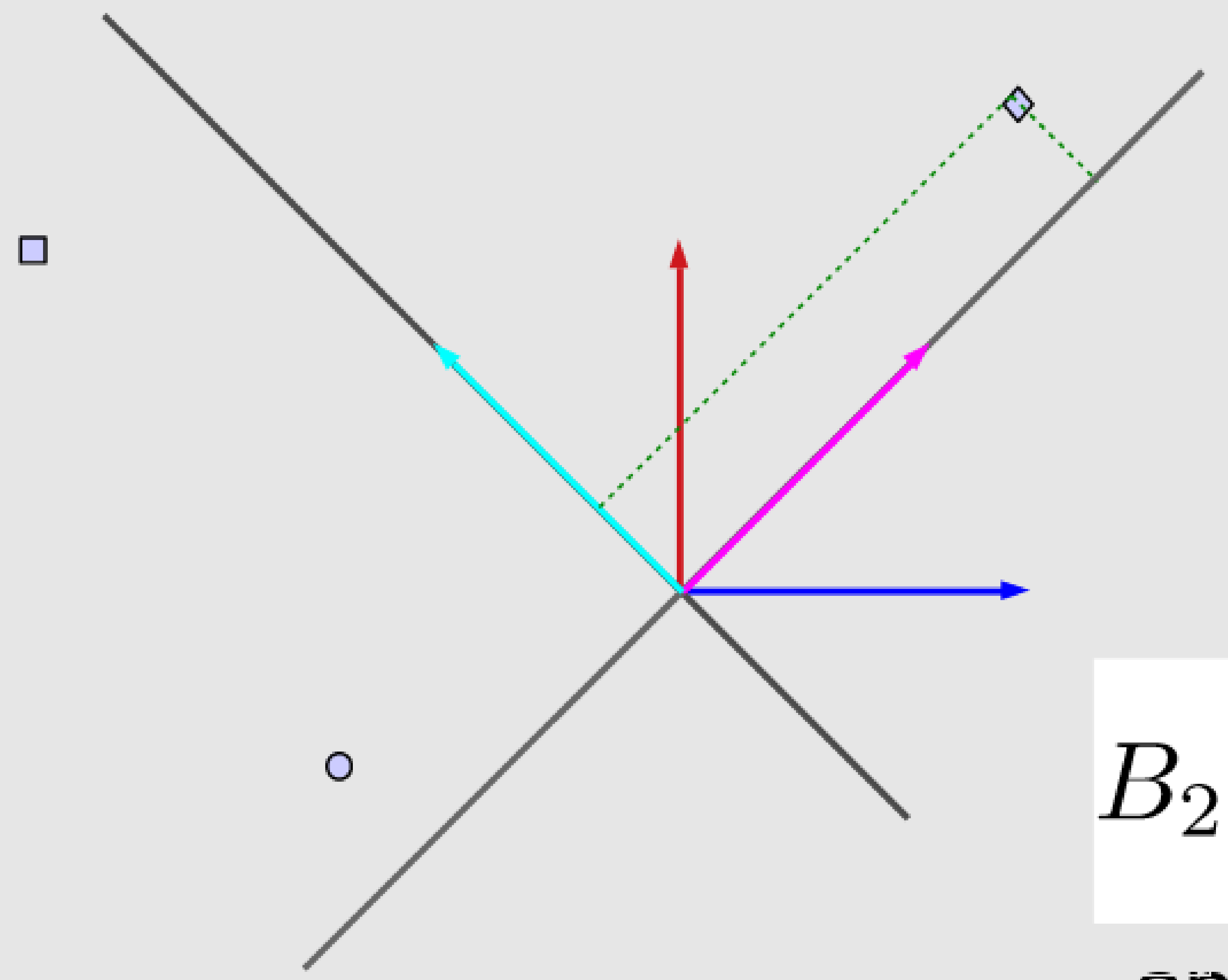
$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\|\vec{p}_i\| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases



orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

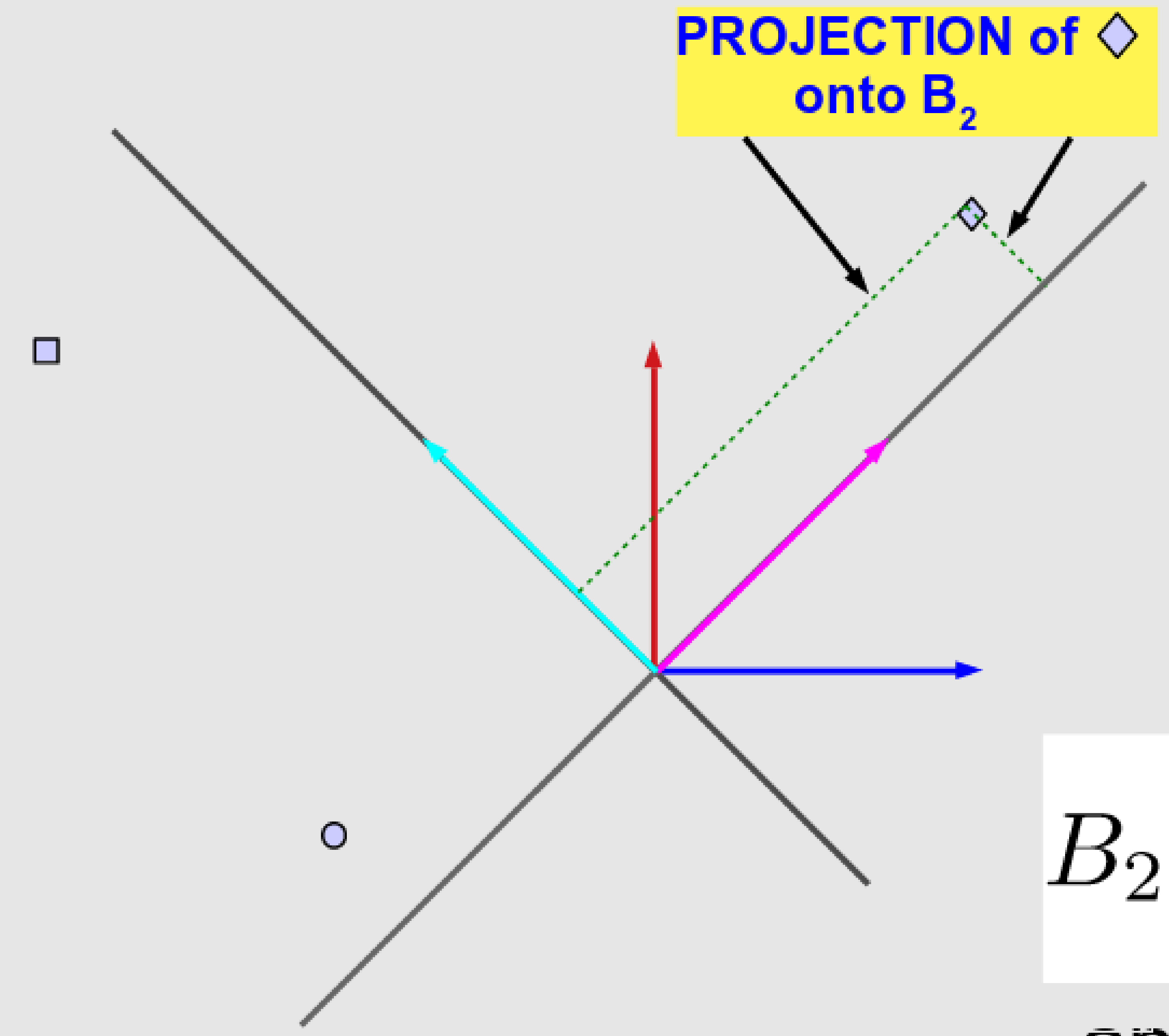
SAMPLES

$$D = \begin{array}{c} \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$B_2 = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad \|\vec{p}_i\| = 1 \qquad \vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

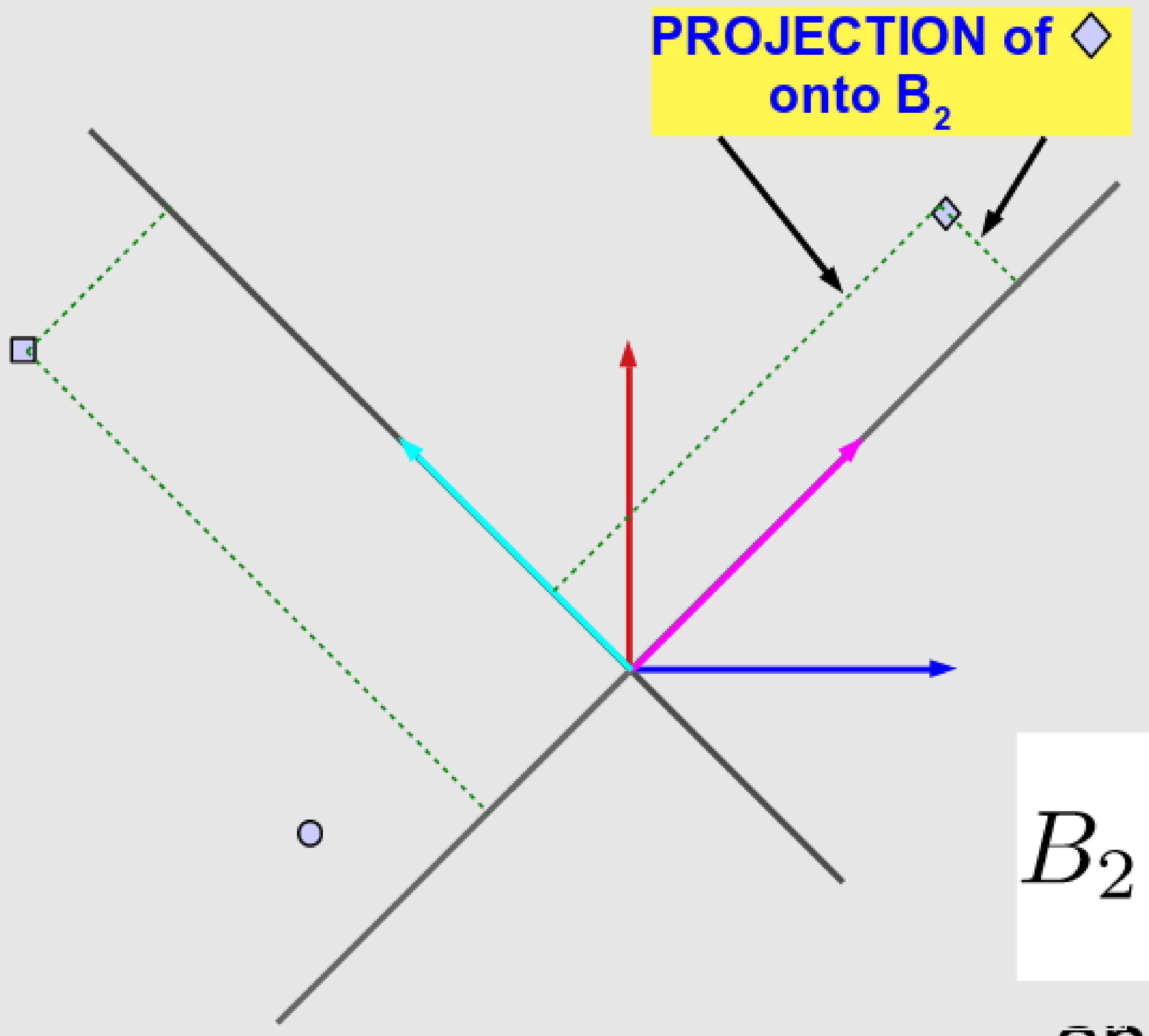
$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$||\vec{p}_i|| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{c} \\ D = \end{array} \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \\ \left[\begin{array}{c} 1 \\ -2 \\ -1 \end{array}\right. & \begin{array}{c} \diamond \\ \square \\ \circ \end{array} & \left.\begin{array}{c} 1.5 \\ 1 \\ -0.5 \end{array}\right] \end{array}$$

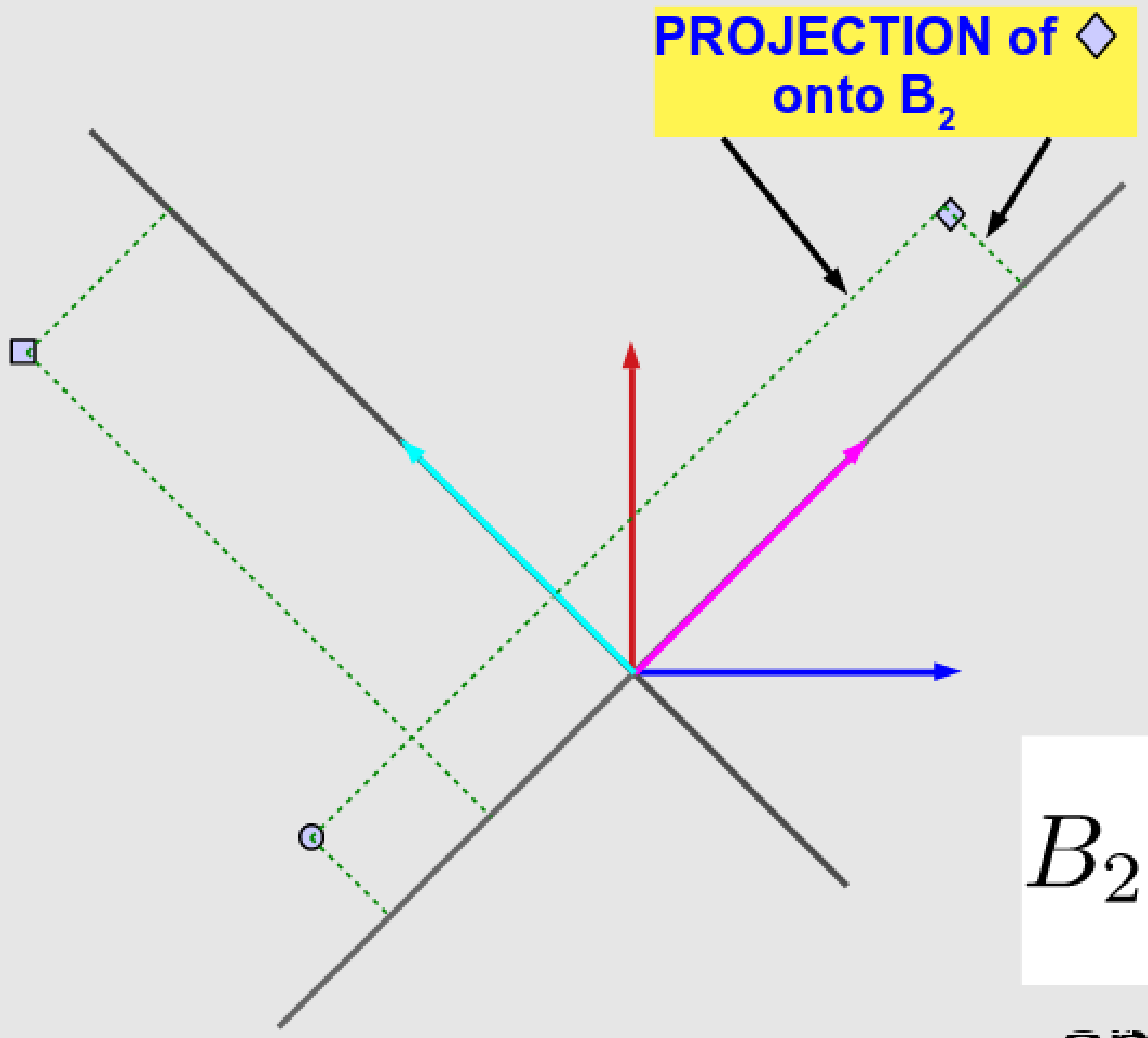
$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$||\vec{p}_i|| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**SAMPLES**

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

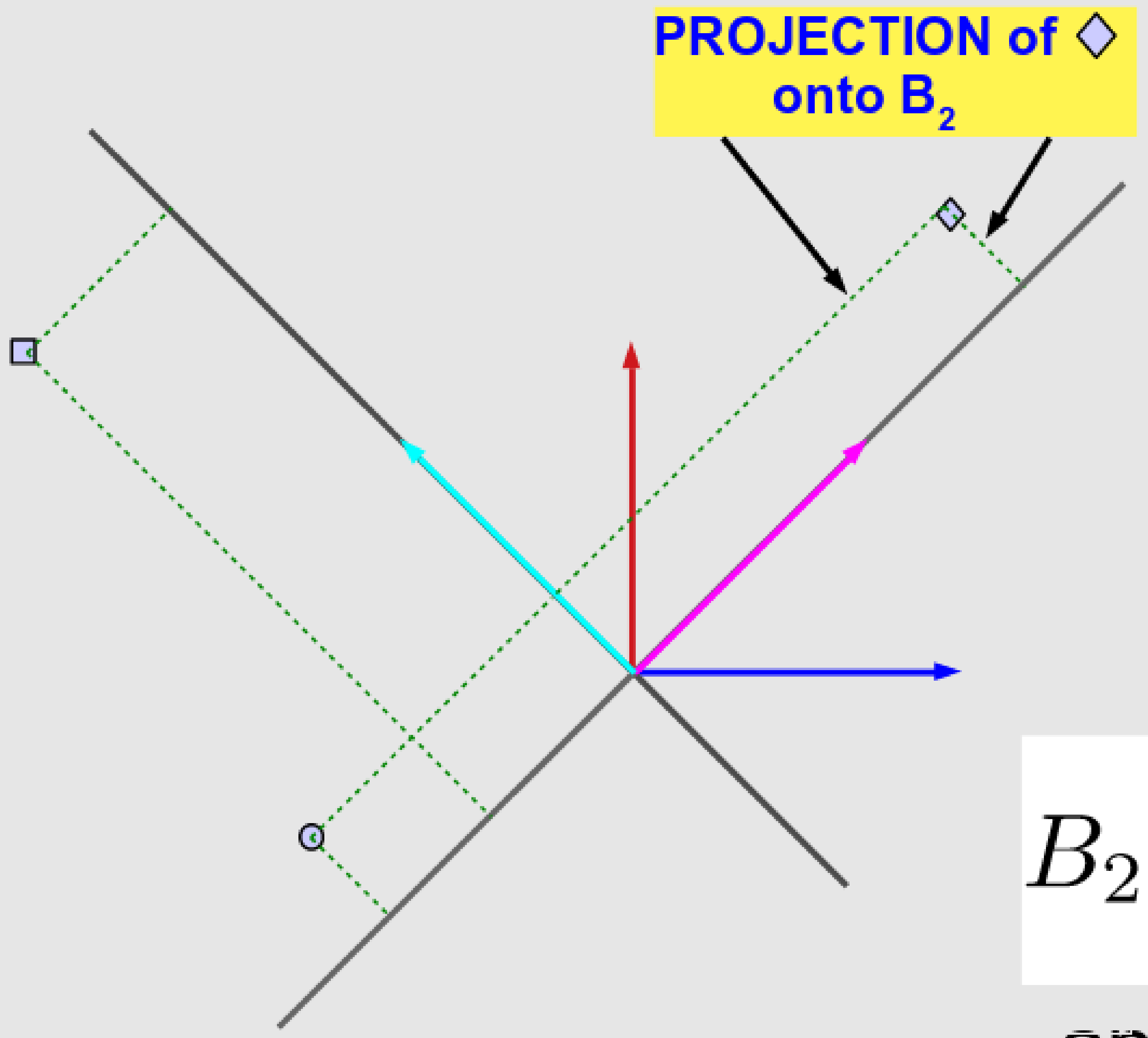
$$\begin{array}{ccc} & & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = & \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$||\vec{p}_i|| = 1$$

$$\vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

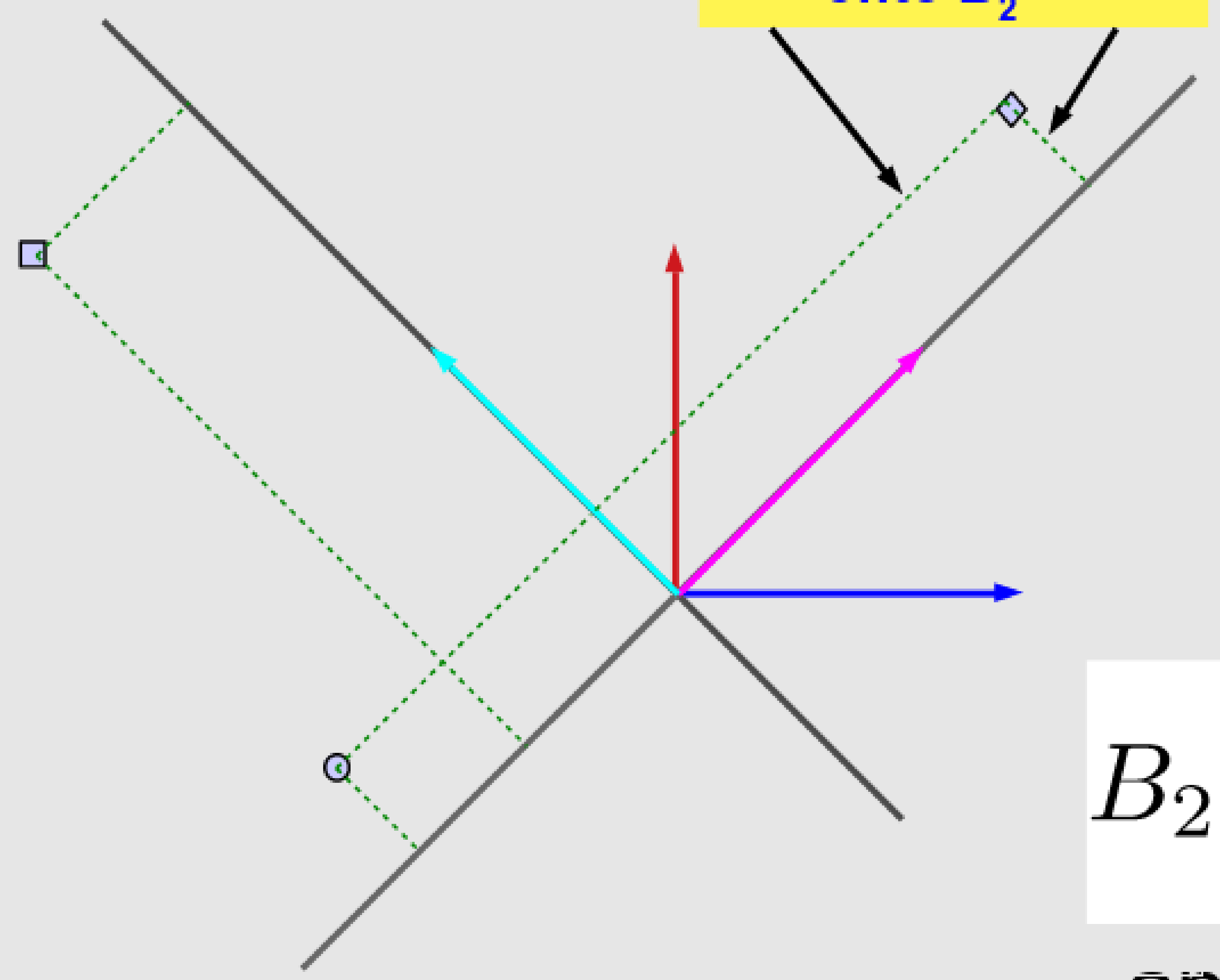
$$D = \begin{array}{c} \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$B_2 = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad ||\vec{p}_i|| = 1 \qquad \vec{p}_1^{\,T} \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

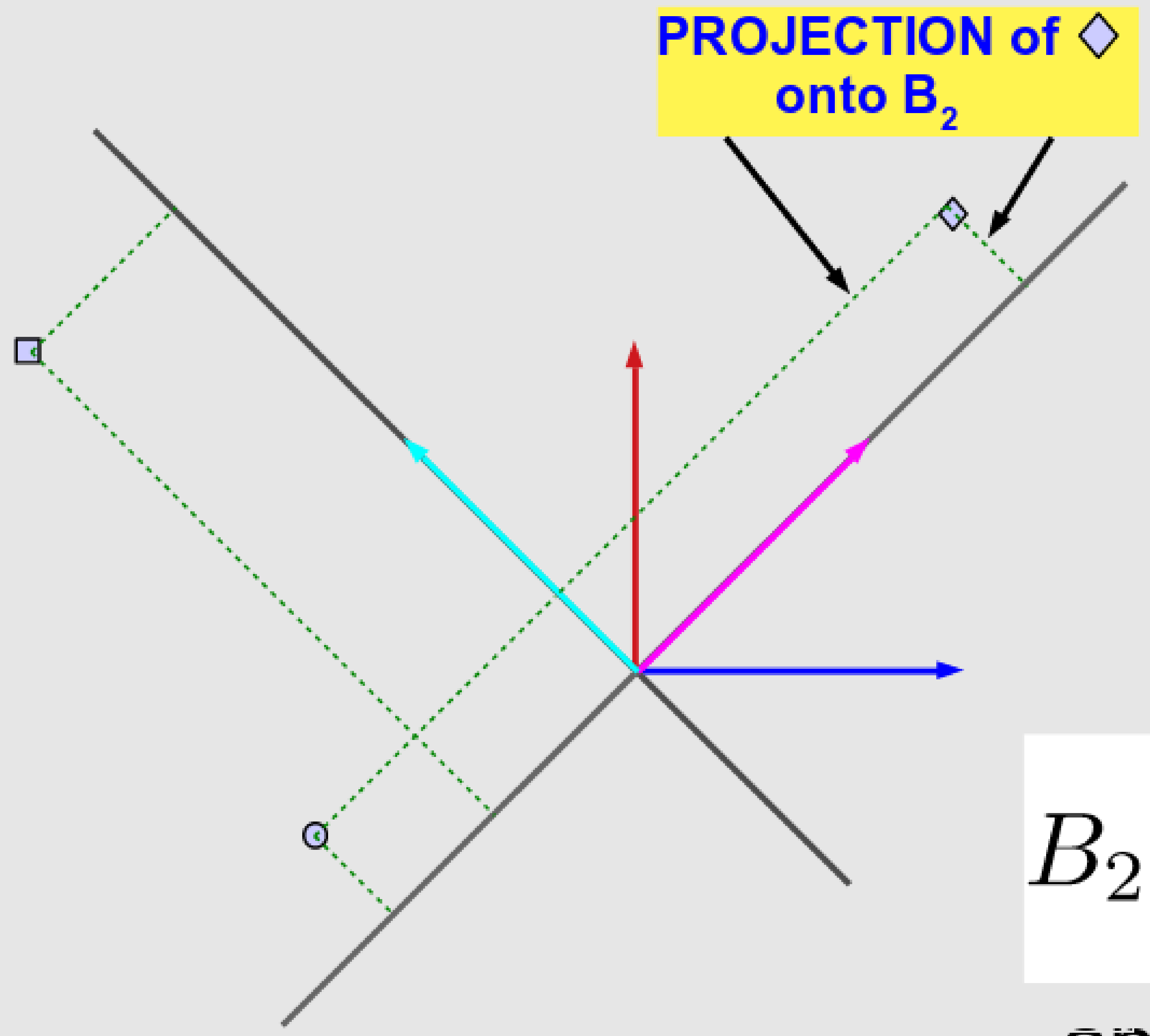
$$\begin{array}{cc} & \begin{array}{ccc} \textbf{x} & & \textbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \quad ||\vec{p}_i|| = 1 \quad \vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$, or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^T + \beta \vec{p}_2^T$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} \text{SAMPLES} & \\ \textbf{x} & \textbf{y} \end{array}$$

$$D = \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix}$$

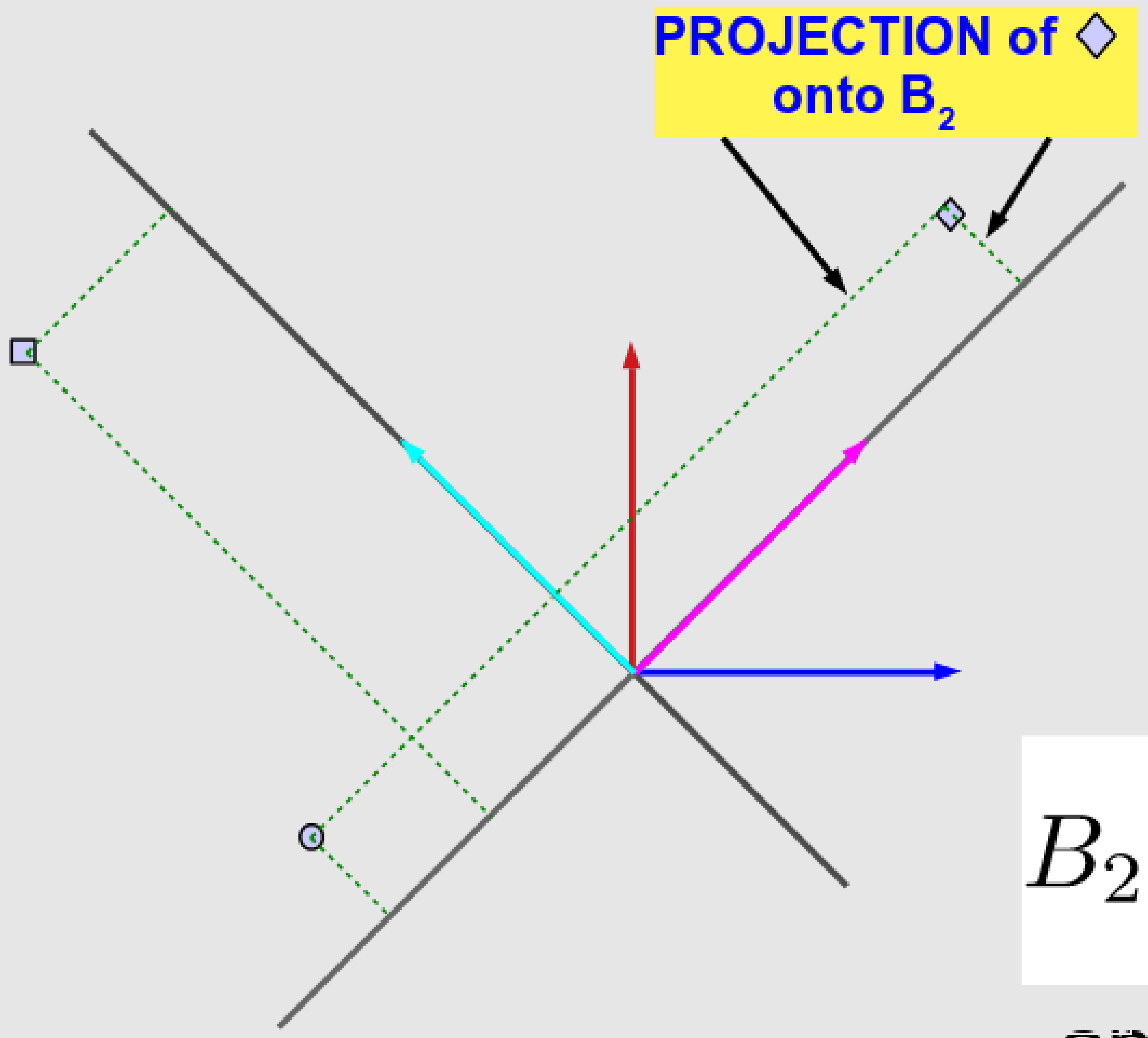
$$\begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array}$$

$$B_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad ||\vec{p}_i|| = 1 \quad \vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$, or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^T + \beta \vec{p}_2^T$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$, $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

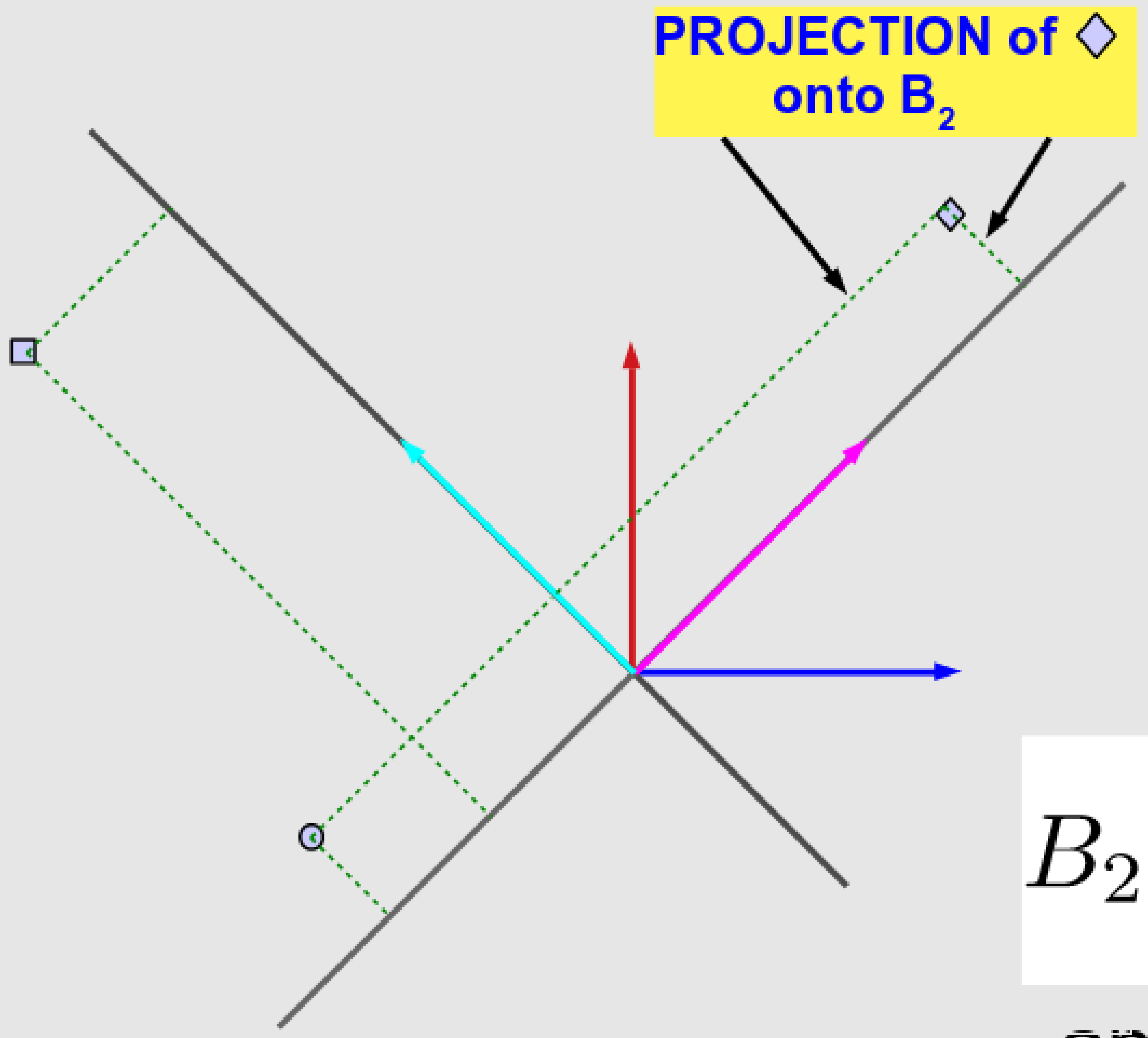
$||\vec{p}_i|| = 1$

$\vec{p}_1^T \vec{p}_2 = 0$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$ , or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^T + \beta \vec{p}_2^T$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$ , $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$
    - ➔ or: $\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} B_2$ ; or, for all the data $D_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = D B_2$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} & \begin{array}{ccc} x & & y \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

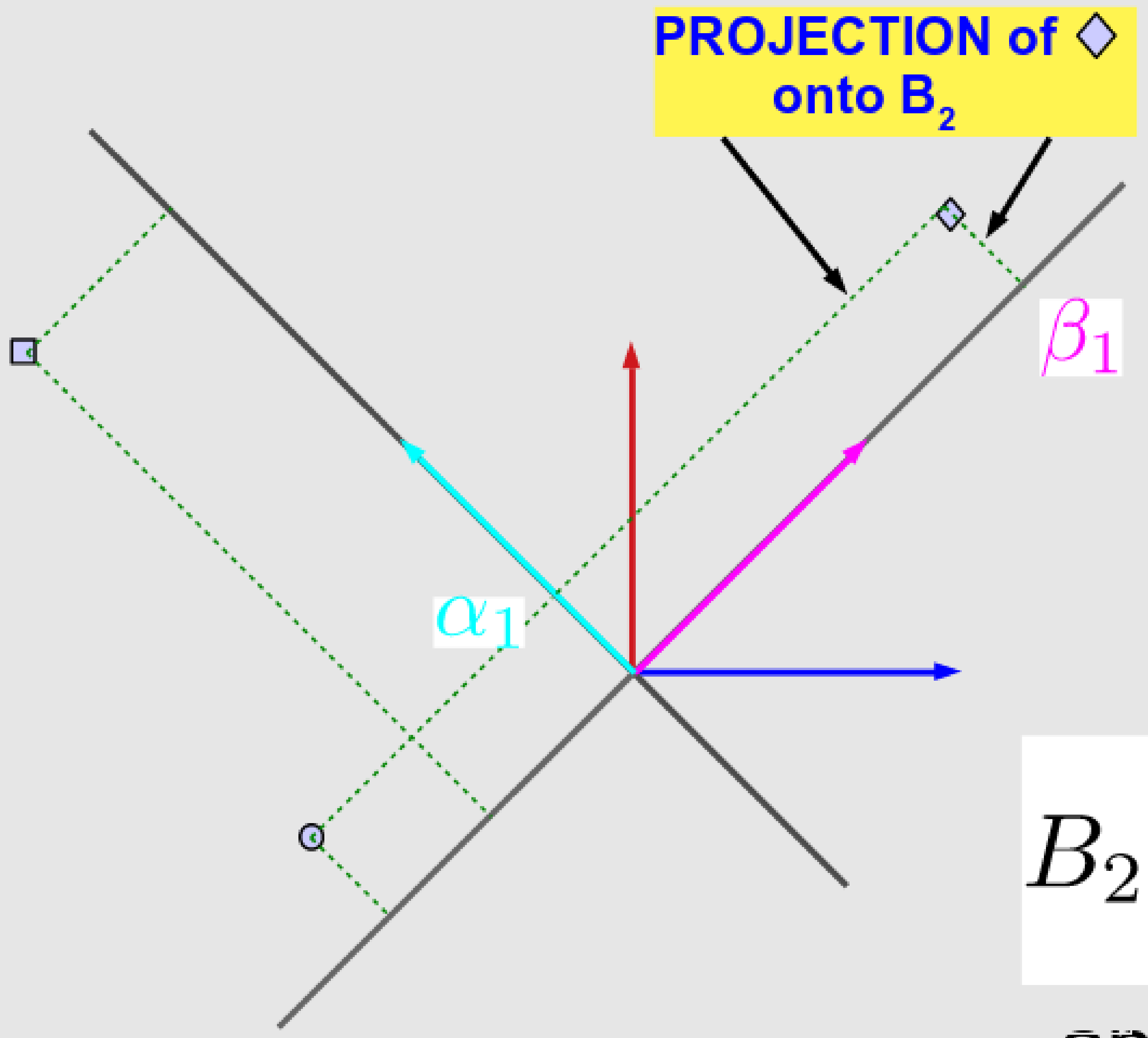
$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad ||\vec{p}_i|| = 1 \qquad \vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$ , or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^T + \beta \vec{p}_2^T$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$ , $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$
    - or: $\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} B_2$ ; or, for all the data $D_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \boxed{DB_2}$

projecting the data D onto the basis $B_2$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$D = \begin{array}{c} \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

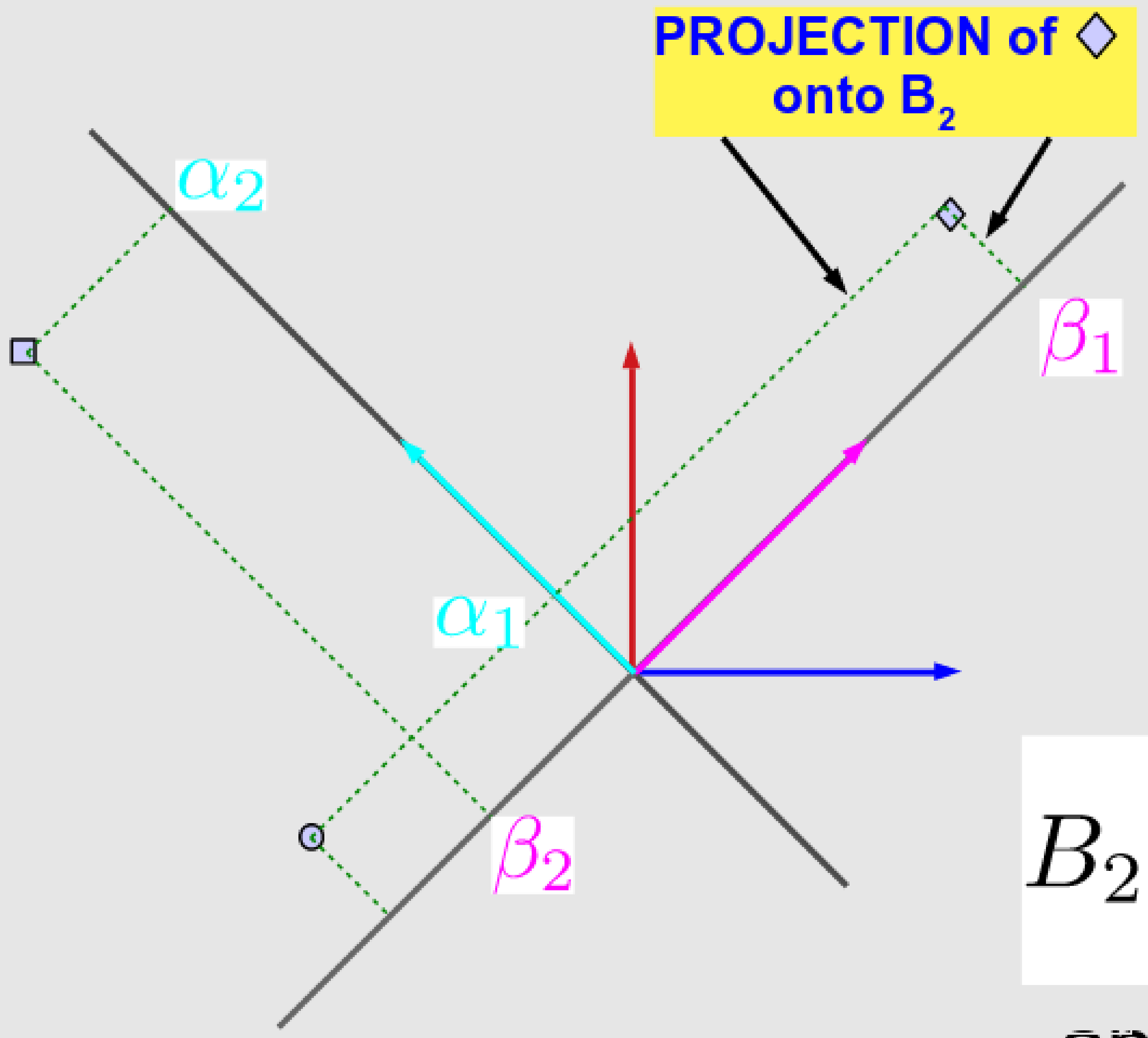
$$B_2 = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad \|\vec{p}_i\| = 1 \qquad \vec{p}_1^{\,T} \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$ , or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^{\,T} + \beta \vec{p}_2^{\,T}$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$ , $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$
    - or: $\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} B_2$ ; or, for all the data $D_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \boxed{DB_2}$

projecting the data D onto the basis $B_2$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} \mathbf{x} & \mathbf{y} \end{array}$$

$$D = \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix}$$

$$\begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array}$$

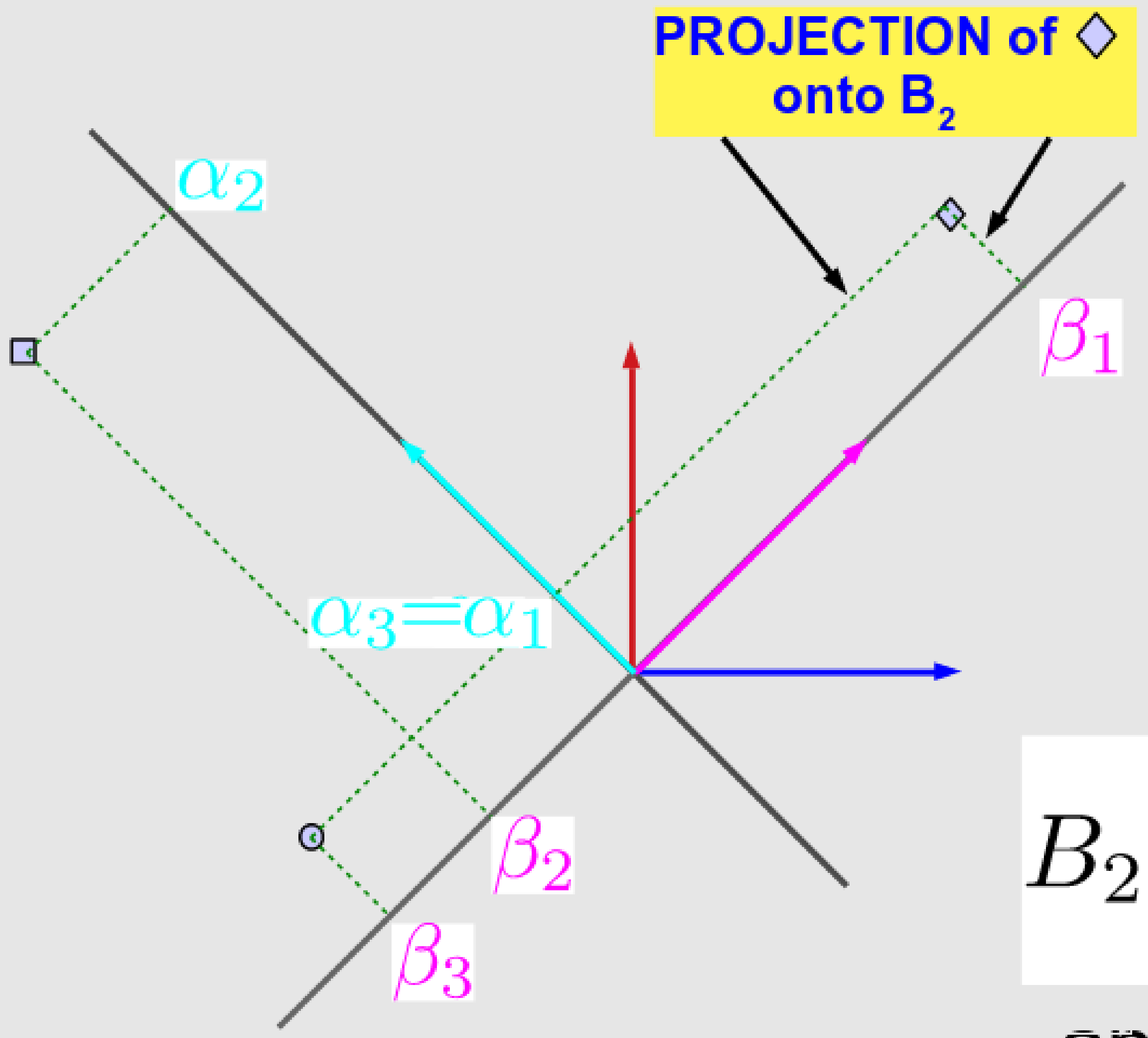
$$B_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad ||\vec{p}_i|| = 1 \qquad \vec{p}_1^T \vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{p}_1 + \beta \vec{p}_2$, or $\begin{bmatrix} x & y \end{bmatrix} = \alpha \vec{p}_1^T + \beta \vec{p}_2^T$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$, $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$
    - ➔ or: $\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} B_2$; or, for all the data $D_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \boxed{DB_2}$

projecting the data D onto the basis $B_2$

# Projection onto Orthonormal Bases




orthonormal basis

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

SAMPLES

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{x} & & \mathbf{y} \end{array} \\ D = & \begin{bmatrix} 1 & \diamond & 1.5 \\ -2 & \square & 1 \\ -1 & \circ & -0.5 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 & \vec{p}_2 \end{array} \\ B_2 = \frac{1}{\sqrt{2}} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad ||\vec{p}_i|| = 1 \qquad \vec{p}_1^{\,T}\vec{p}_2 = 0$$

another orthonormal basis

- How can we calculate the projections?
  - data point: $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\vec{p}_1 + \beta\vec{p}_2$, or $\begin{bmatrix} x & y \end{bmatrix} = \alpha\vec{p}_1^{\,T} + \beta\vec{p}_2^{\,T}$
  - post-multiply by basis vectors: $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_1 = \alpha$, $\begin{bmatrix} x & y \end{bmatrix} \vec{p}_2 = \beta$
    - ➔ or: $\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} B_2$; or, for all the data $D_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \boxed{DB_2}$

projecting the data D onto the basis $B_2$

# Using the SVD for Data Analysis, Feature Extraction and Clustering

# Matrices Representing Ratings

- Movies rated by Users (eg, Netflix, Amazon Video)

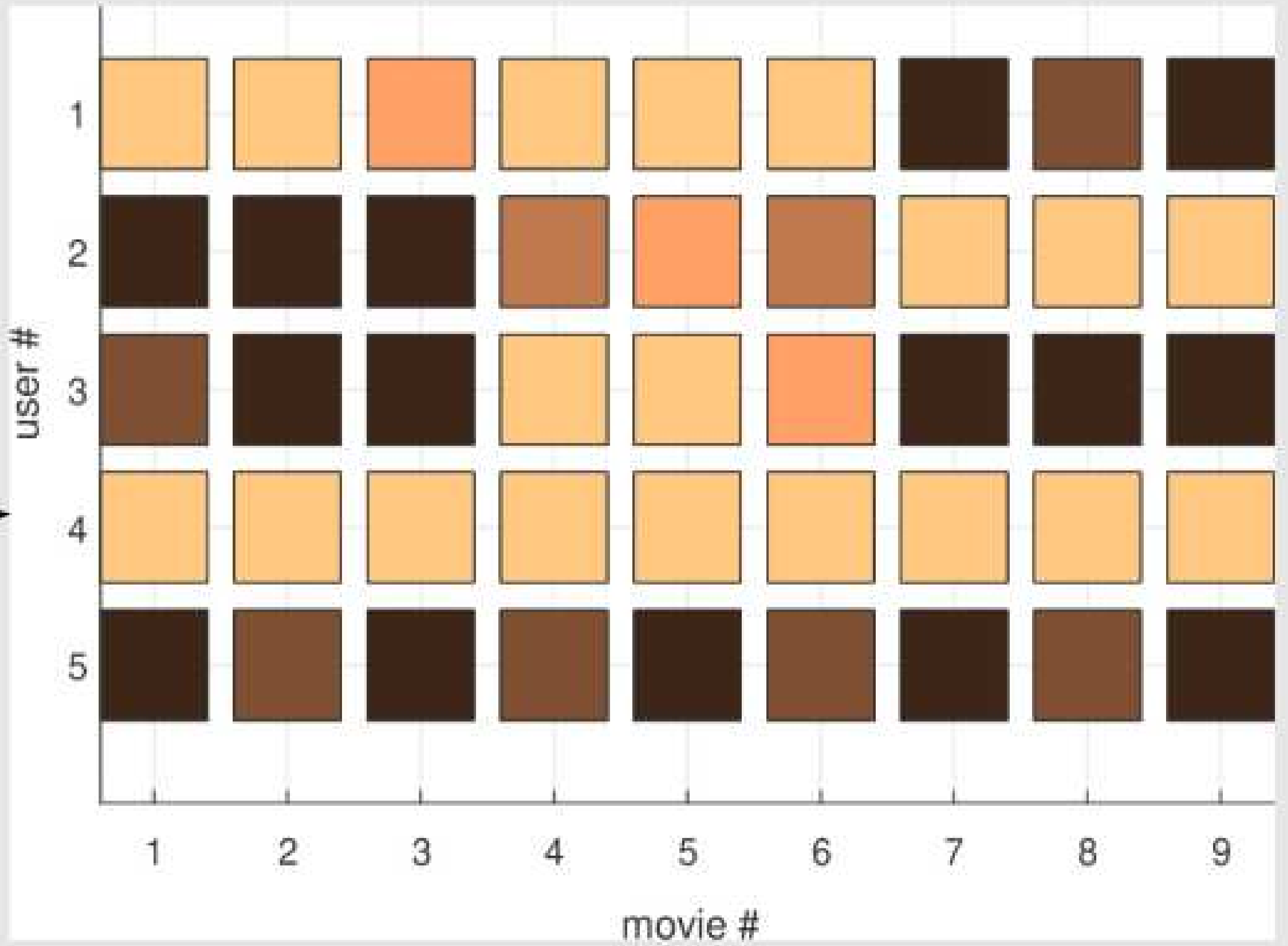
| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

# Matrices Representing Ratings

- Movies rated by Users (eg, Netflix, Amazon Video)

| Movie → User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

**lighter colours = stronger ratings**




$$= U\,\Sigma\,V^T$$

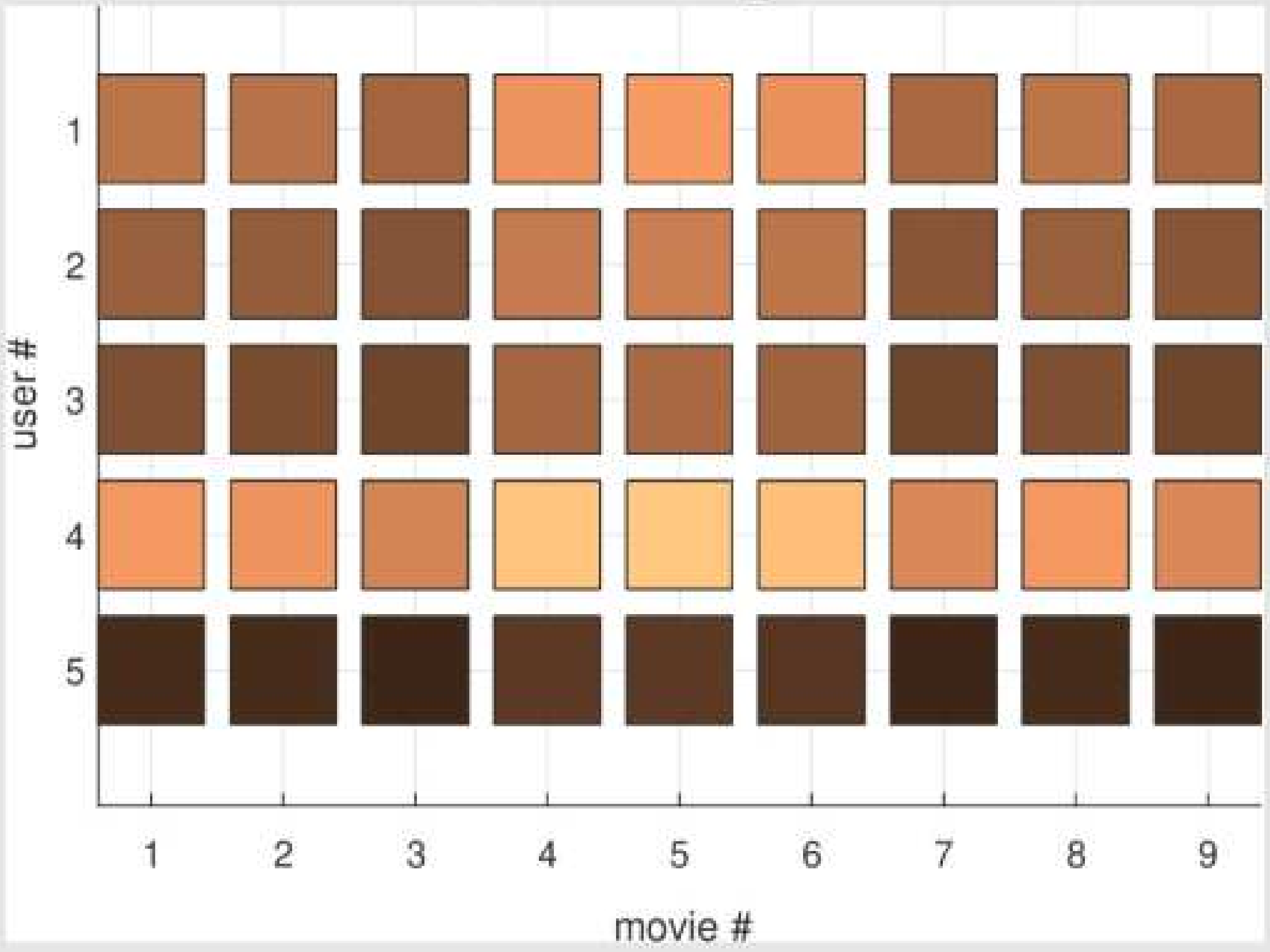
# Features of Rating Matrices

| Movie → User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

# Features of Rating Matrices

| Movie → / User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

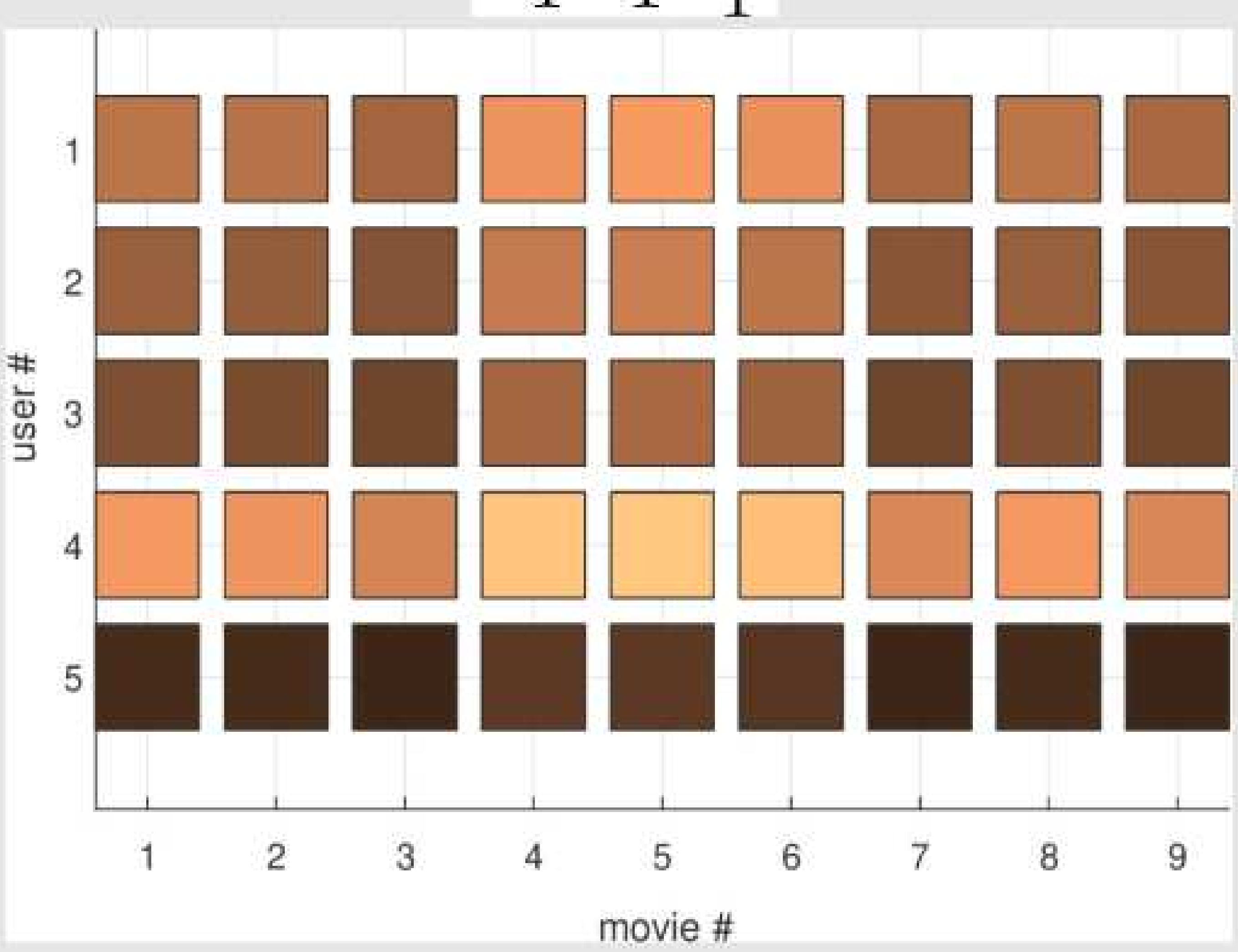
$\sigma_1 \vec{u}_1 \vec{v}_1^T$



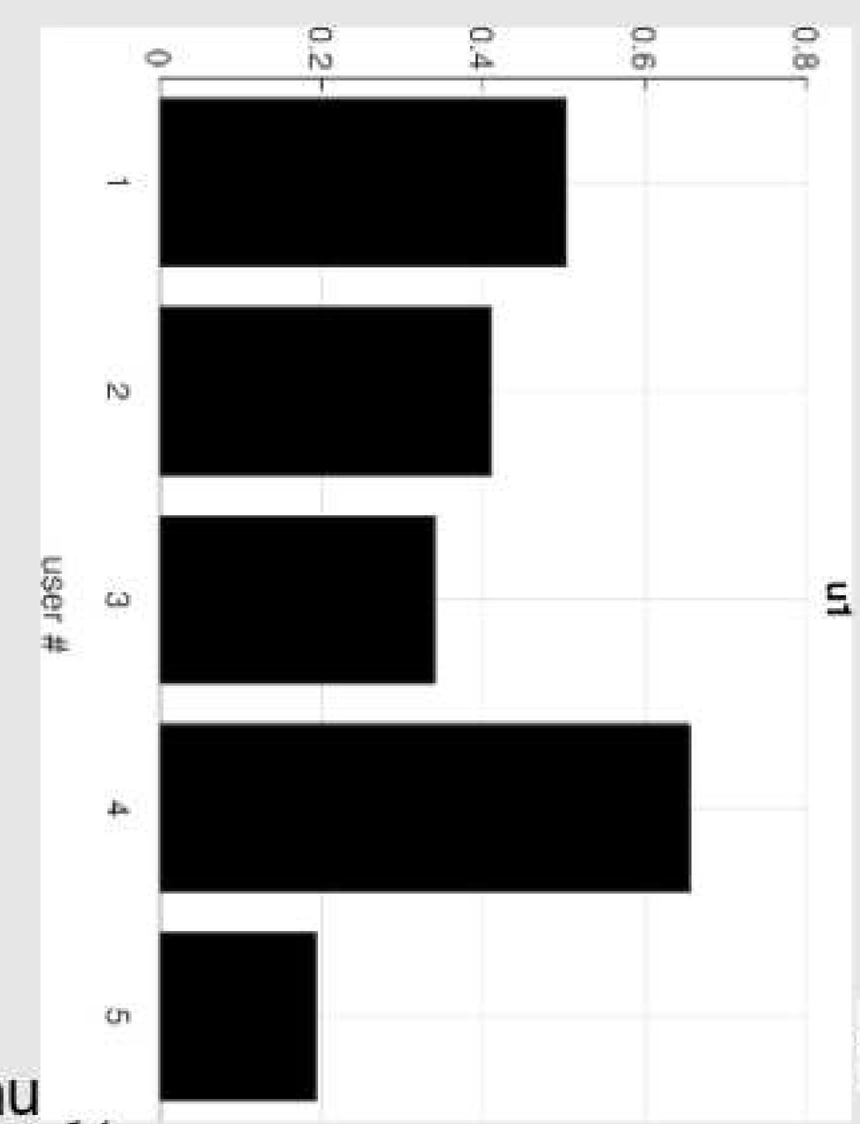
# Features of Rating Matrices

| Movie → / User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

$\sigma_1 \vec{u}_1 \vec{v}_1^T$




$=$

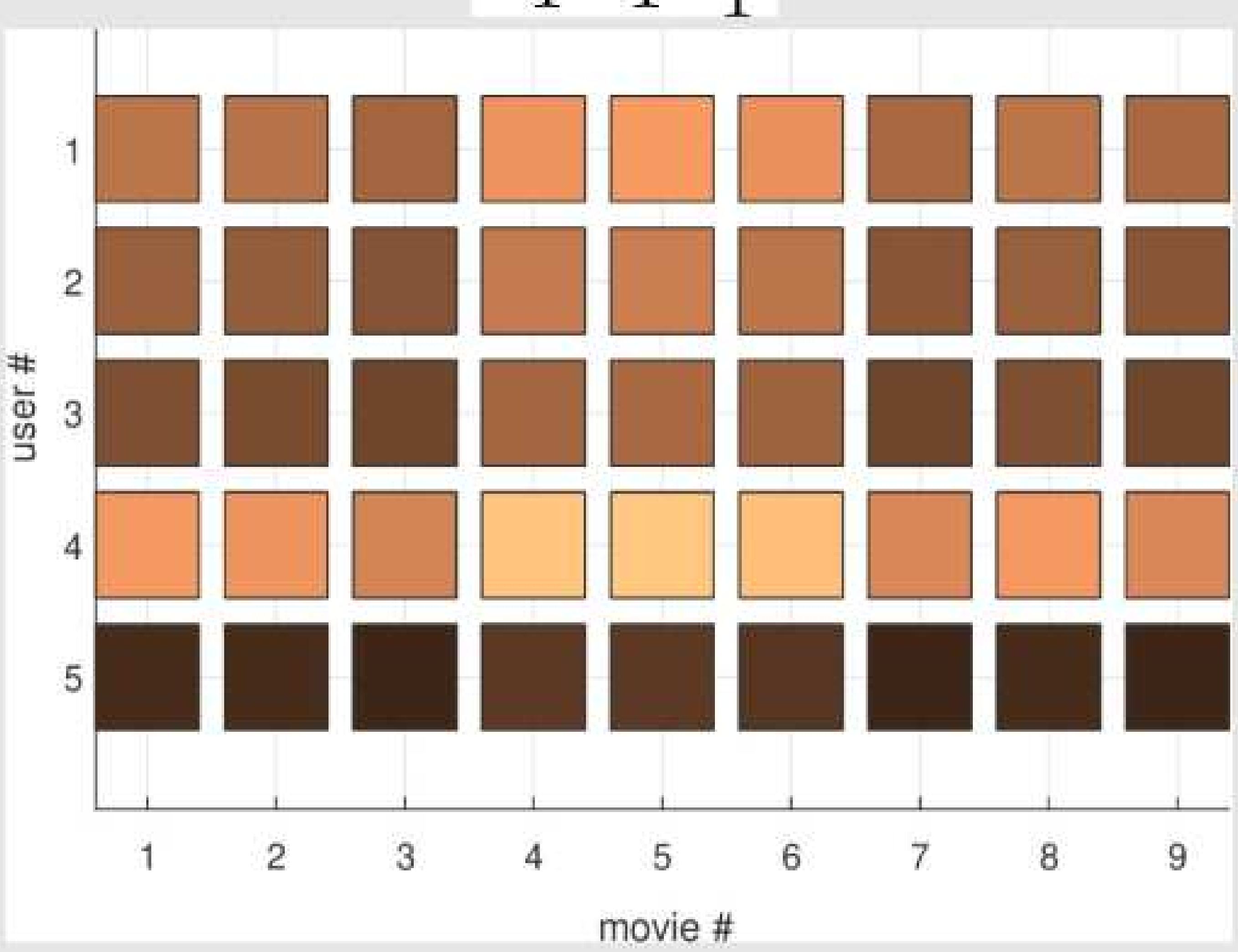



$\vec{u}_1$

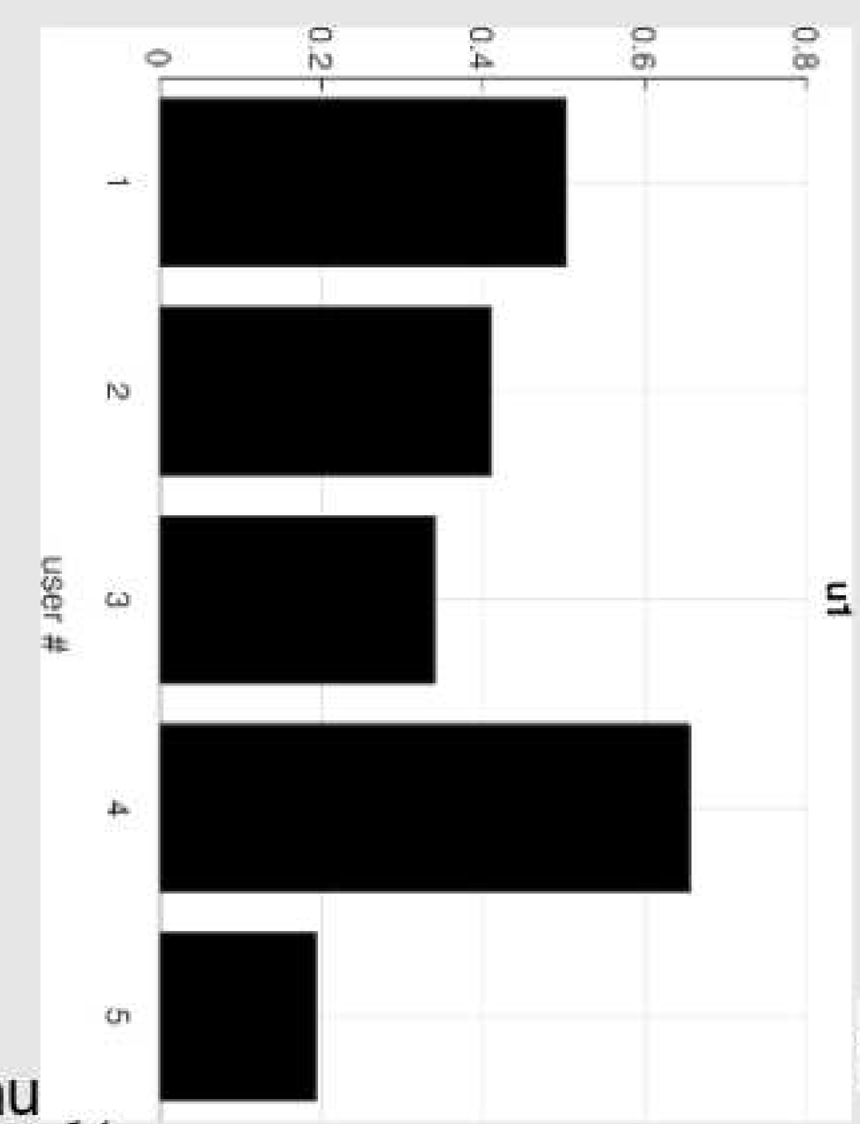
# Features of Rating Matrices

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

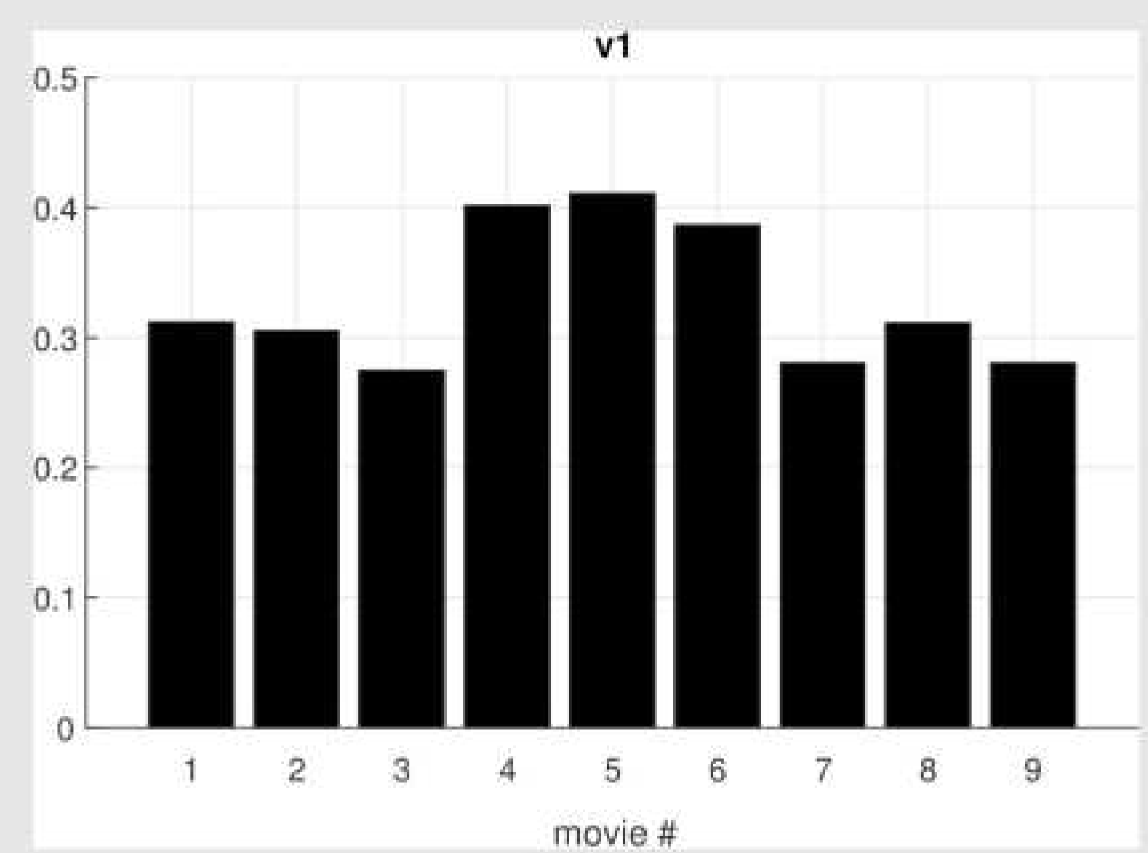
$\sigma_1 \vec{u}_1 \vec{v}_1^T$




=




$\vec{u}_1$

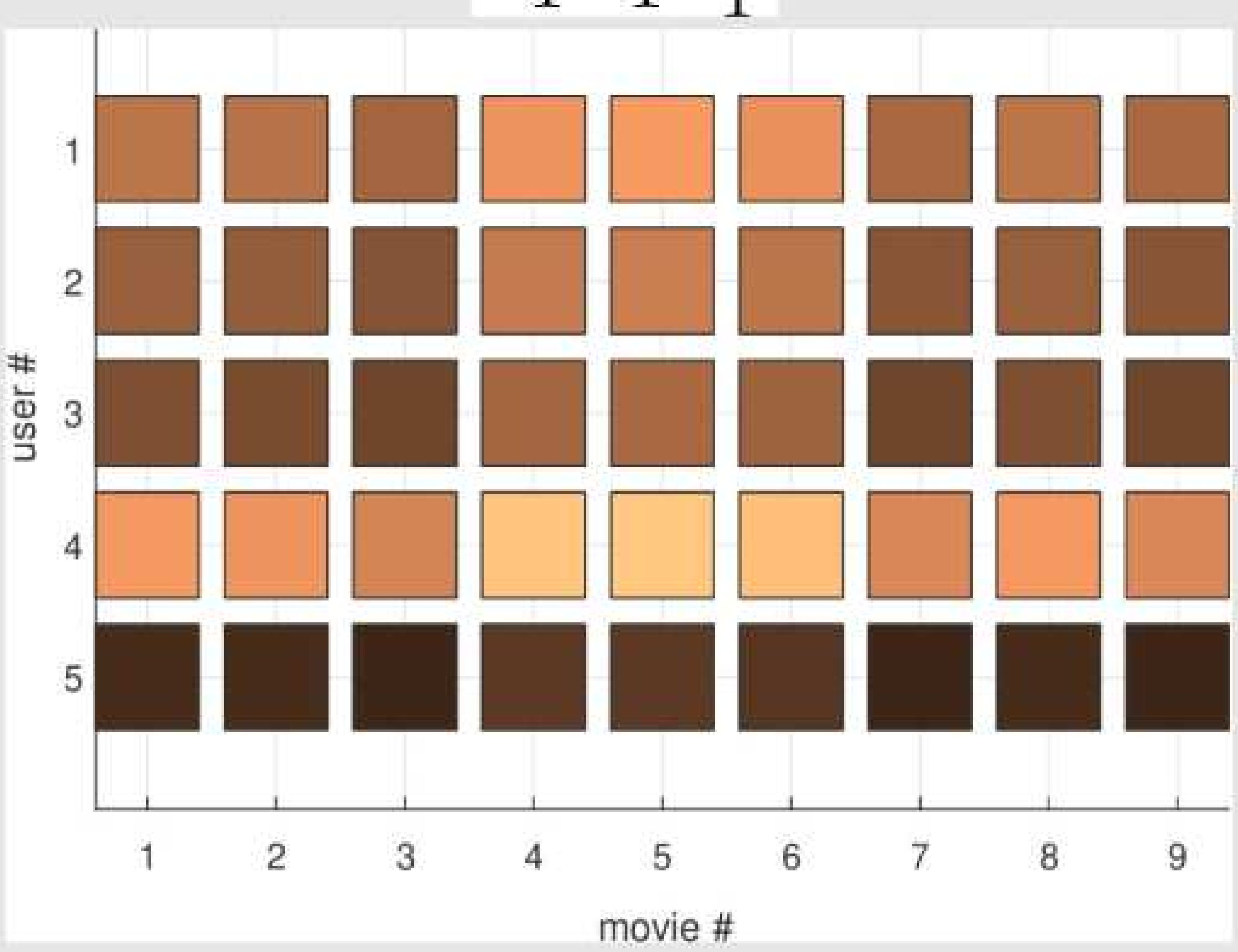



$\vec{v}_1^T$

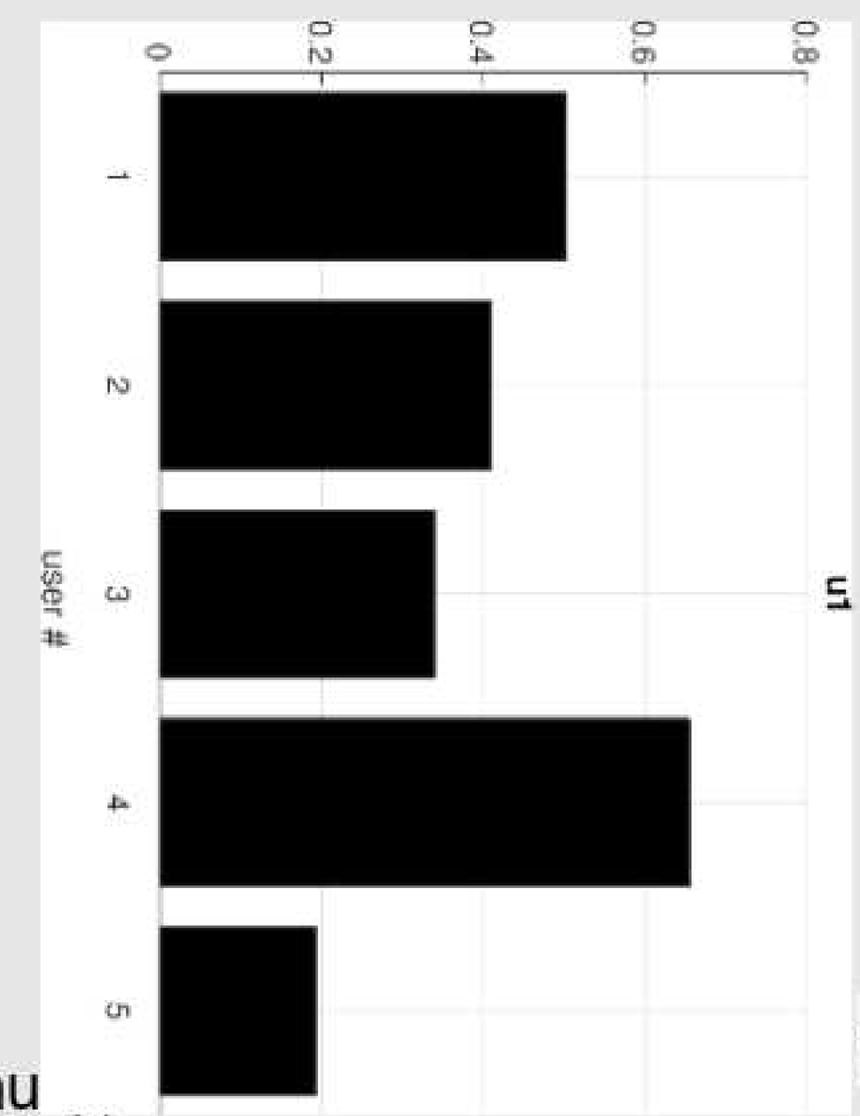
# Features of Rating Matrices

| Movie → User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

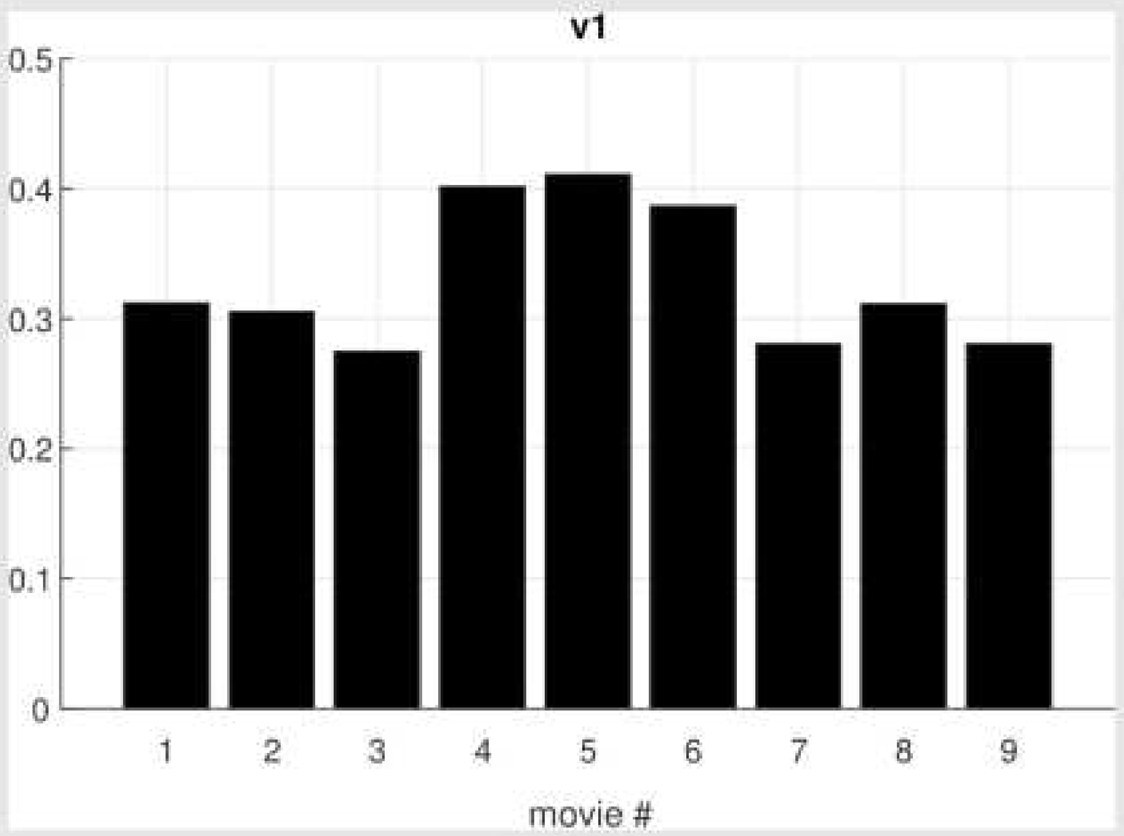
$\sigma_1 \vec{u}_1 \vec{v}_1^T$




$=$




$\vec{u}_1$




$\vec{v}_1^T$

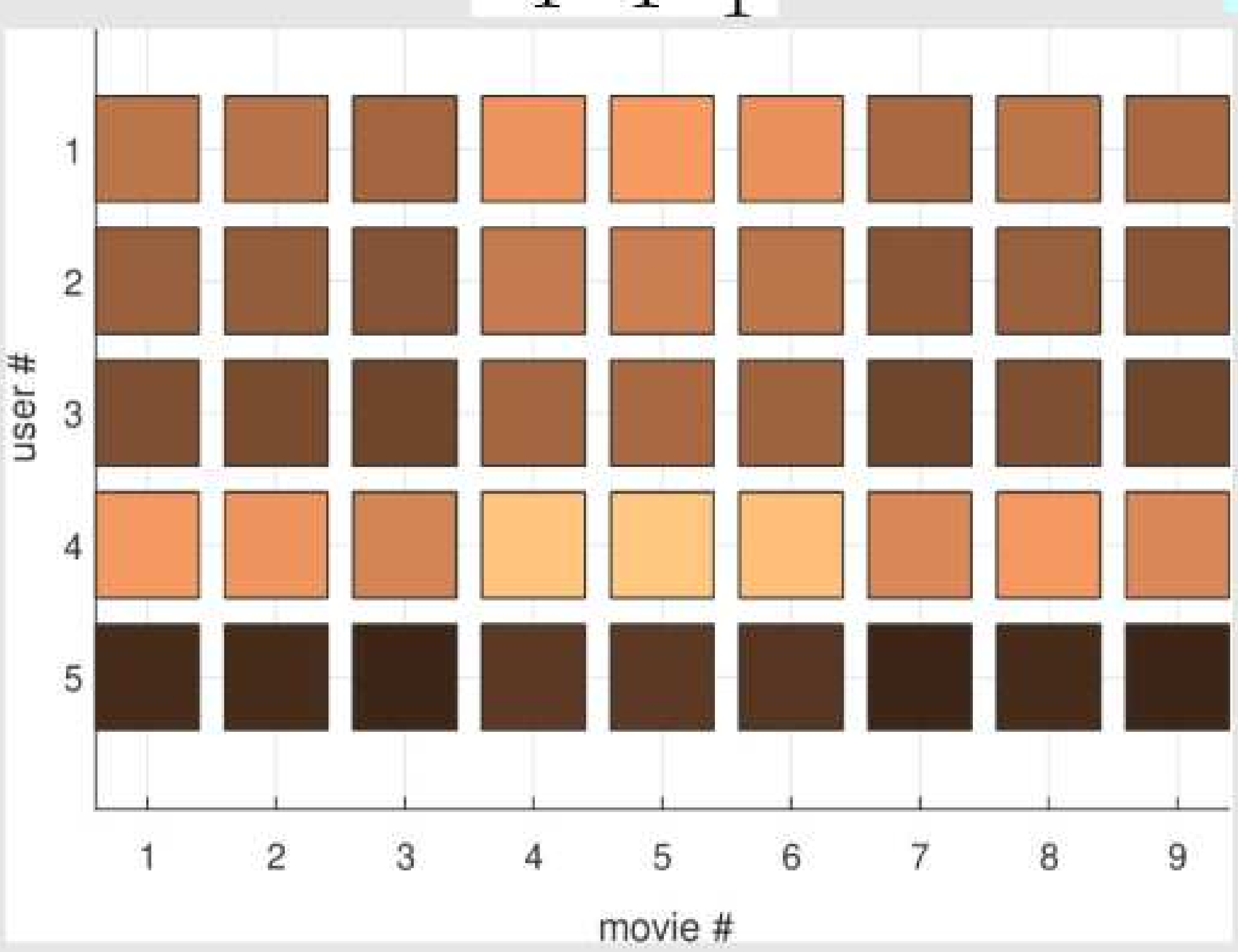
$\sigma_1 = 22.6$

# Features of Rating Matrices

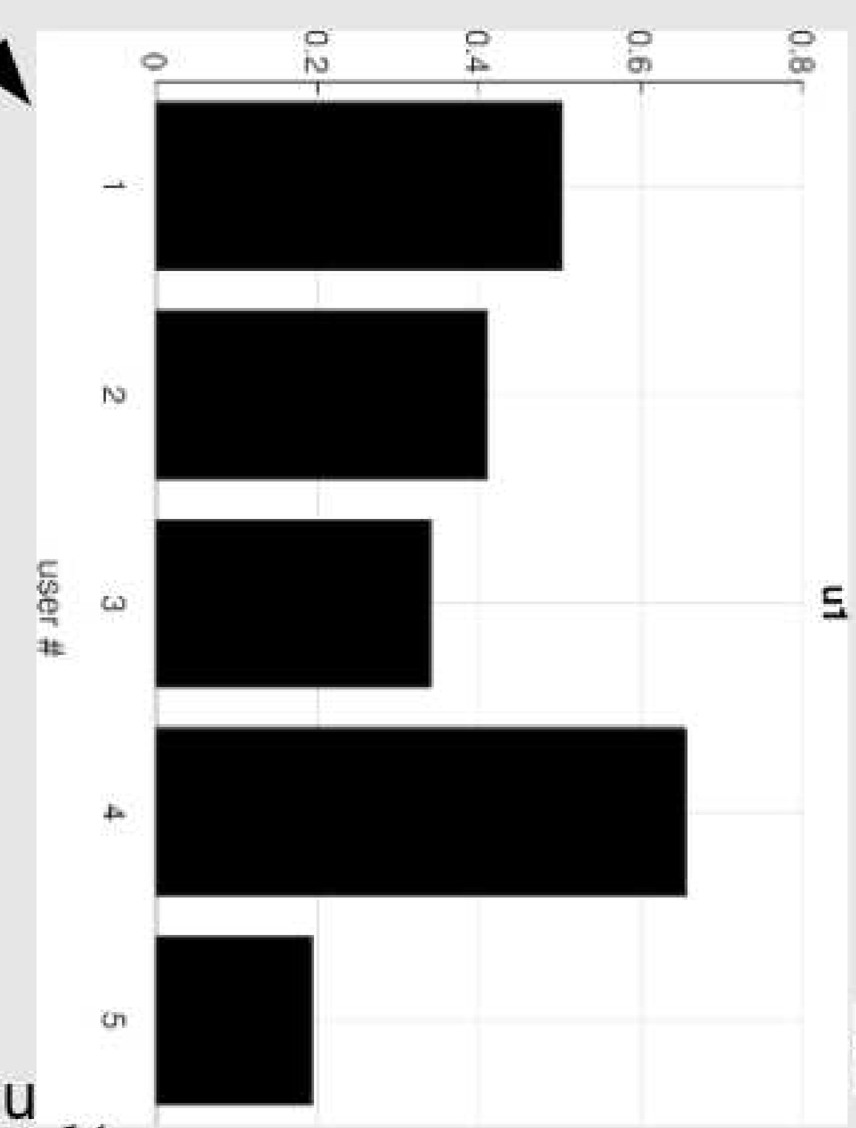
| Movie → / User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

"most typical" col (movie) feature:
65% like D's choices, 50% like A's,
40% like B's, 35% like C's
20% E's

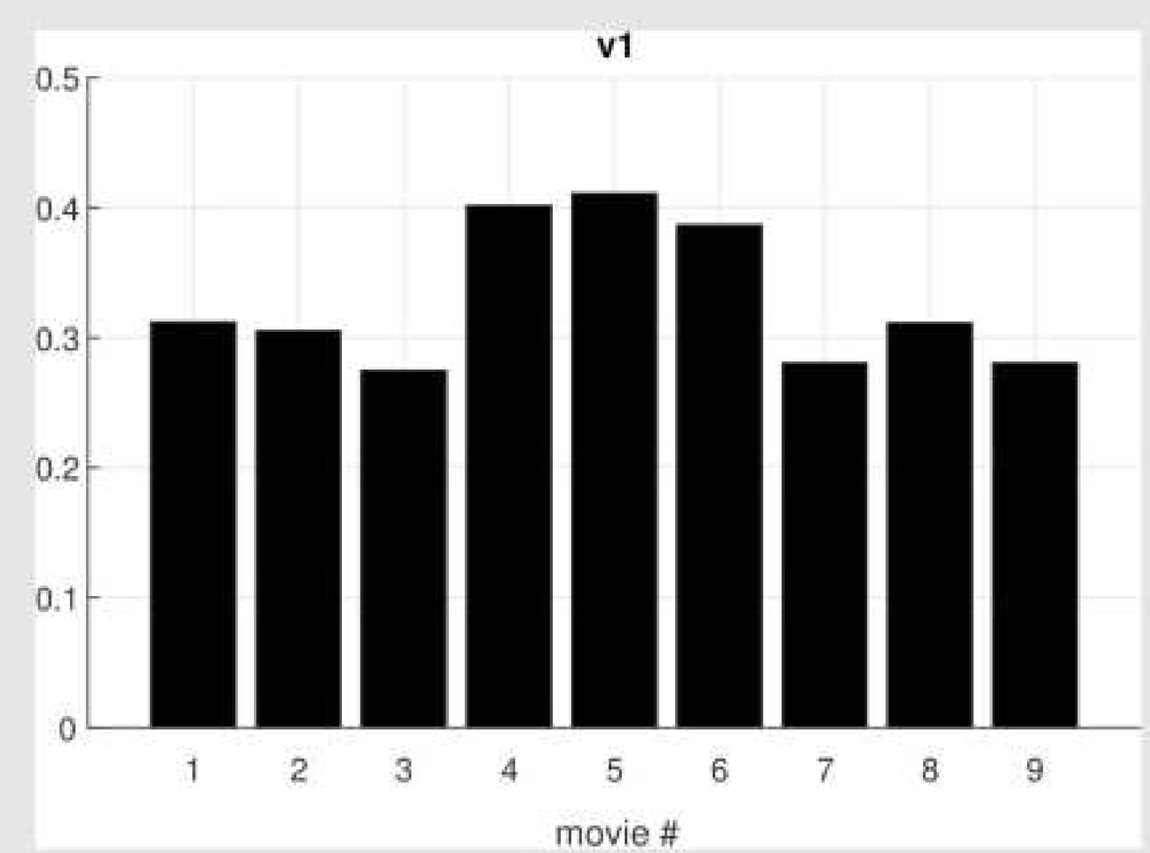
$\sigma_1 \vec{u}_1 \vec{v}_1^T$




=




$\vec{u}_1$



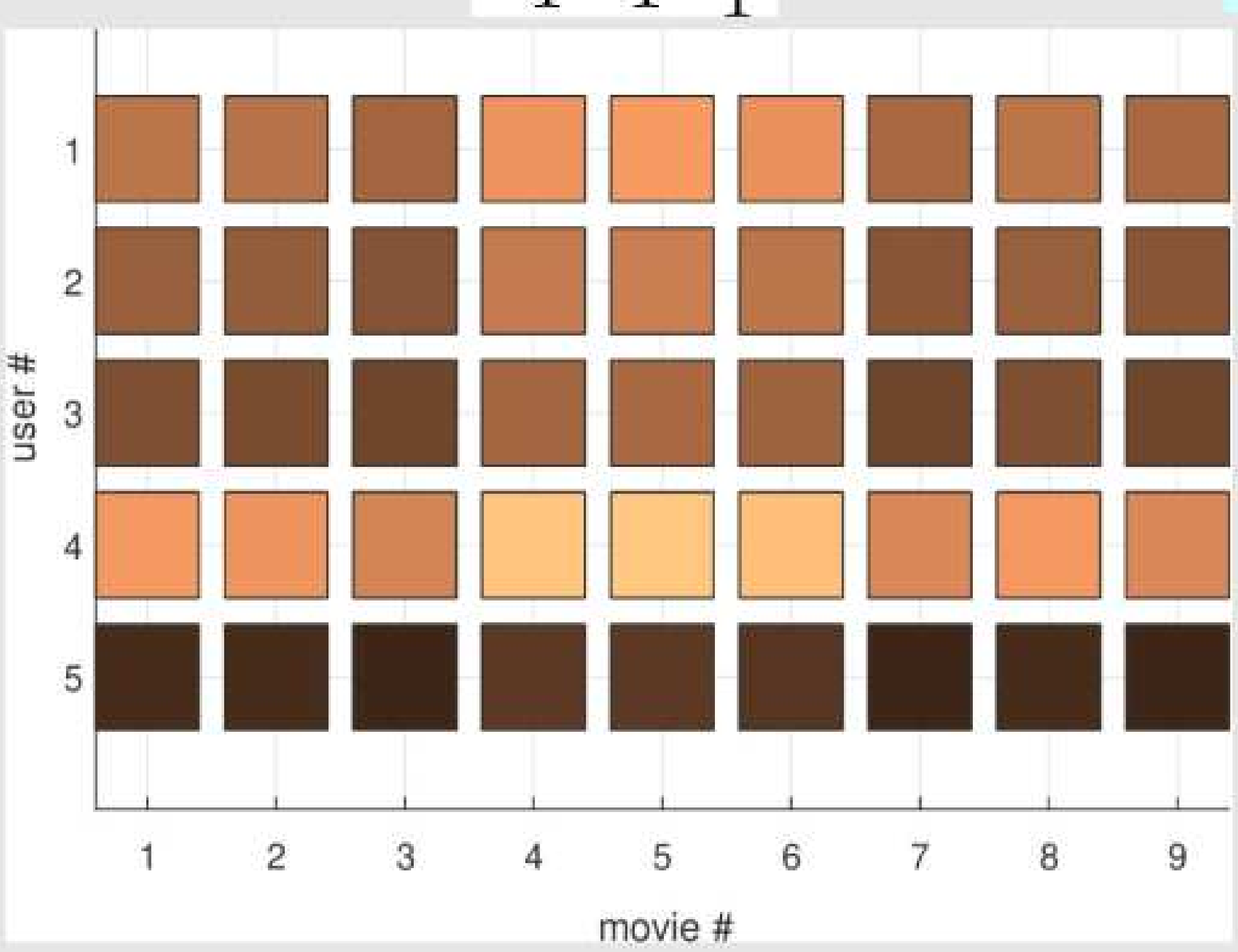

$\vec{v}_1^T$

$\sigma_1 = 22.6$

# Features of Rating Matrices

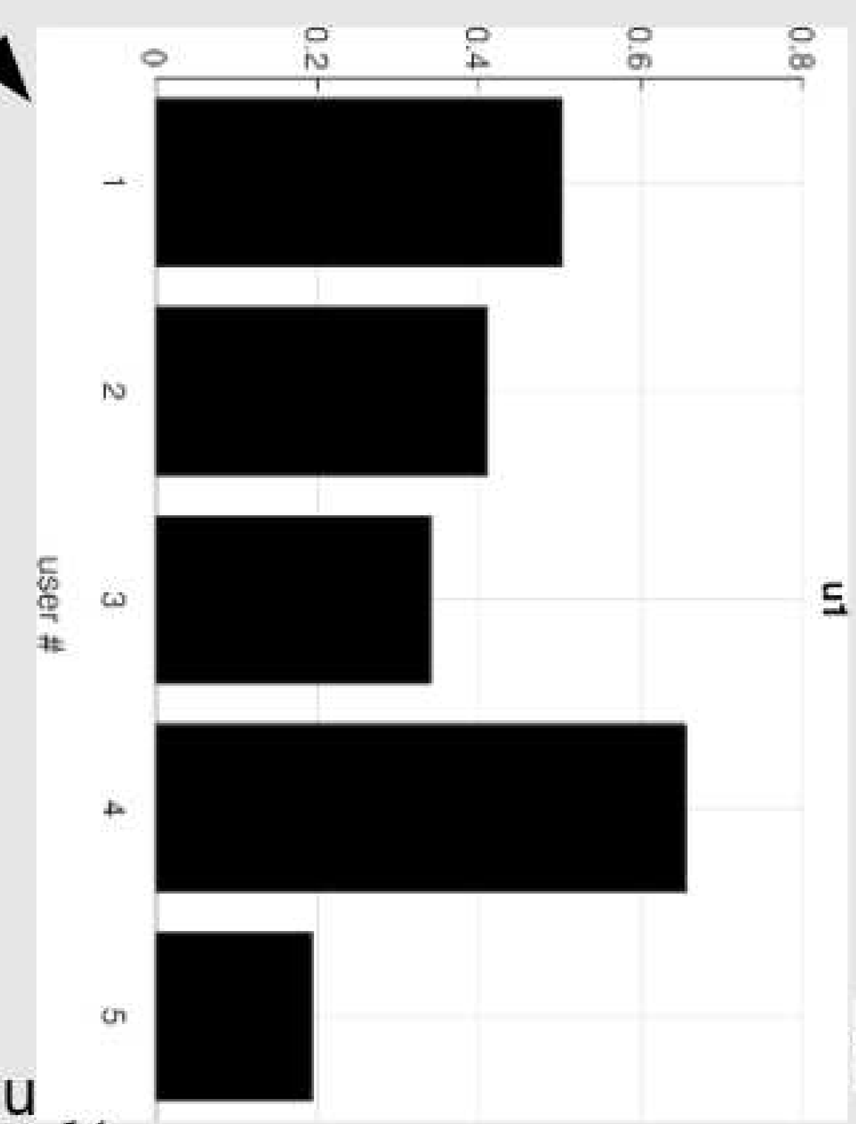
| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

$\sigma_1 \vec{u}_1 \vec{v}_1^T$



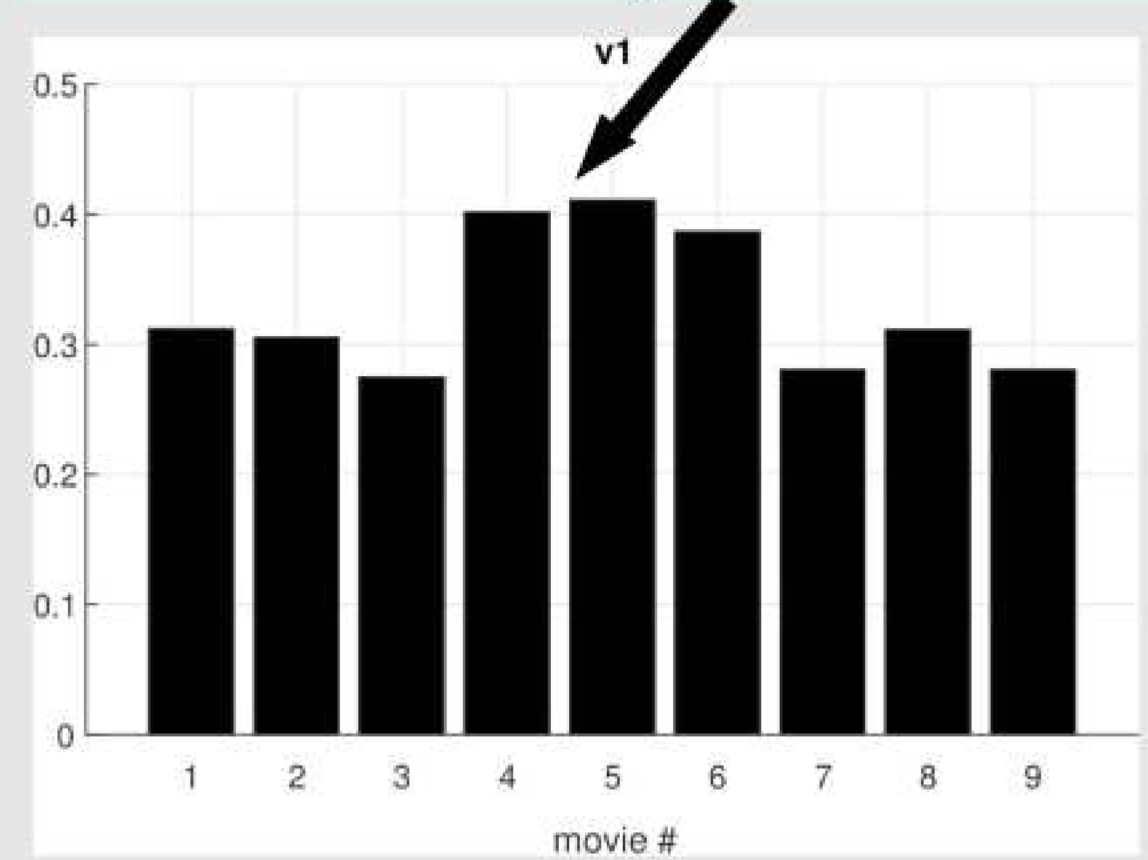

=

"most typical" col (movie) feature: 65% like D's choices, 50% like A's, 40% like B's, 35% like C's 20% E's




$\vec{u}_1$

"most typical" row (user) feature: likes SF more; action somewhat less; and comedy even less



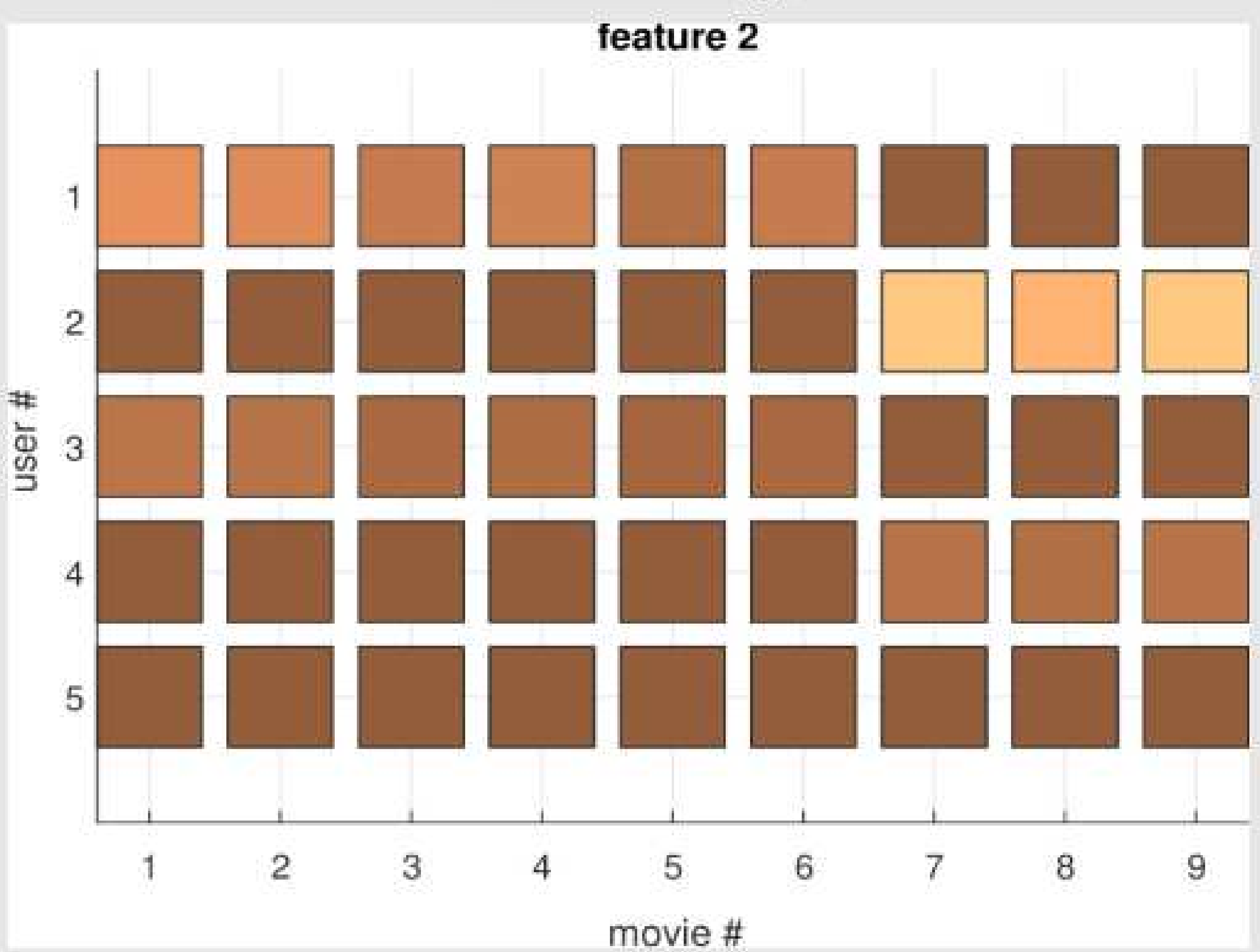

$\vec{v}_1^T$

$\sigma_1 = 22.6$

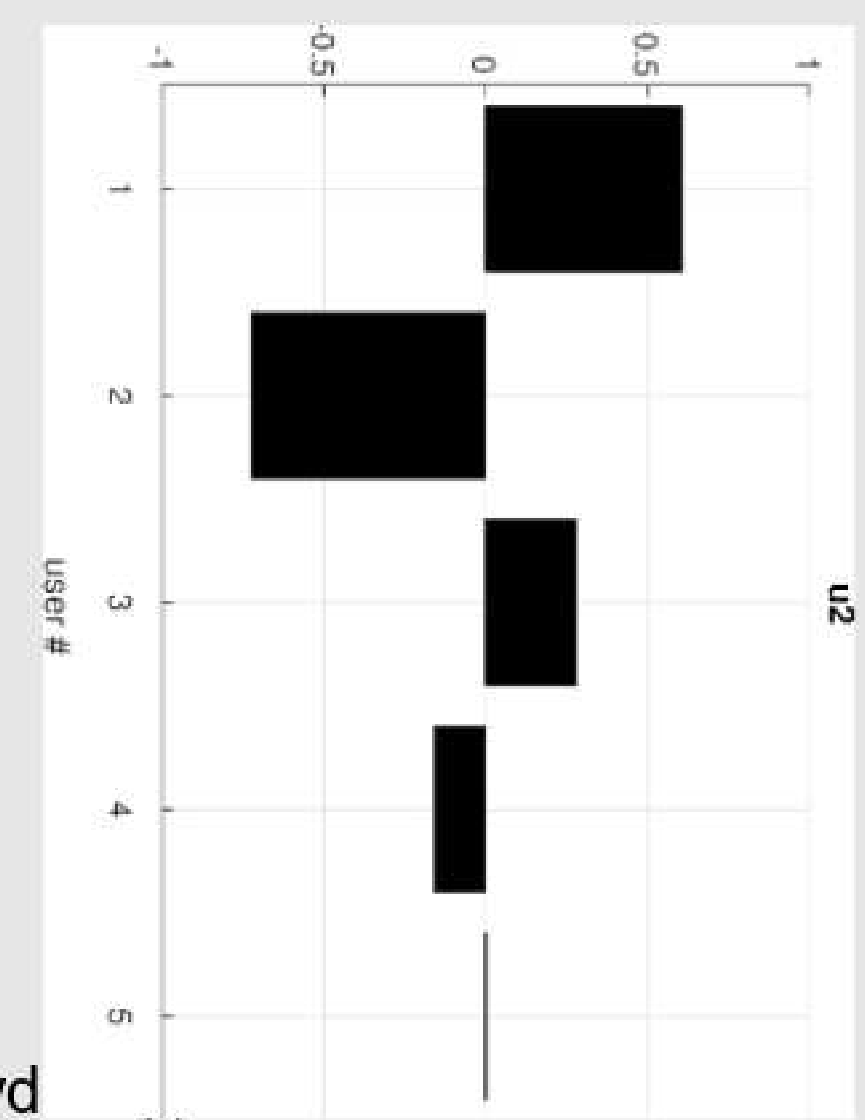
# Features of Rating Matrices (contd.)

| Movie → User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

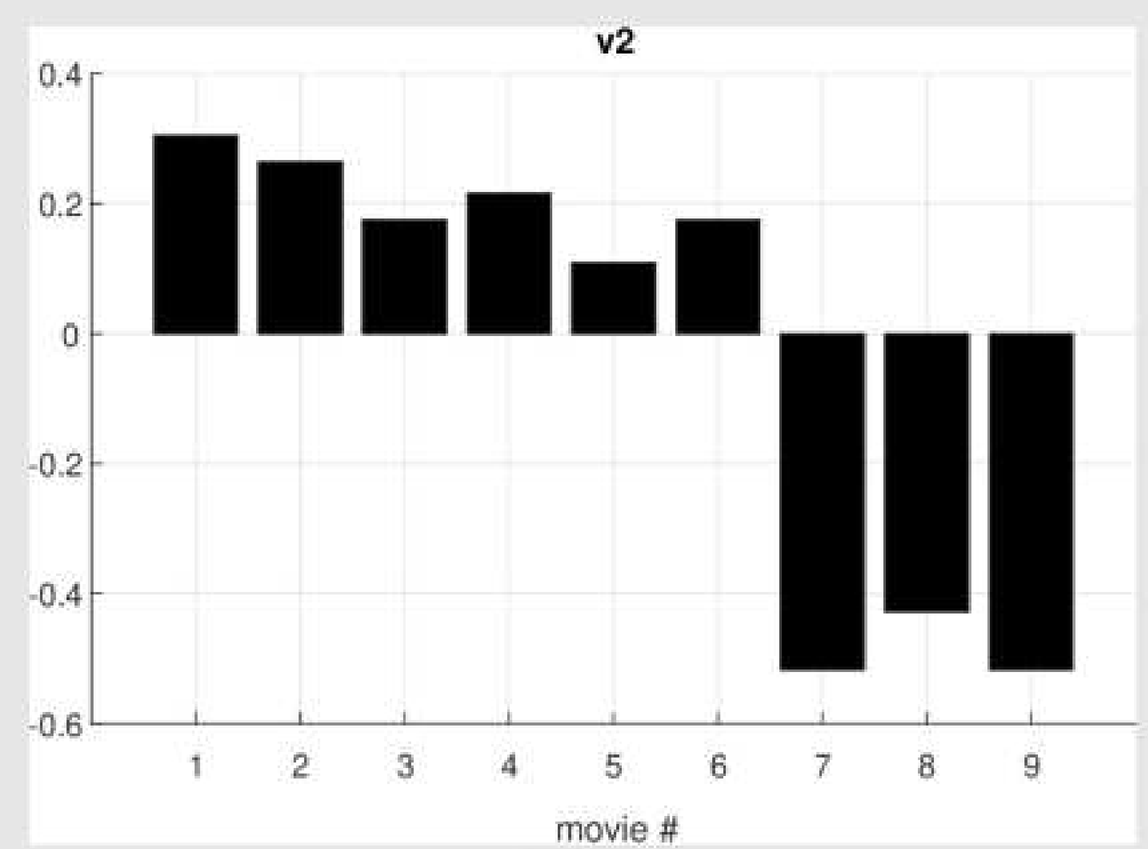
$\sigma_2 \vec{u}_2 \vec{v}_2^T$




=




$\vec{u}_2$




$\vec{v}_2^T$

$\sigma_2 = 6.8$

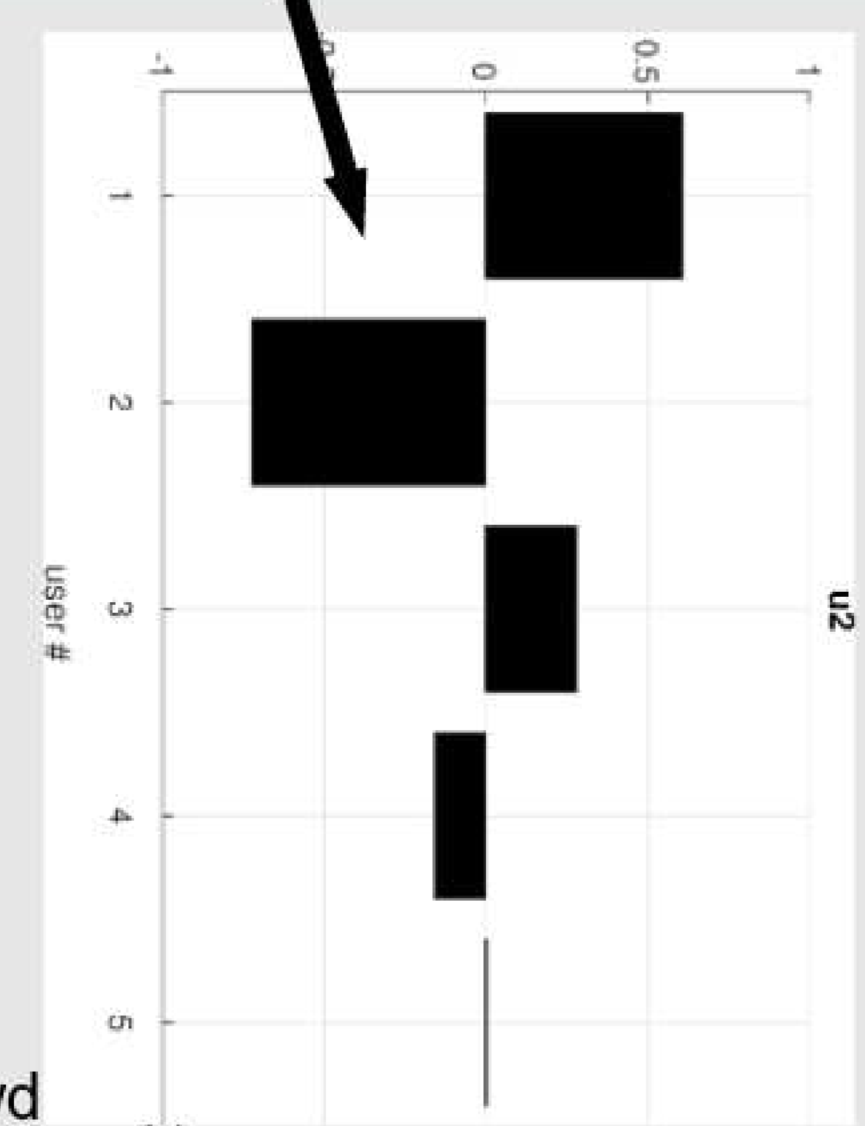
# Features of Rating Matrices (contd.)

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

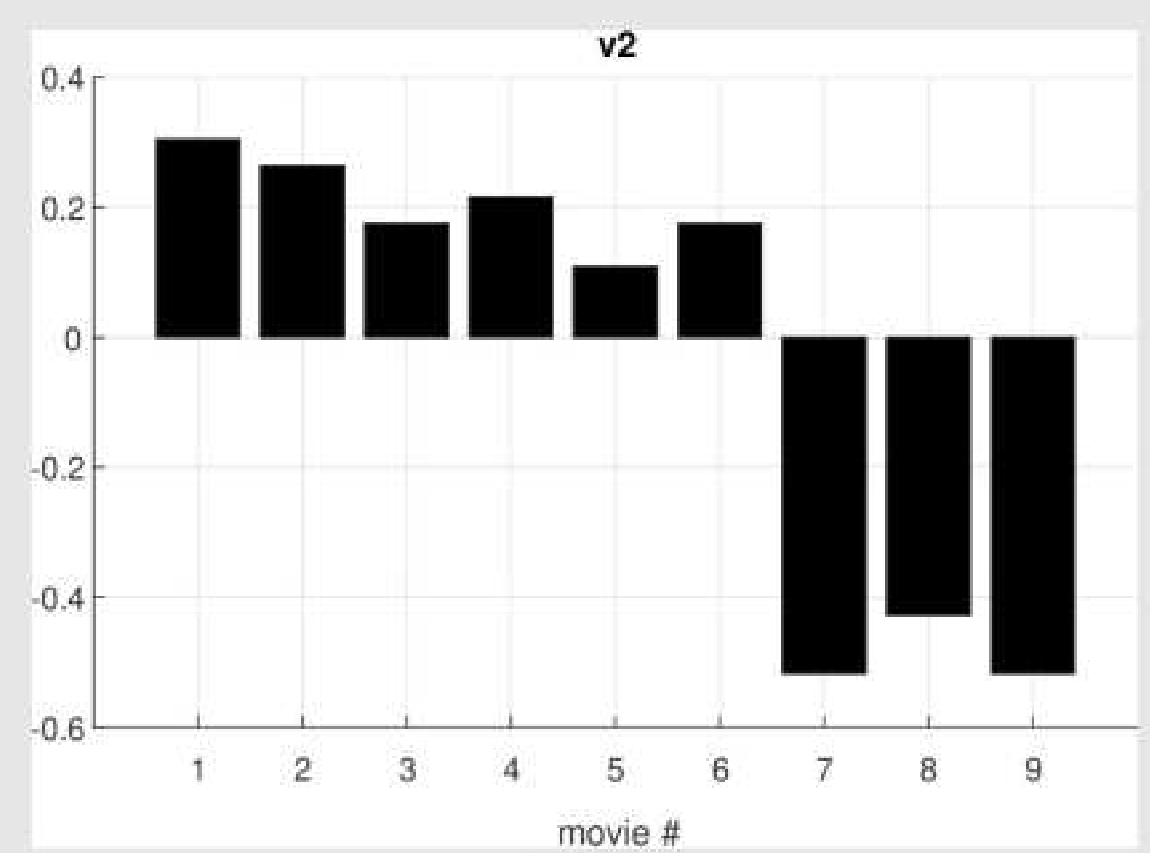
2nd most typical col (movie) feature:
55% like A's choices, 70% unlike B's,
35% like C's, 15% unlike D's,
negligibly like E's

$\sigma_2 \vec{u}_2 \vec{v}_2^T$


=




$\vec{u}_2$



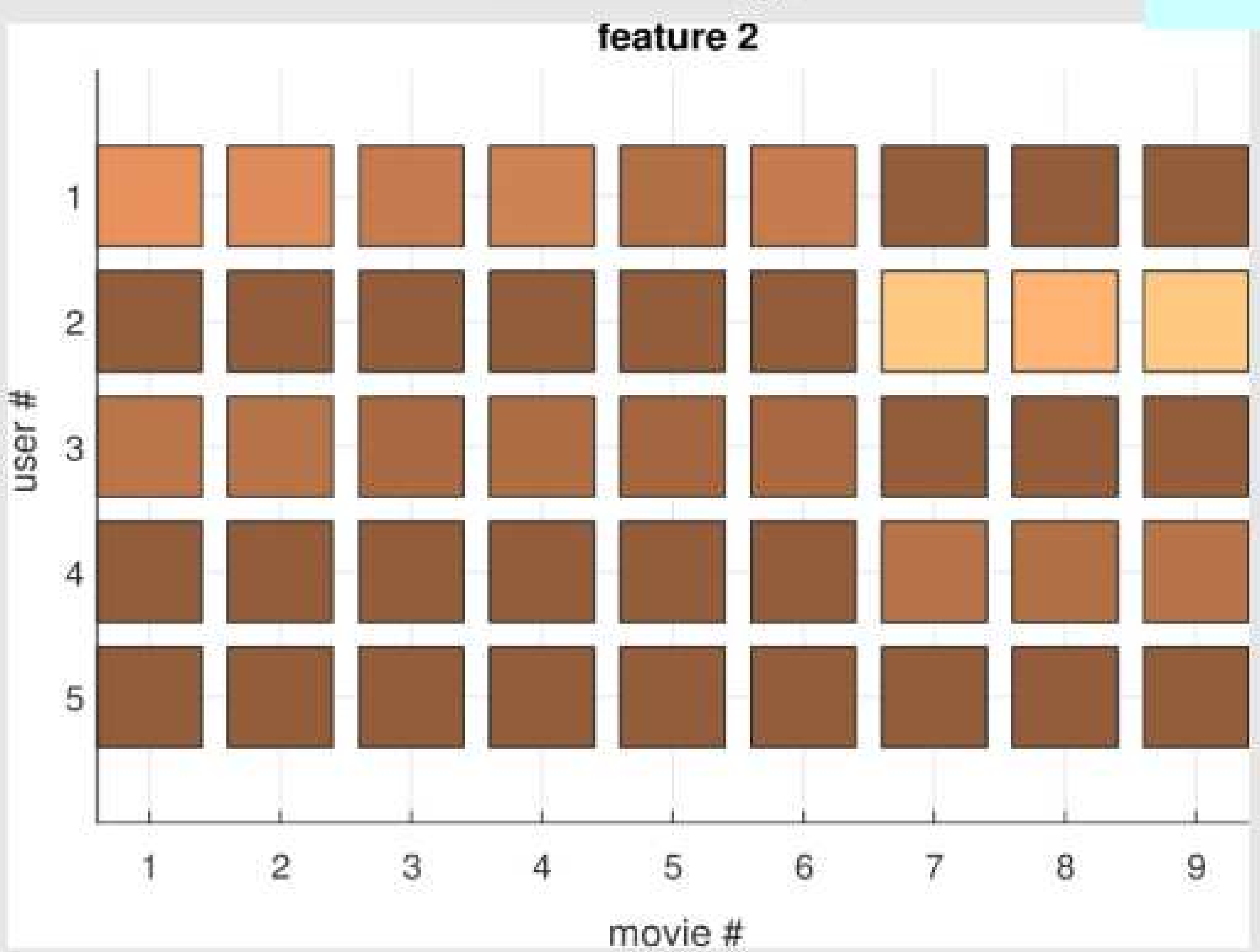

$\vec{v}_2^T$

$\sigma_2 = 6.8$

# Features of Rating Matrices (contd.)

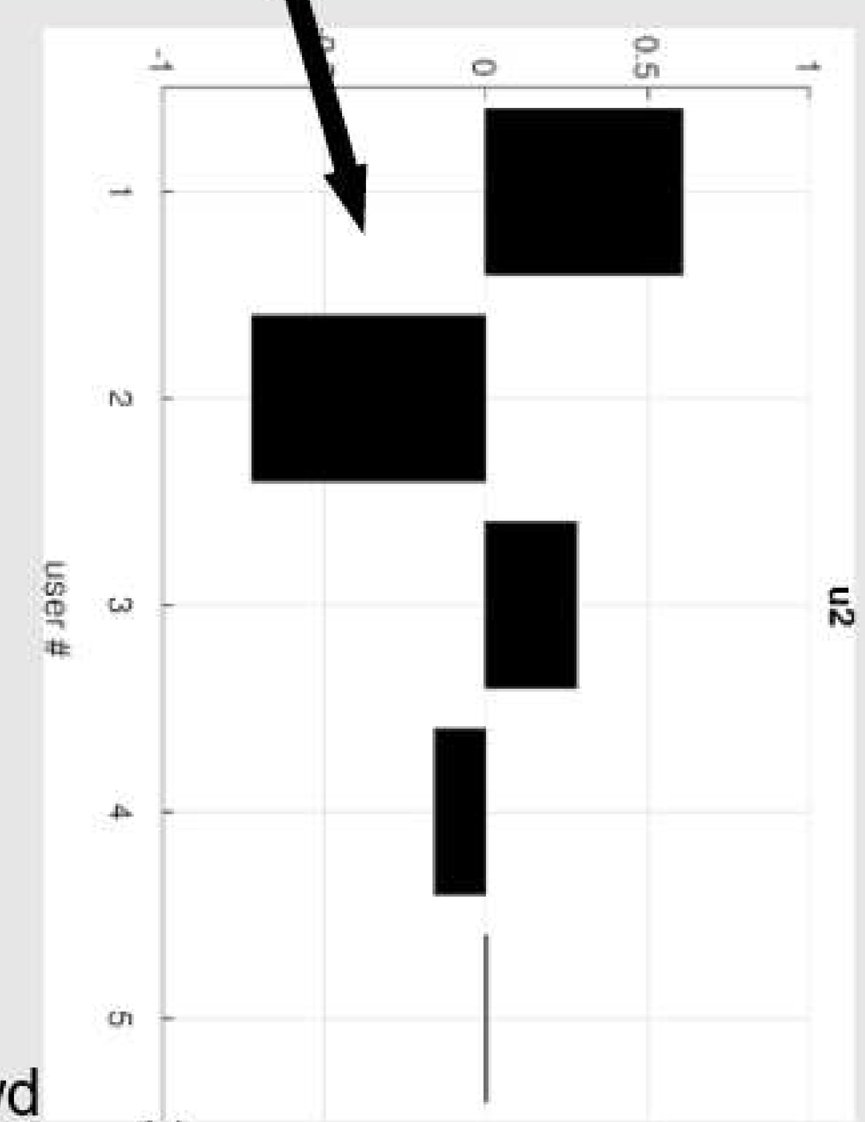
| Movie → / User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

$\sigma_2 \vec{u}_2 \vec{v}_2^T$



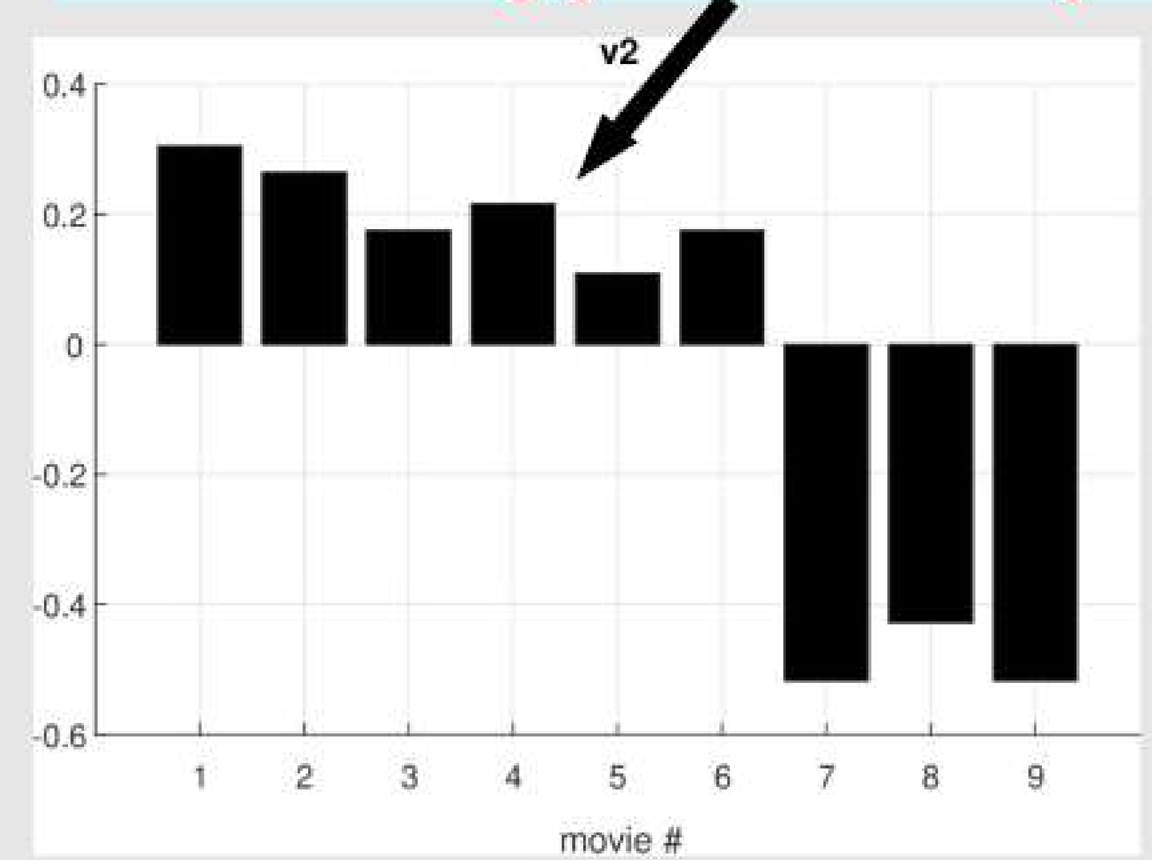

=

2nd most typical col (movie) feature:
55% like A's choices, 70% unlike B's, 35% like C's, 15% unlike D's, negligibly like E's




$\vec{u}_2$

2nd most typical row (user) feature:
likes mostly action;
a bit less SF;
strongly anti-comedy




$\vec{v}_2^T$

$\sigma_2 = 6.8$

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features

# Projection in the Feature Basis

| Movie → / User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

$$\begin{bmatrix}5\\1\\2\\5\\1\end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

col.(movie) features

$$\begin{bmatrix} 5 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

$$\begin{bmatrix} 5 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

col.(movie) features

"how much FMJ is like the most typical movie"

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

col.(movie) features

$$\begin{bmatrix}5\\1\\2\\5\\1\end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

"how much FMJ is like the most typical movie"

"how much FMJ is like the 3rd most typical movie"

# Projection in the Feature Basis

| Movie → <br> User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

col.(movie) features

$$\begin{bmatrix} 5 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

"how much FMJ is like the most typical movie"

"how much FMJ is like the 3rd most typical movie"

➔ $$\alpha_{i1} = \vec{u}_i^T \begin{bmatrix} 5 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \quad i = 1, \cdots, 5$$

# Projection in the Feature Basis

| Movie → User Name ↓ | Full Metal Jacket | Die Hard | Yojimbo | 2001: A Space Odyssey | The Quiet Earth | On The Beach | Would I Lie to You | Dr. Strangelove | Hokkabaz |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 5 | 4 | 5 | 5 | 5 | 1 | 2 | 1 |
| B | 1 | 1 | 1 | 3 | 4 | 3 | 5 | 5 | 5 |
| C | 2 | 1 | 1 | 5 | 5 | 4 | 2 | 1 | 1 |
| D | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| E | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 |

- Express each col of A (movie column) as a linear combination of col. features
  - e.g., **Full Metal Jacket** column:

col.(movie) features

$$\begin{bmatrix}5\\1\\2\\5\\1\end{bmatrix} = \alpha_{11}\vec{u}_1 + \alpha_{21}\vec{u}_2 + \alpha_{31}\vec{u}_3 + \cdots + \alpha_{51}\vec{u}_5$$

"how much FMJ is like the most typical movie"

"how much FMJ is like the 3rd most typical movie"

➔ $$\alpha_{i1} = \vec{u}_i^T \begin{bmatrix}5\\1\\2\\5\\1\end{bmatrix}, \quad i = 1, \cdots, 5$$

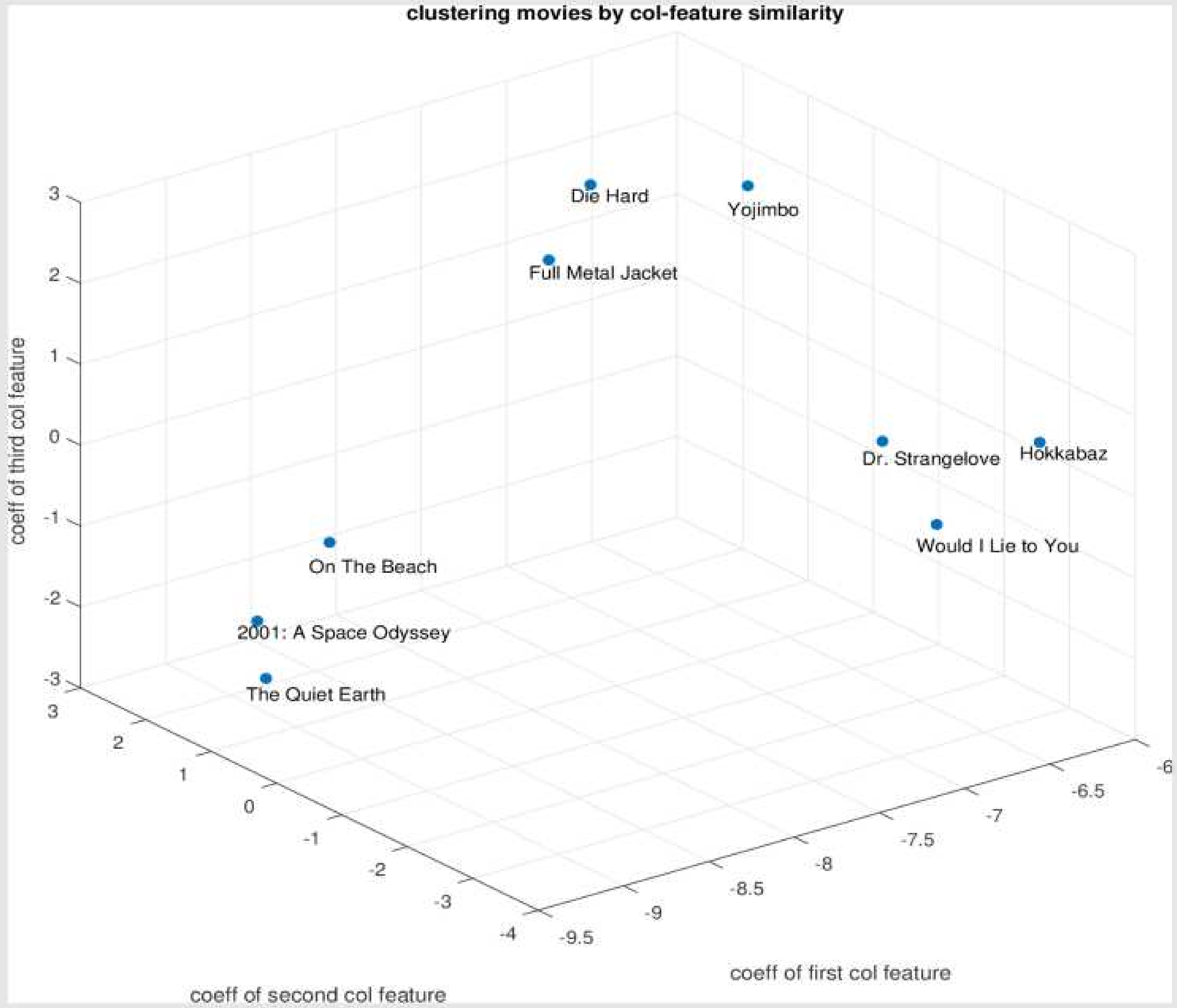
**Projections of 1st col (FMJ) onto column feature basis**

# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features

row (user) features

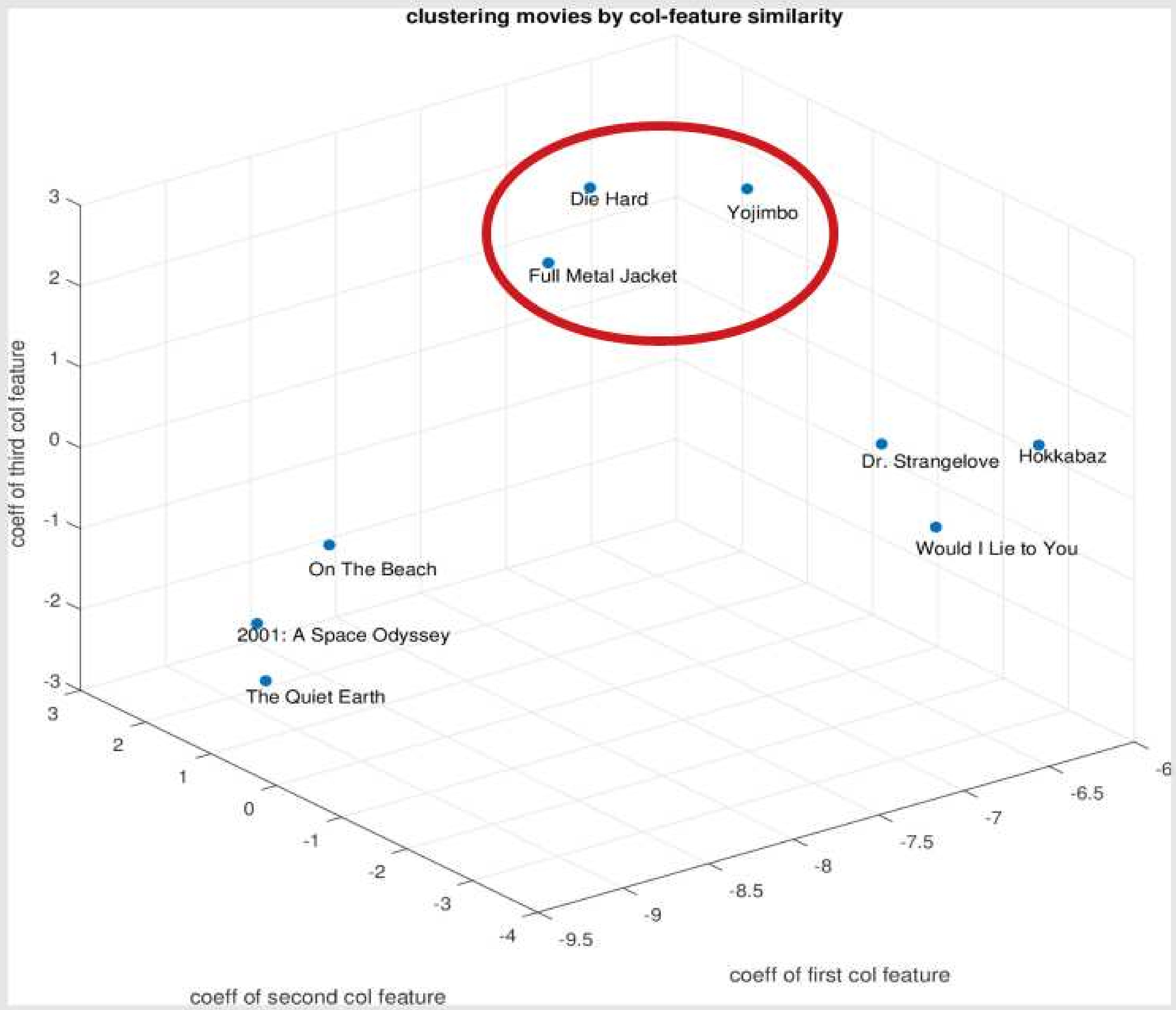


# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features

row (user) features

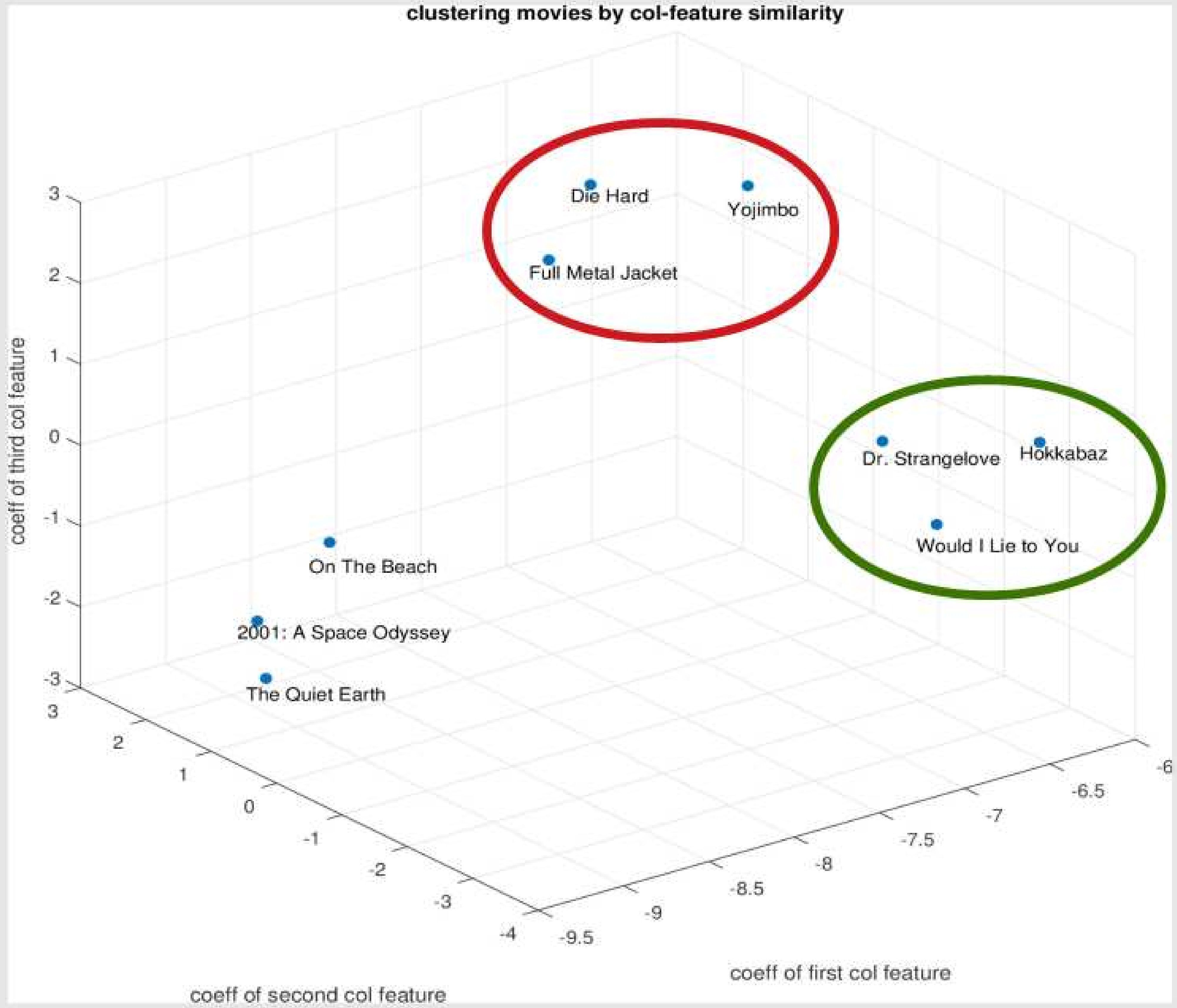


# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features

row (user) features

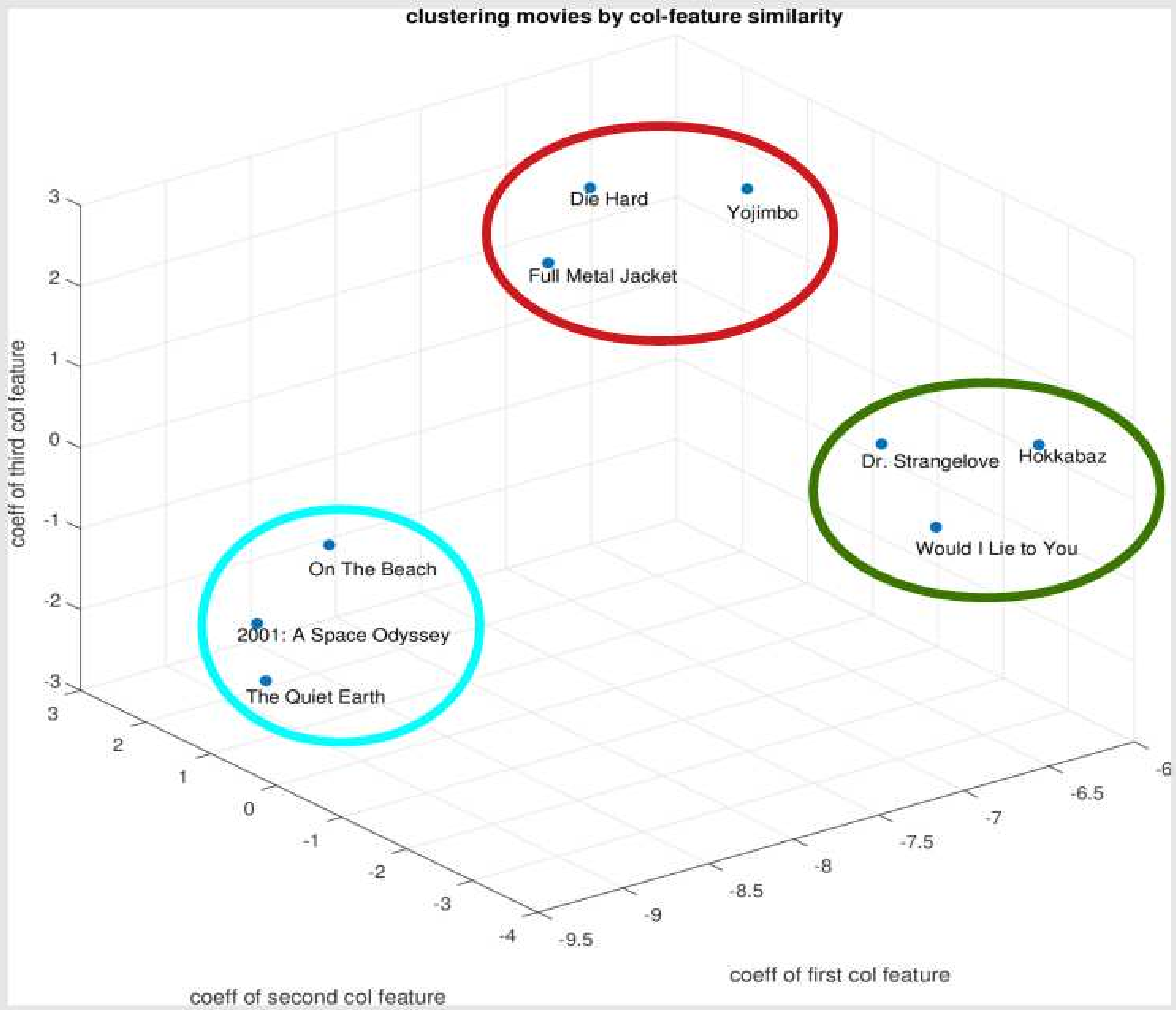


# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features

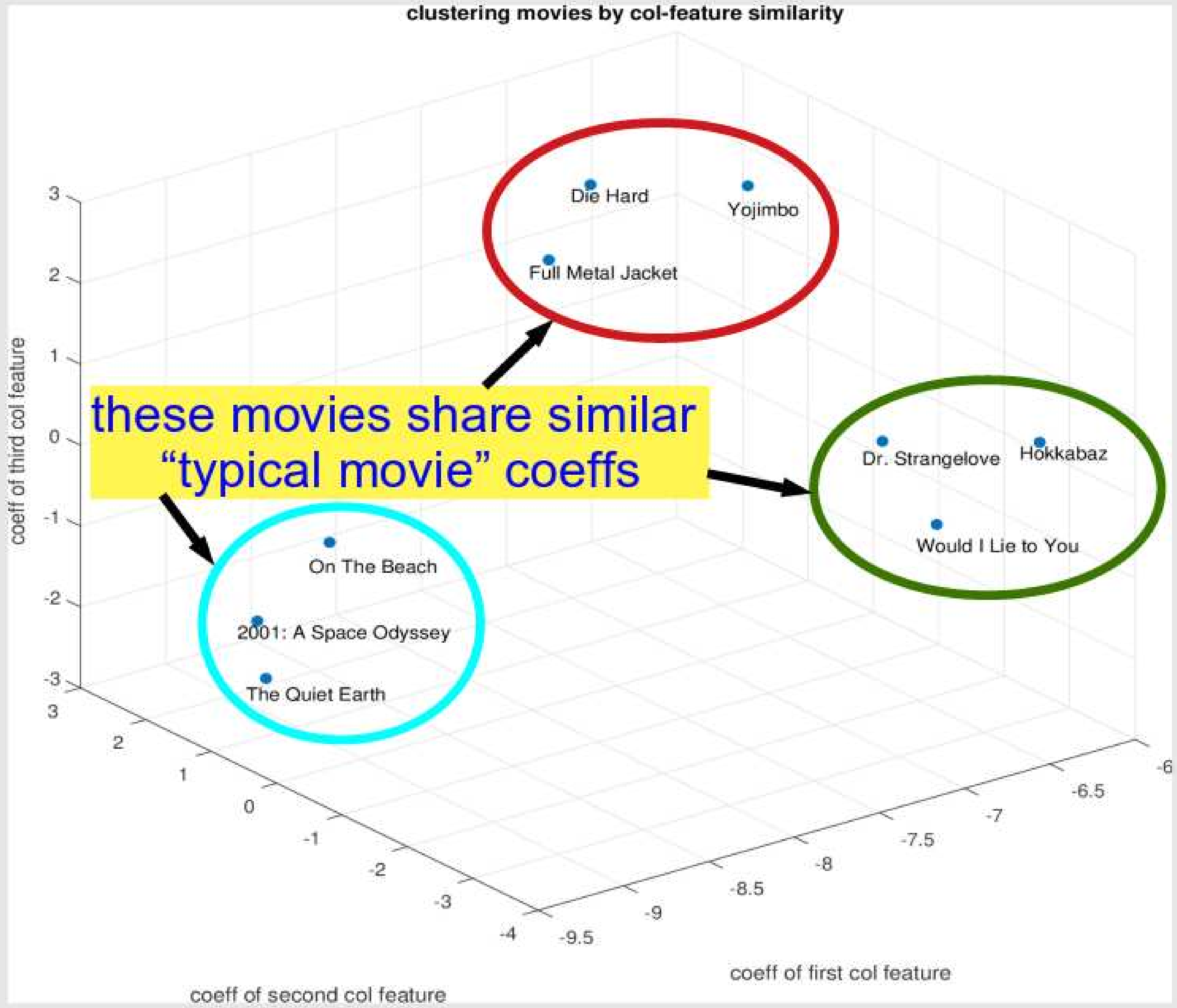
row (user) features



# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features

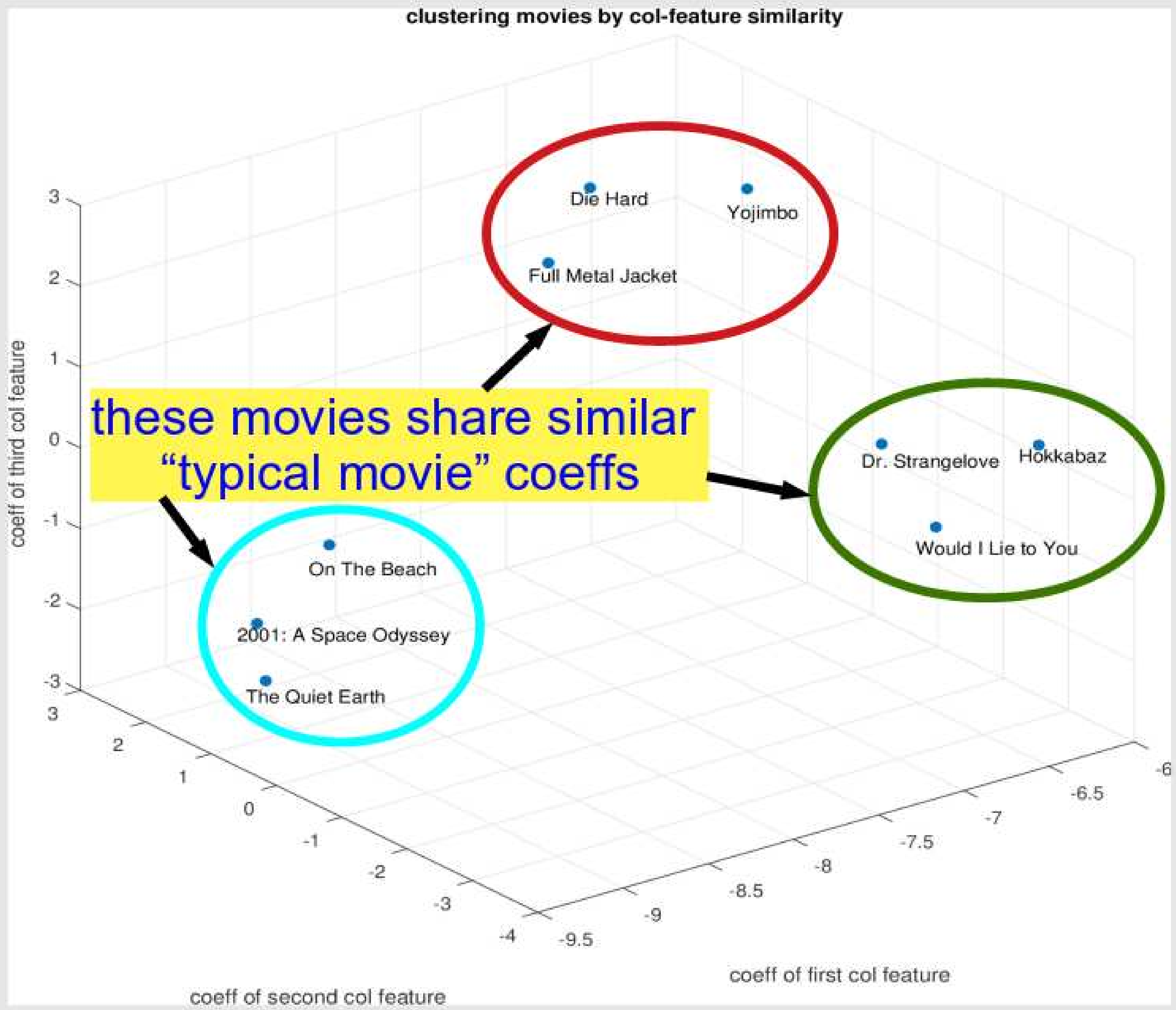



row (user) features

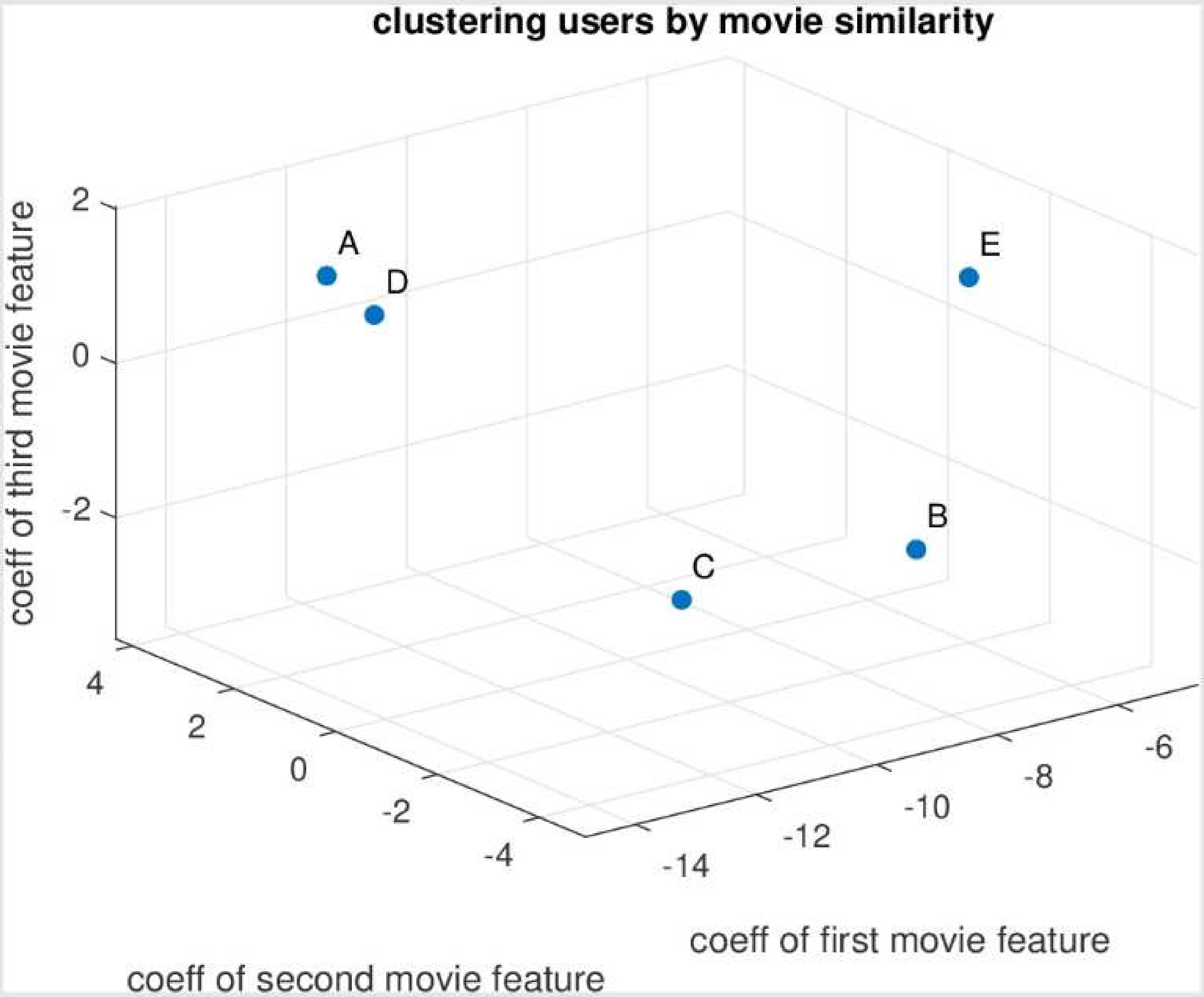
# Clustering in Feature Bases

- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

Movies classified by projections on column (movie) features




Users classified by projection on row (user) features



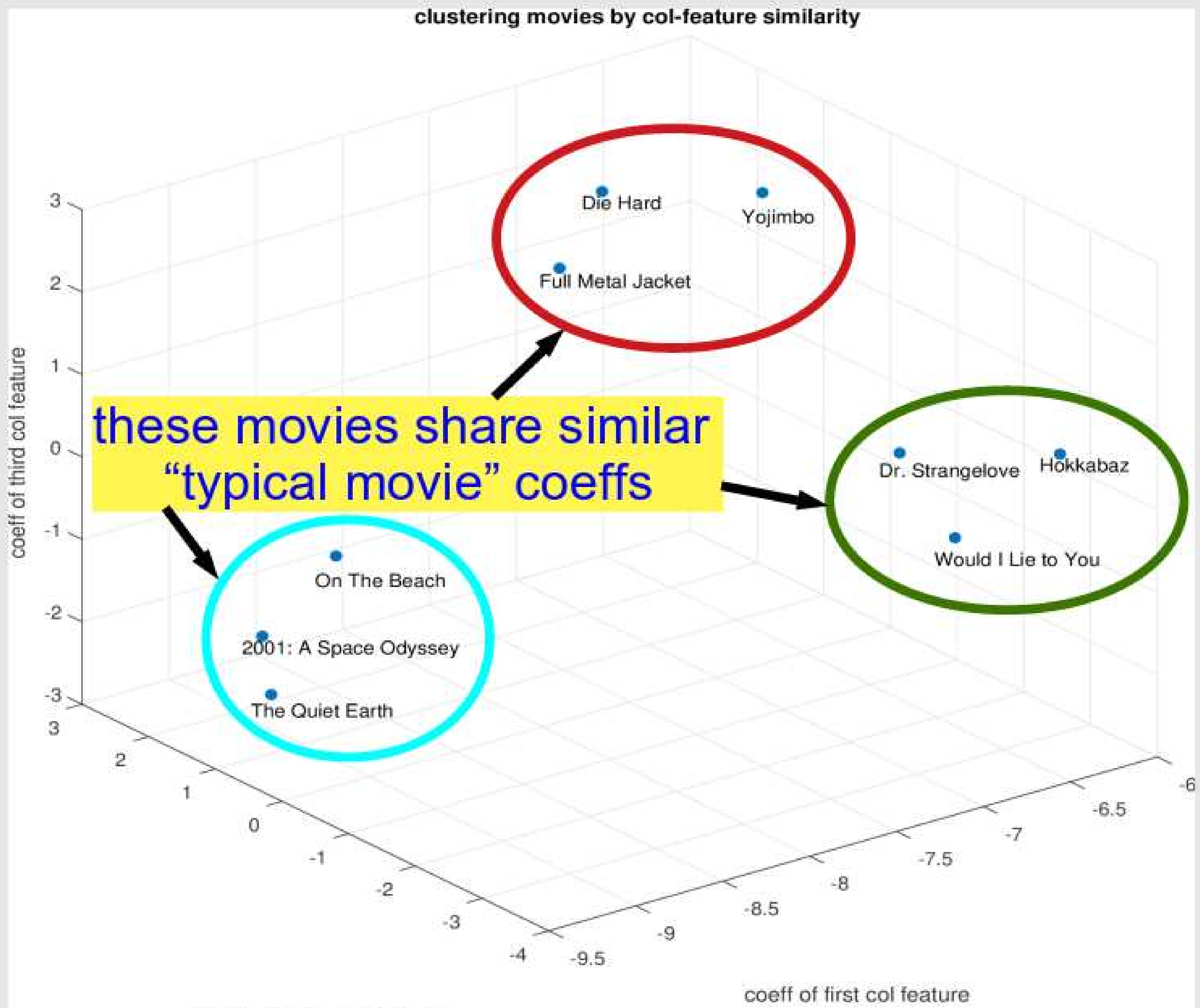
# Clustering in Feature Bases

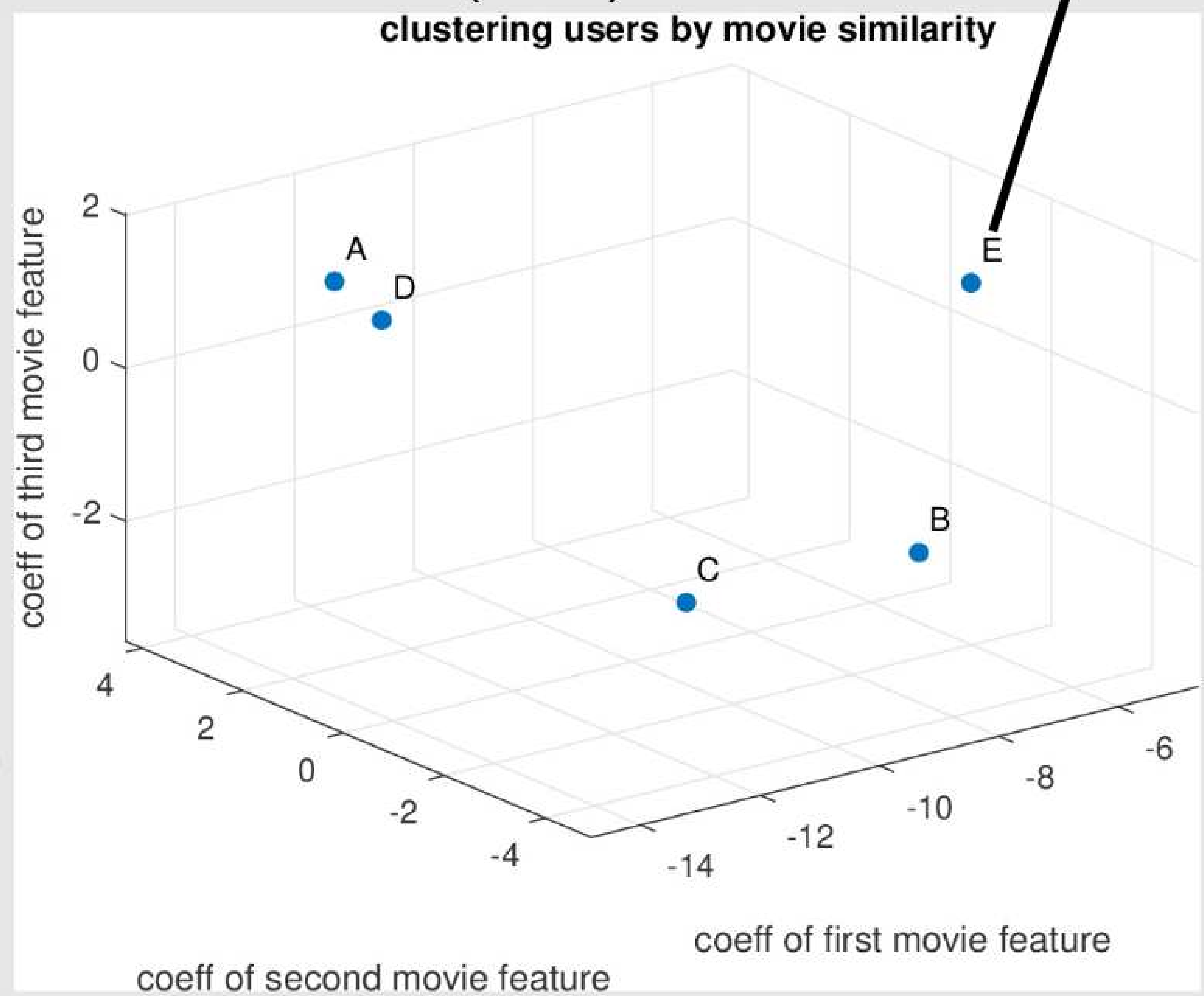
- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

$$\text{user E row} = \begin{bmatrix}1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1\end{bmatrix}$$
$$= \beta_{51}\vec{v}_1^T + \beta_{52}\vec{v}_2^T + \beta_{53}\vec{v}_3^T + \cdots + \beta_{55}\vec{v}_5^T$$

Movies classified by projections on column (movie) features

Users classified by projection on row (user) features



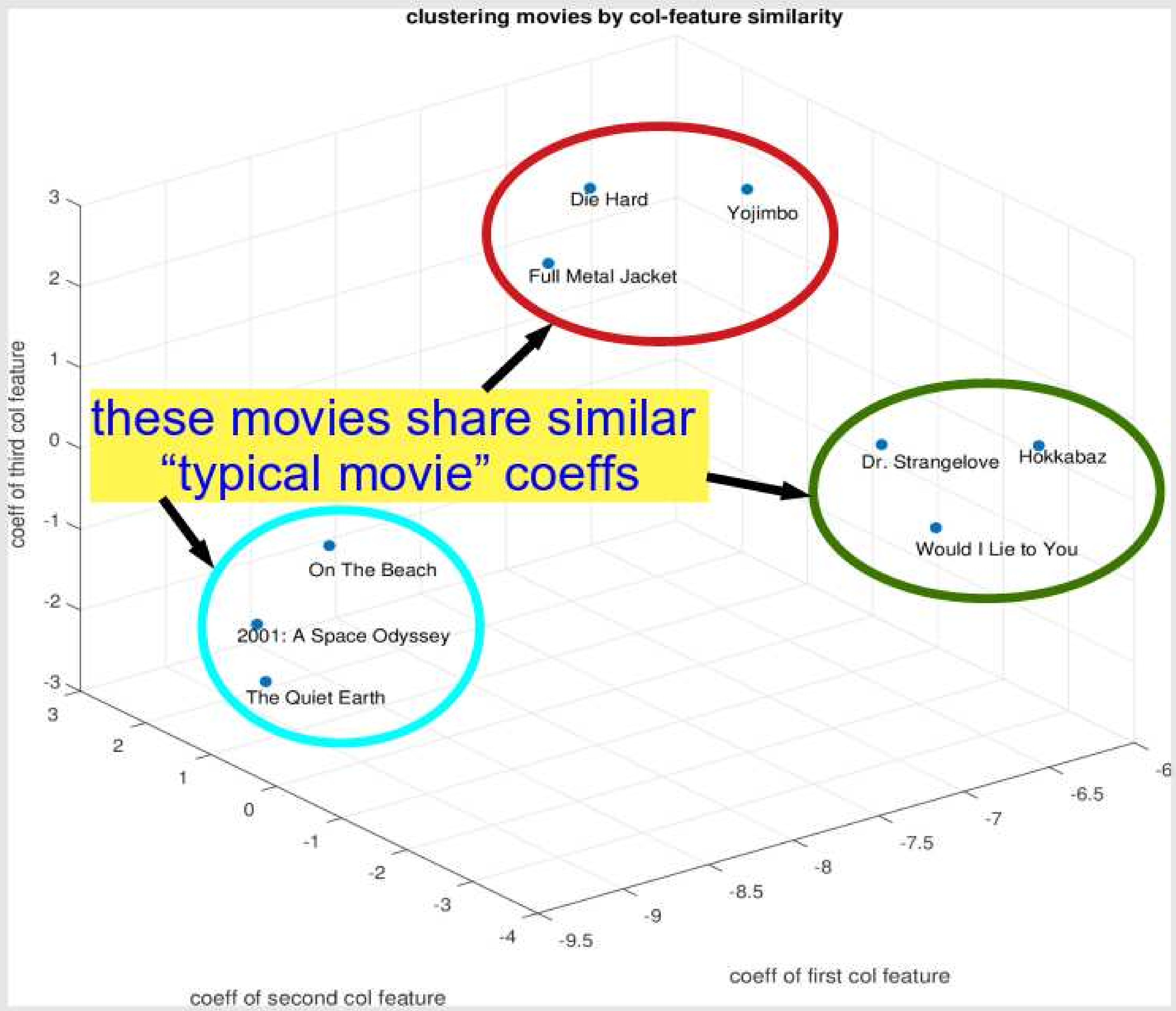



# Clustering in Feature Bases

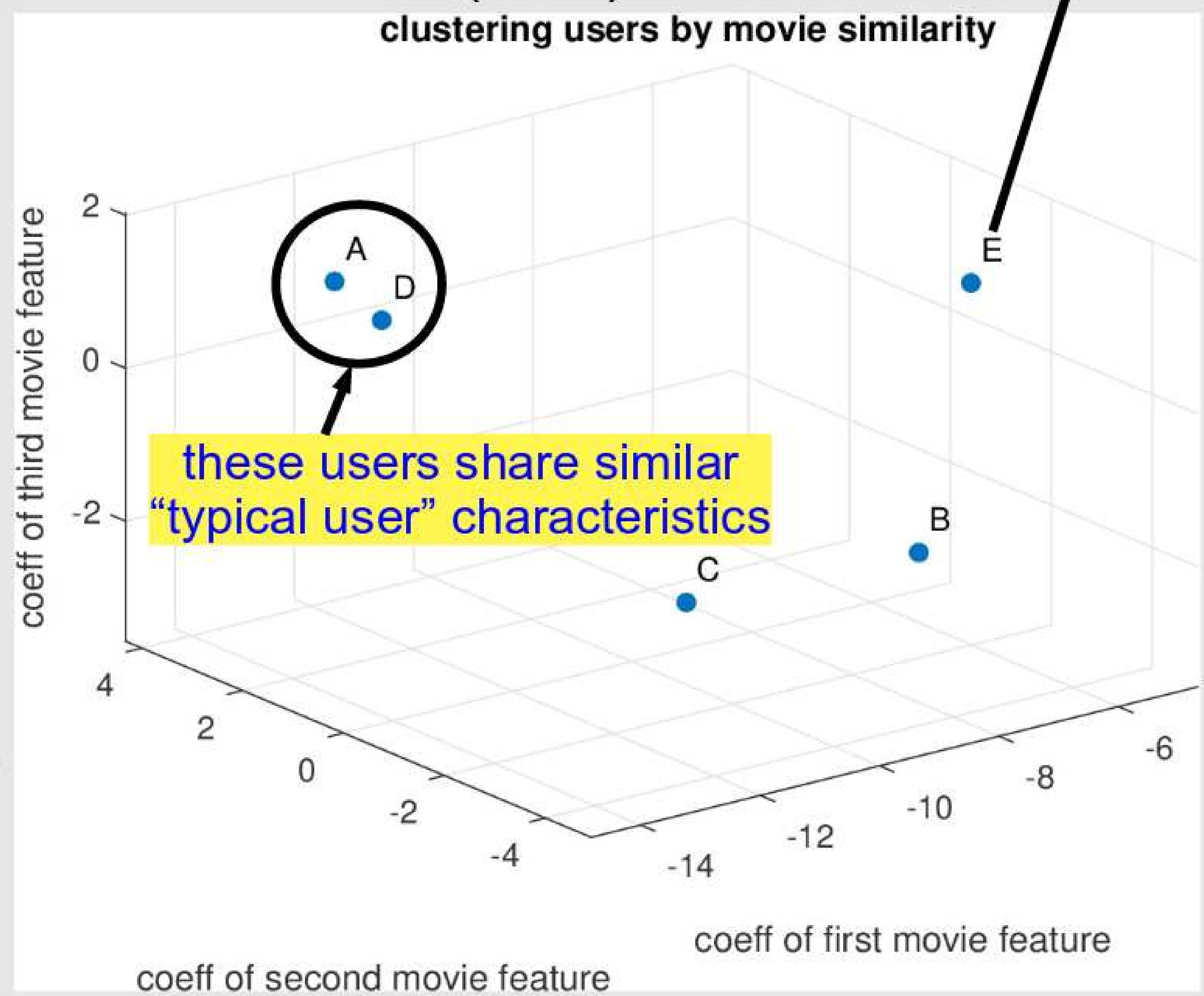
- Scatter plot of $\alpha_{11}$, $\alpha_{21}$, and $\alpha_{31}$ for all movies

$$\text{user E row} = \begin{bmatrix}1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1\end{bmatrix}$$
$$= \beta_{51}\vec{v}_1^T + \beta_{52}\vec{v}_2^T + \beta_{53}\vec{v}_3^T + \cdots + \beta_{55}\vec{v}_5^T$$

Movies classified by projections on column (movie) features

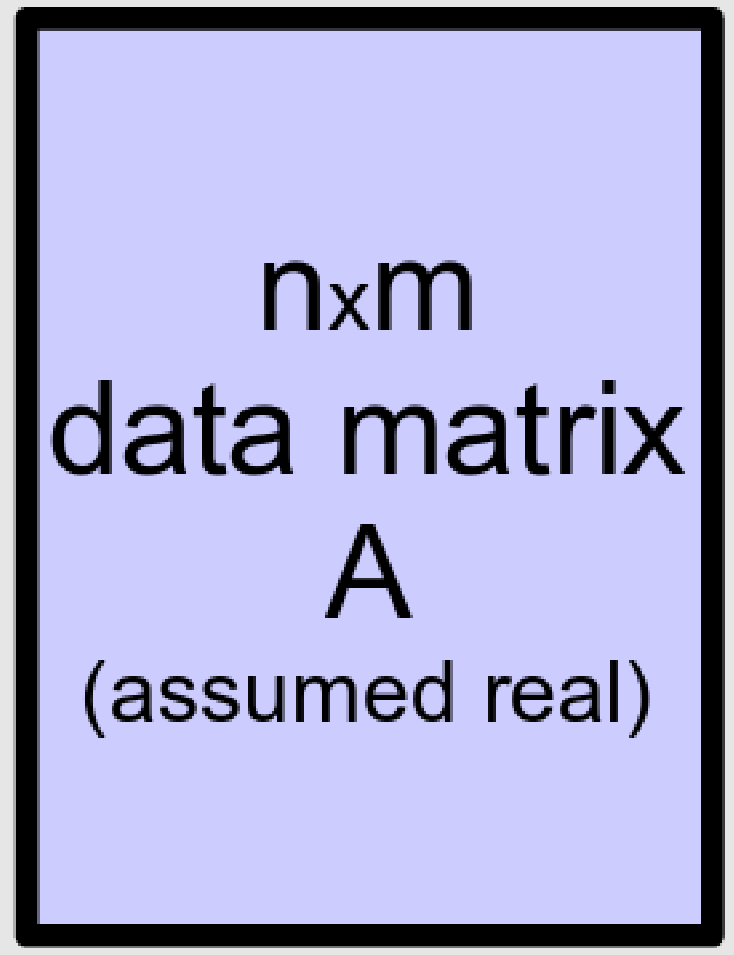



Users classified by projection on row (user) features

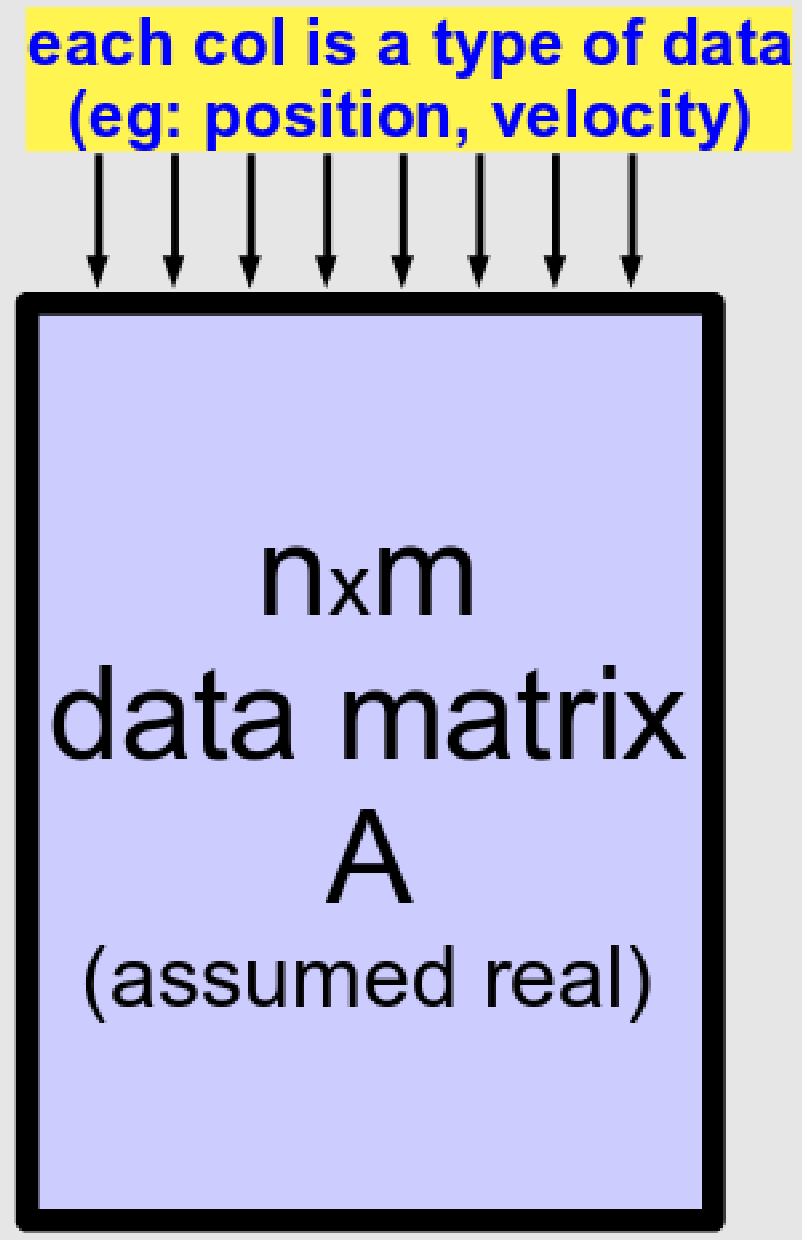


# Principal Component Analysis (PCA)

# Covariance Matrices

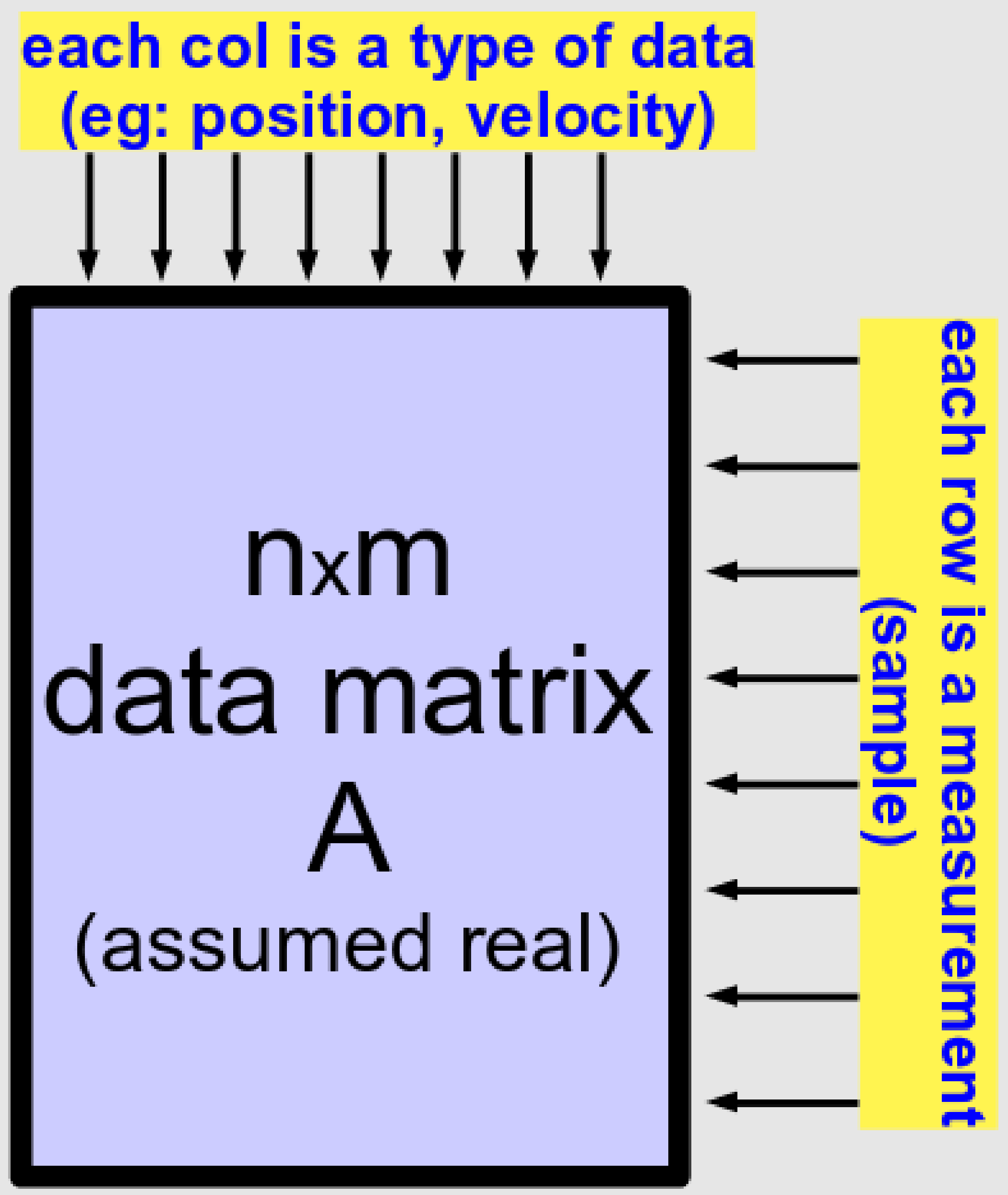


# Covariance Matrices



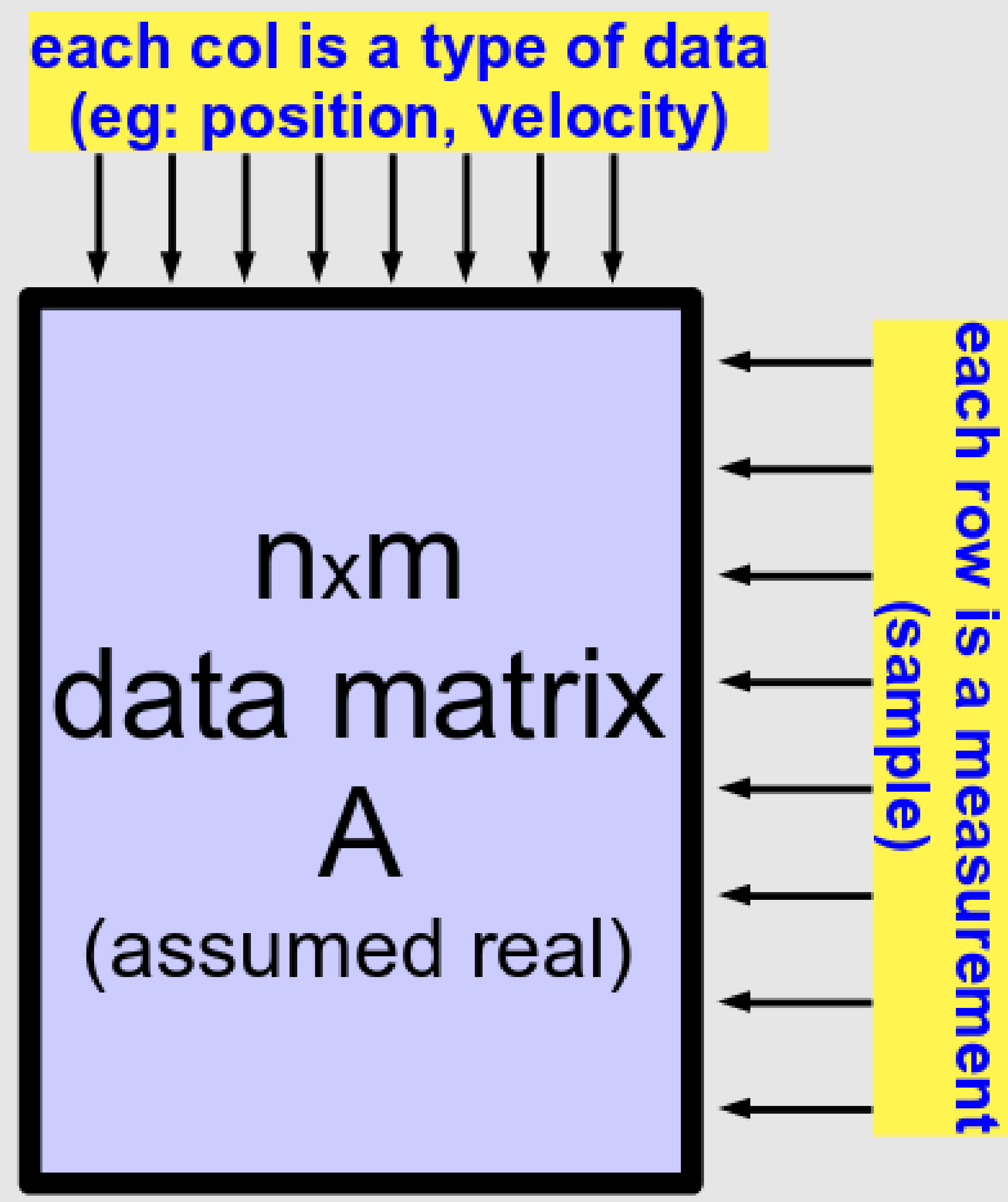

•

# Covariance Matrices



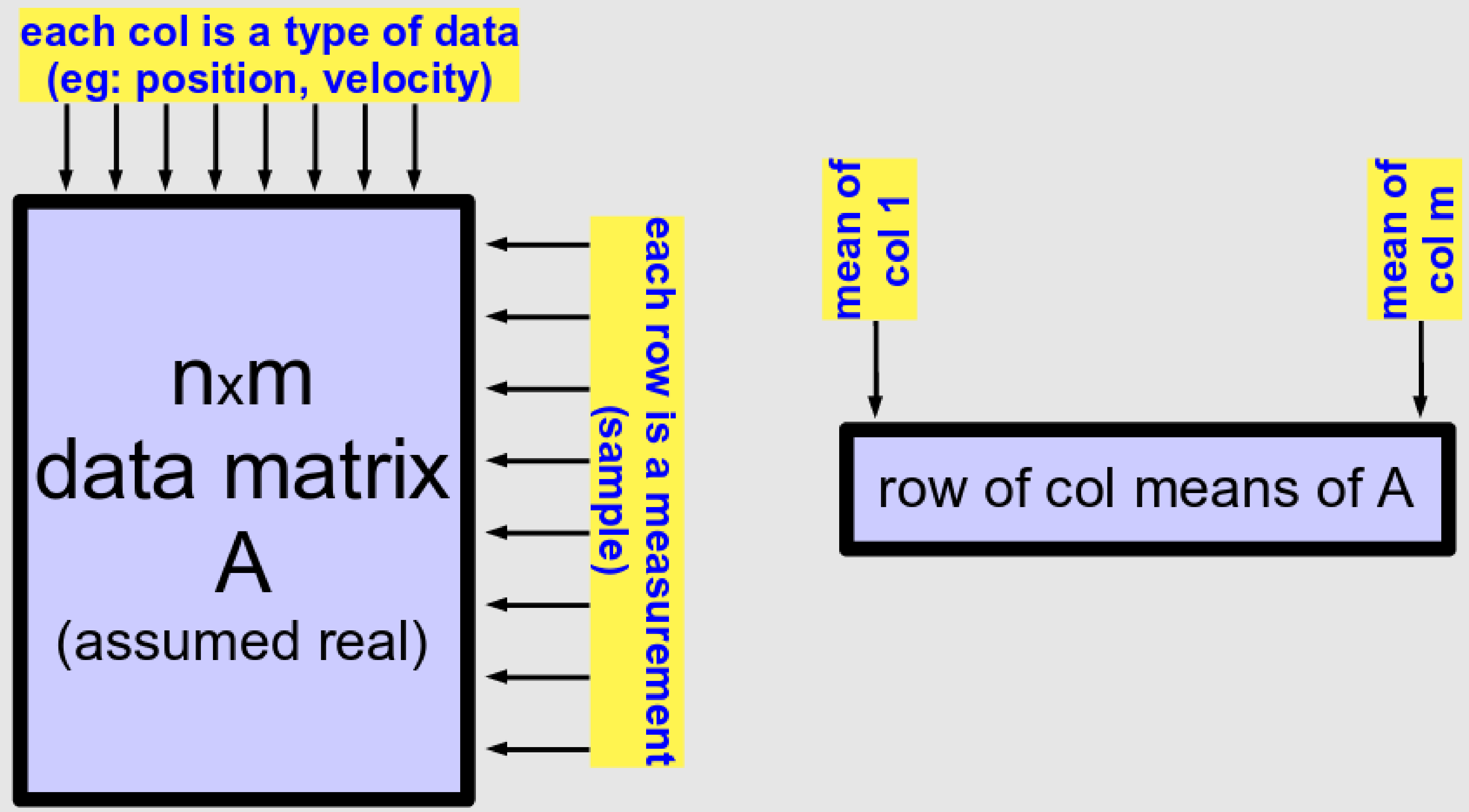

- 

# Covariance Matrices



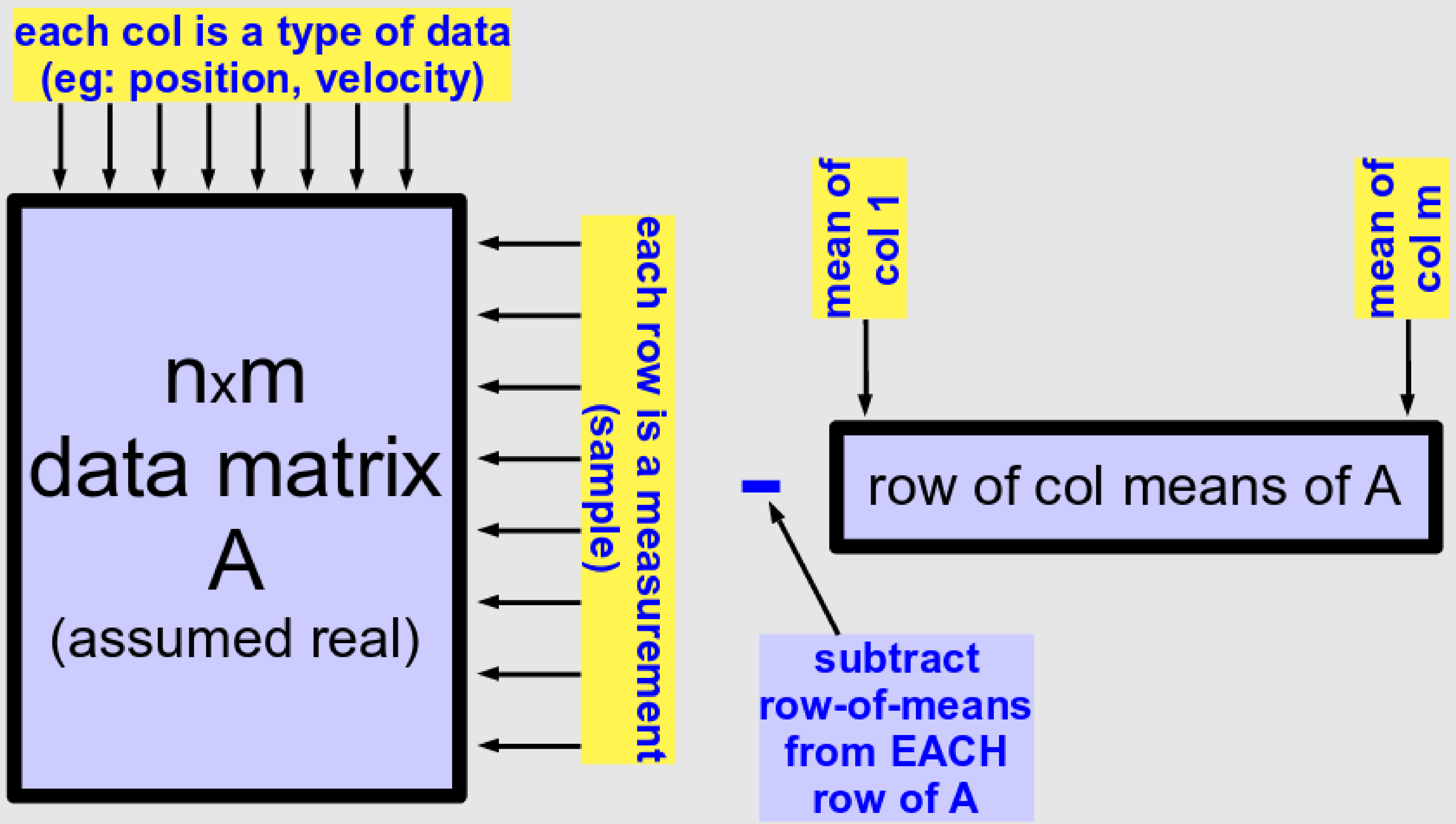

row of col means of A

# Covariance Matrices

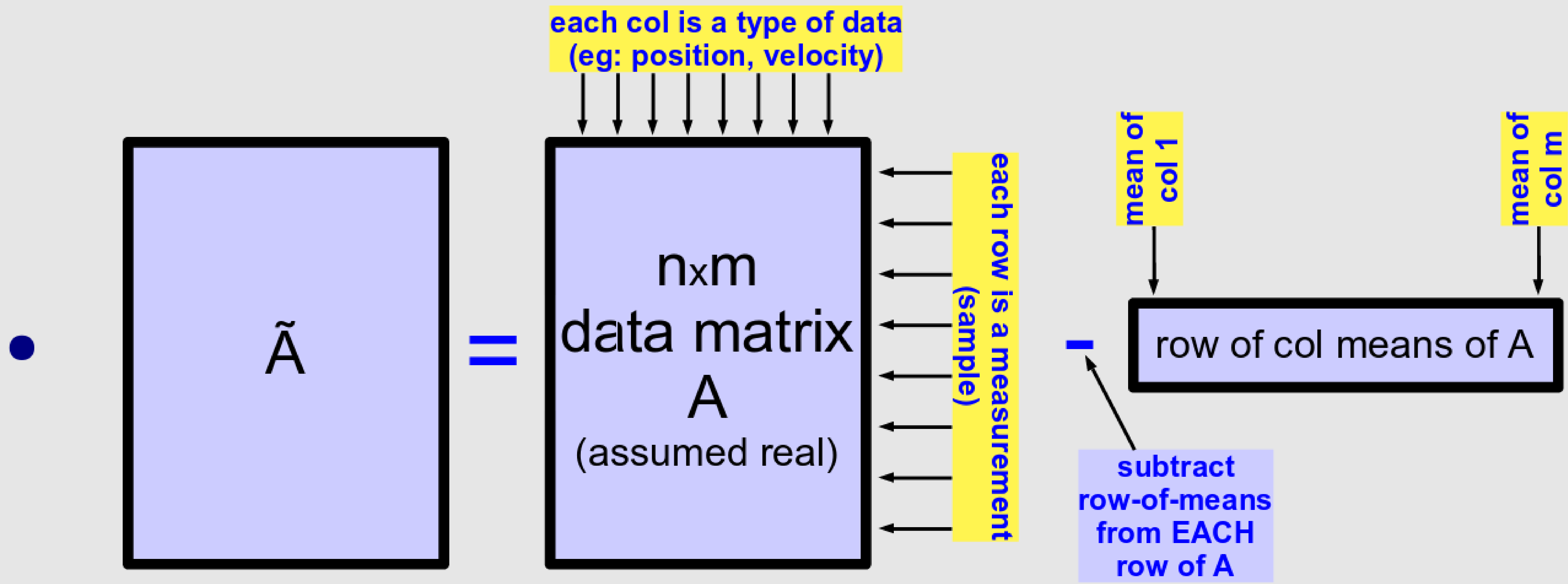



•

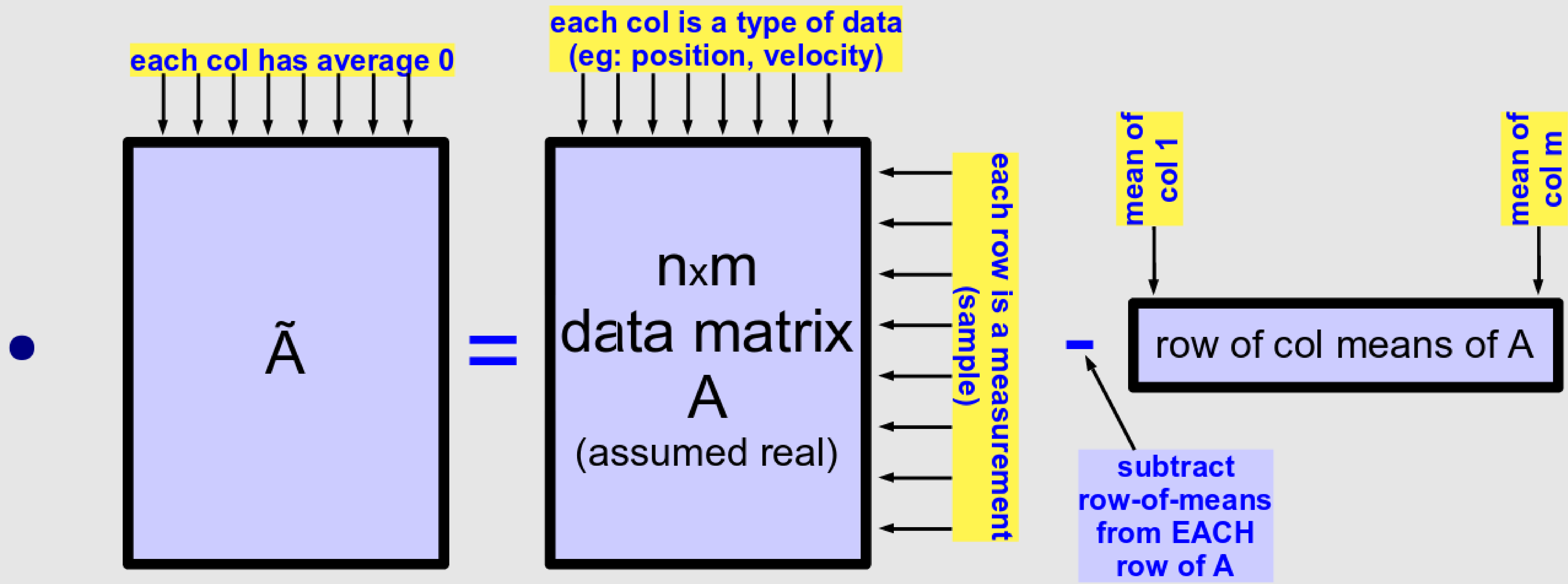
# Covariance Matrices



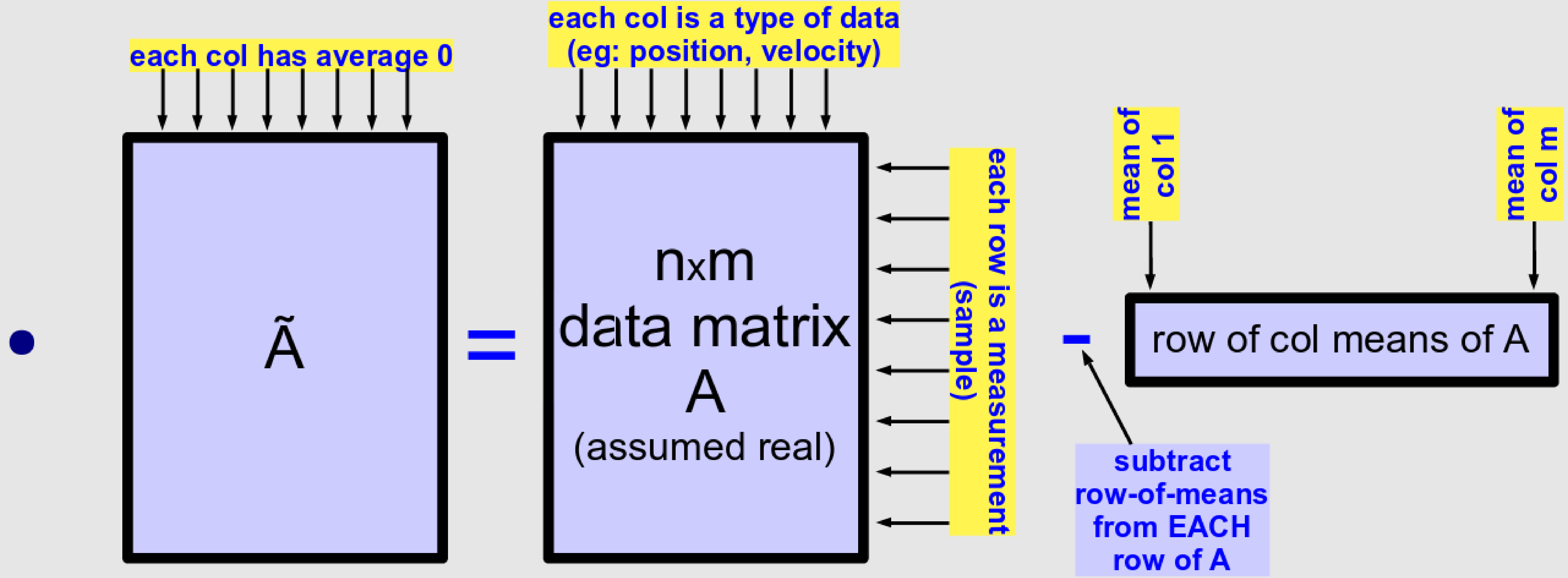
# Covariance Matrices

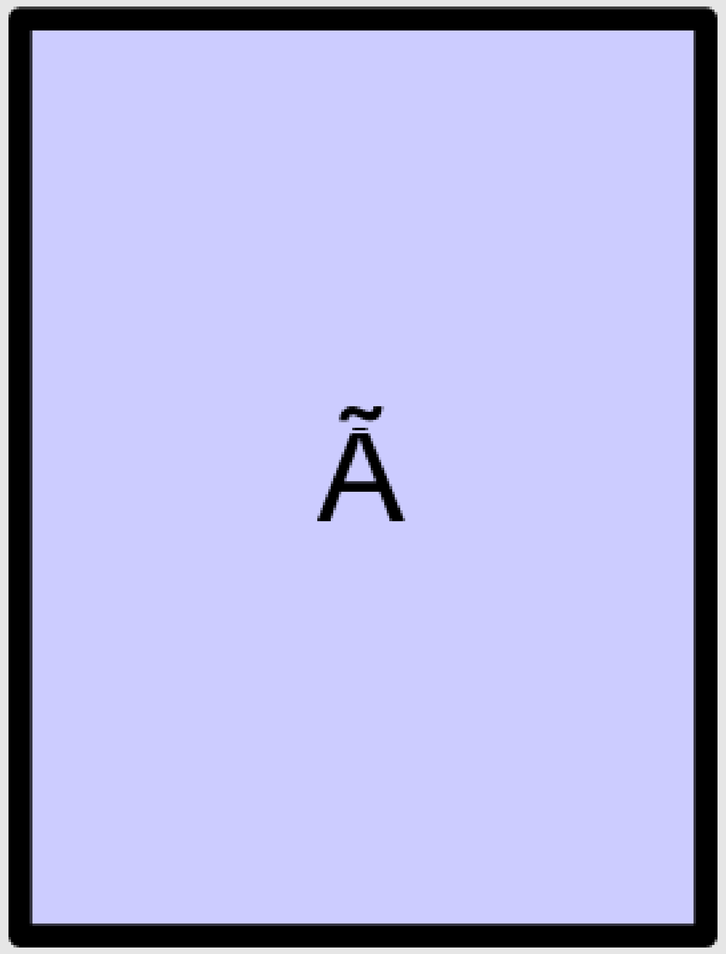


# Covariance Matrices

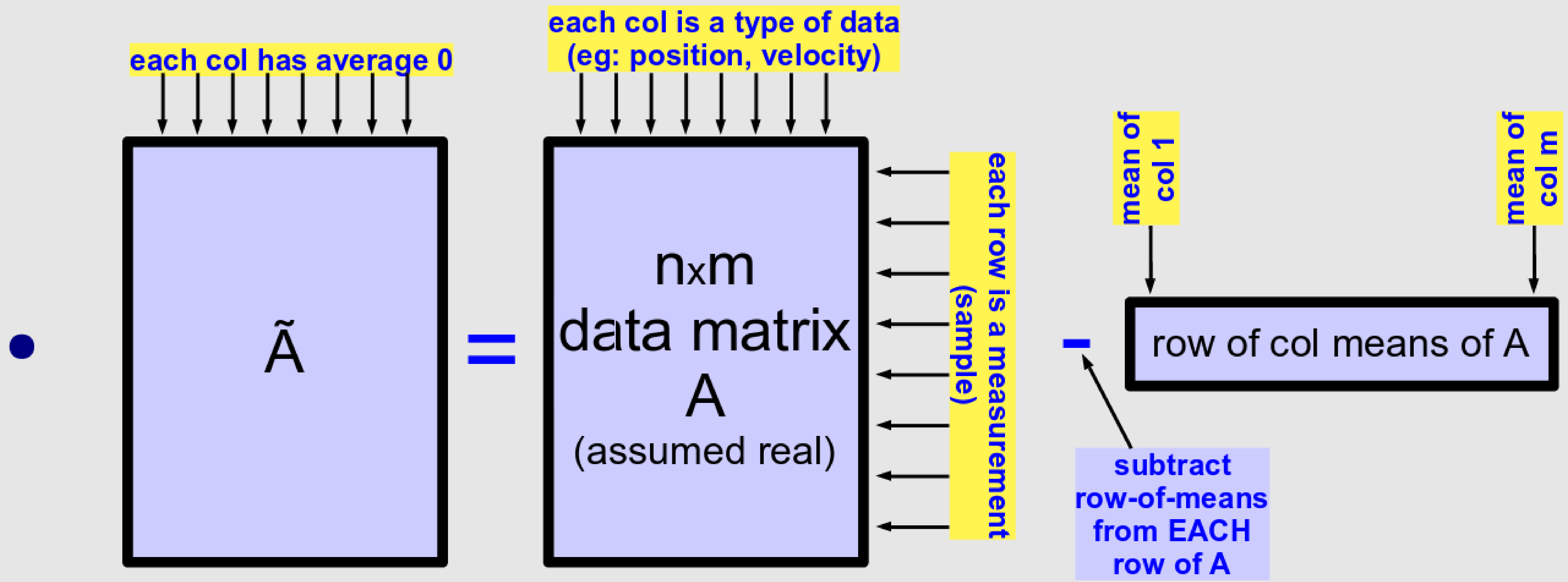


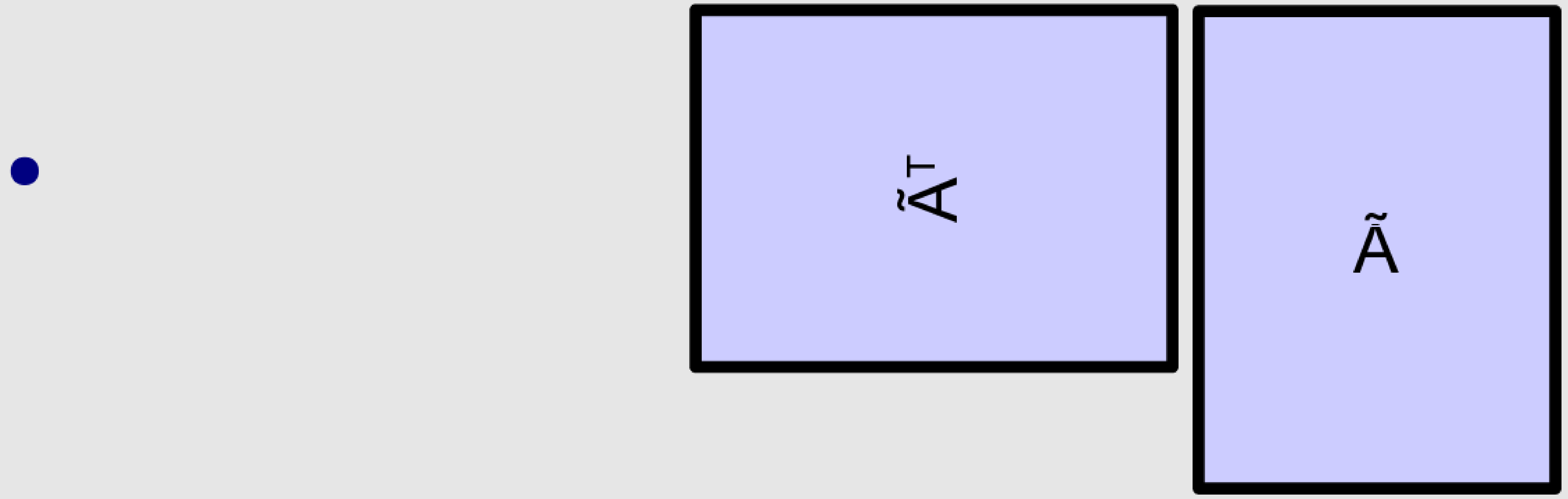
# Covariance Matrices



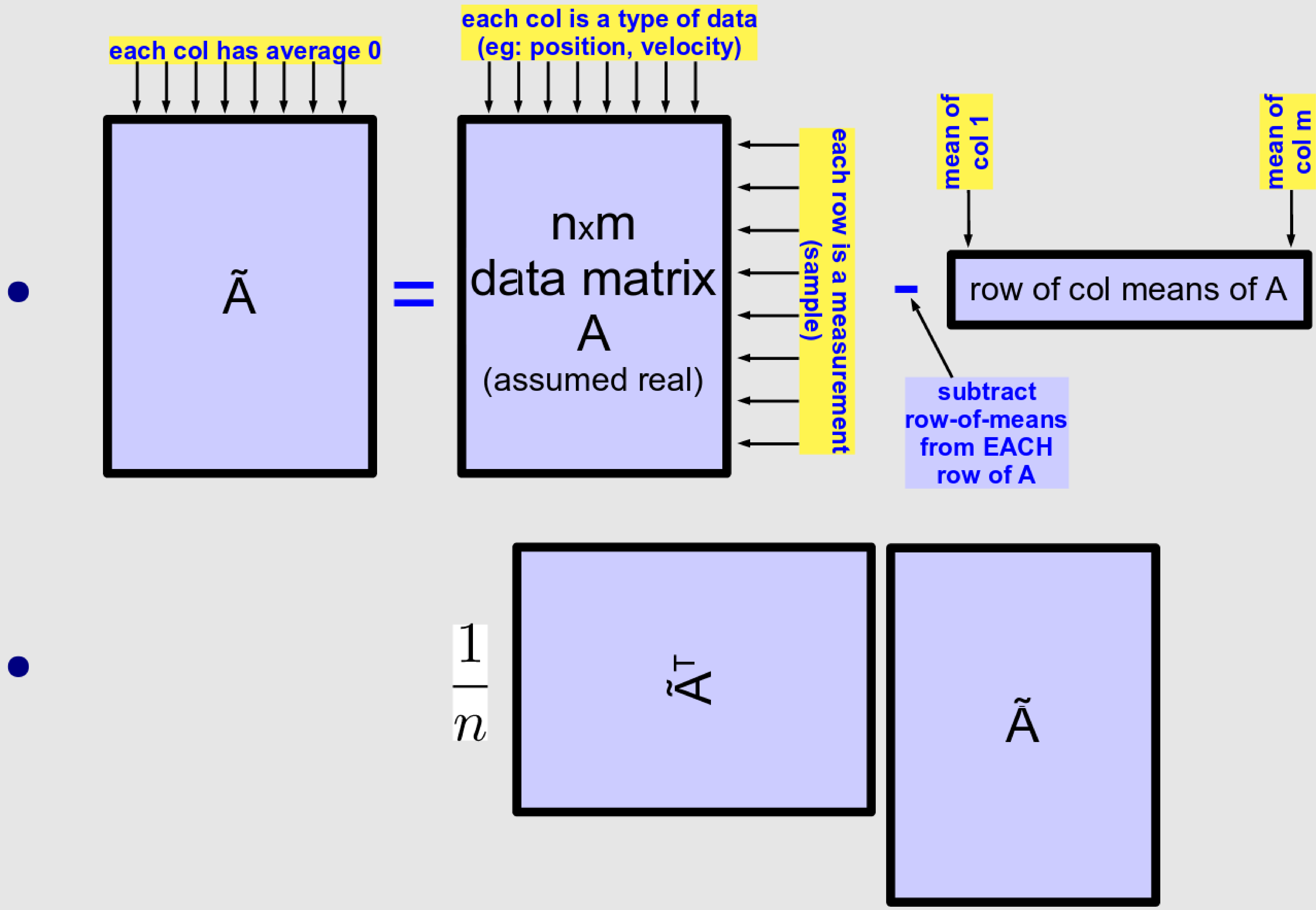



# Covariance Matrices

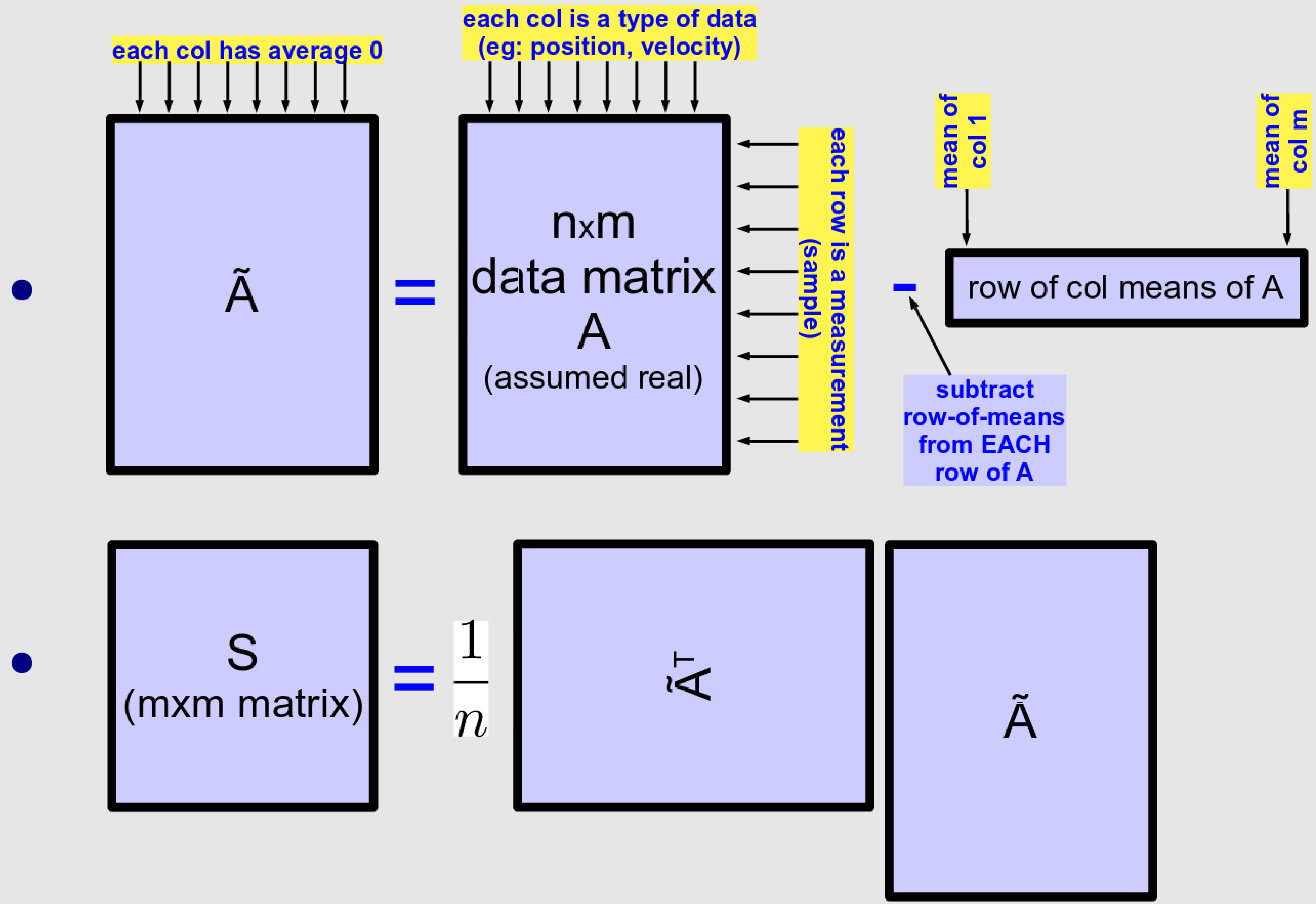


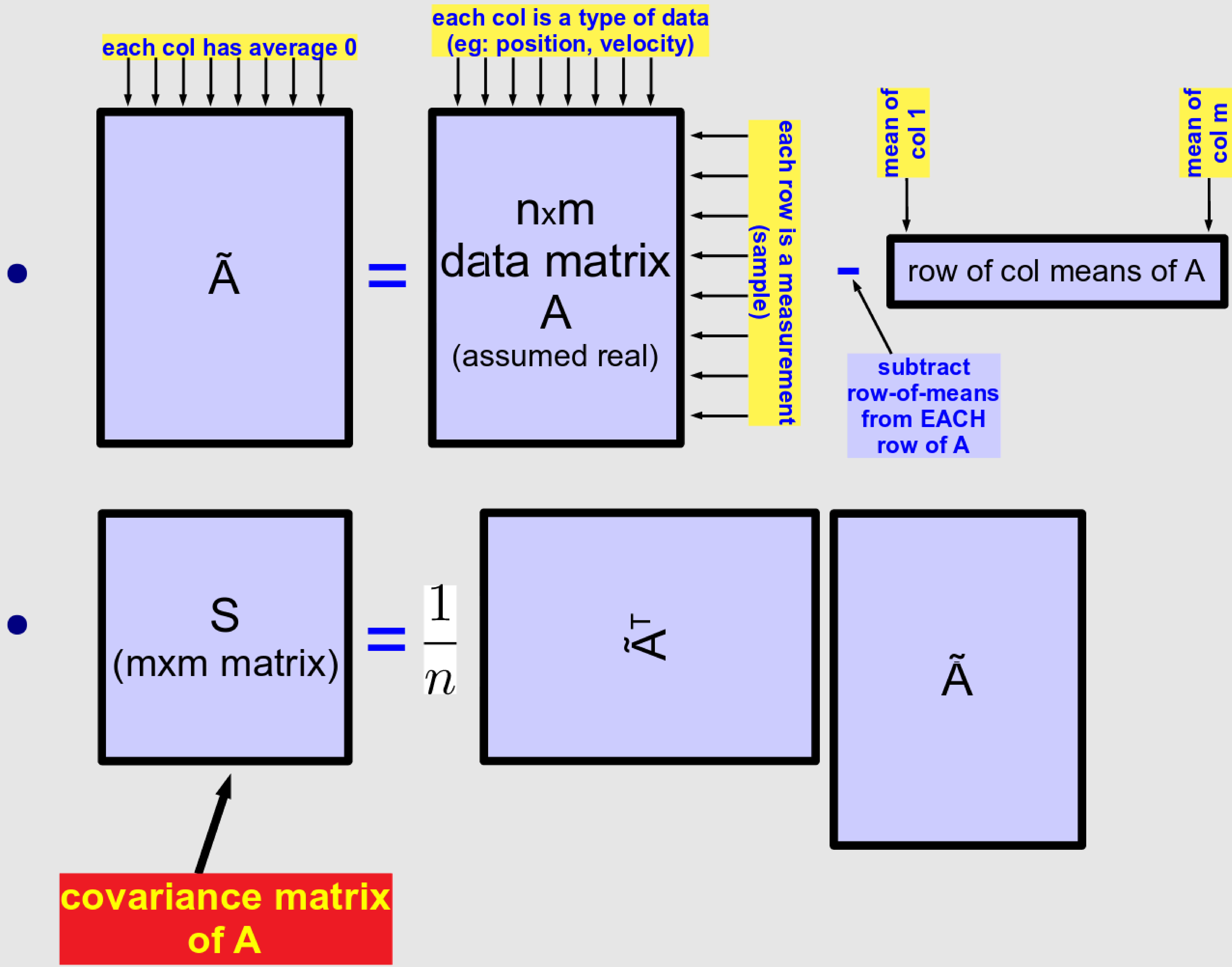
# Covariance Matrices



# Covariance Matrices



# Covariance Matrices



# Covariance Matrices: Properties

- $S \triangleq \dfrac{1}{n}\tilde{A}^T\tilde{A}$

# Covariance Matrices: Properties

- $S \triangleq \dfrac{1}{n} \tilde{A}^T \tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$

# Covariance Matrices: Properties

- $S \triangleq \dfrac{1}{n}\tilde{A}^T\tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**

# Covariance Matrices: Properties

- $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**
  - $s_{ii} = \frac{1}{n}\sum_{j=1}^{n} \tilde{a}_{ij}^2 \geq 0$ : **variance of i^th^ row of A**

# Covariance Matrices: Properties

- $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**
  - $s_i^2 \triangleq s_{ii} = \frac{1}{n}\sum_{j=1}^{n} \tilde{a}_{ij}^2 \geq 0$ : **variance of i$^{\text{th}}$ row of A**

# Covariance Matrices: Properties

- $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**
  - $s_i^2 \triangleq s_{ii} = \frac{1}{n}\sum_{j=1}^{n}\tilde{a}_{ij}^2 \geq 0$ : **variance of i^th row of A**
  - $S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix}$

# Covariance Matrices: Properties

- $S \triangleq \frac{1}{n} \tilde{A}^T \tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**
  - $s_i^2 \triangleq s_{ii} = \frac{1}{n} \sum_{j=1}^{n} \tilde{a}_{ij}^2 \geq 0$ **: variance of i<sup>th</sup> row of A**
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix}$$
  - **can also show:** $|s_{ij}| \leq s_i s_j$
    - using the **Cauchy-Schwartz inequality**

# Covariance Matrices: Properties

- $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A}$
- S is square and **symmetric**: $S = S^T$ or $s_{ij} = s_{ji}$
- The **diagonal entries of S** are **real and ≥ 0**
  - $s_i^2 \triangleq s_{ii} = \frac{1}{n}\sum_{j=1}^{n}\tilde{a}_{ij}^2 \geq 0$ : **variance of i^th row of A**
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix}$$

**covariance matrix**

- **can also show:** $|s_{ij}| \leq s_i s_j$
  - ➔ using the **Cauchy-Schwartz inequality**

# The Correlation Matrix

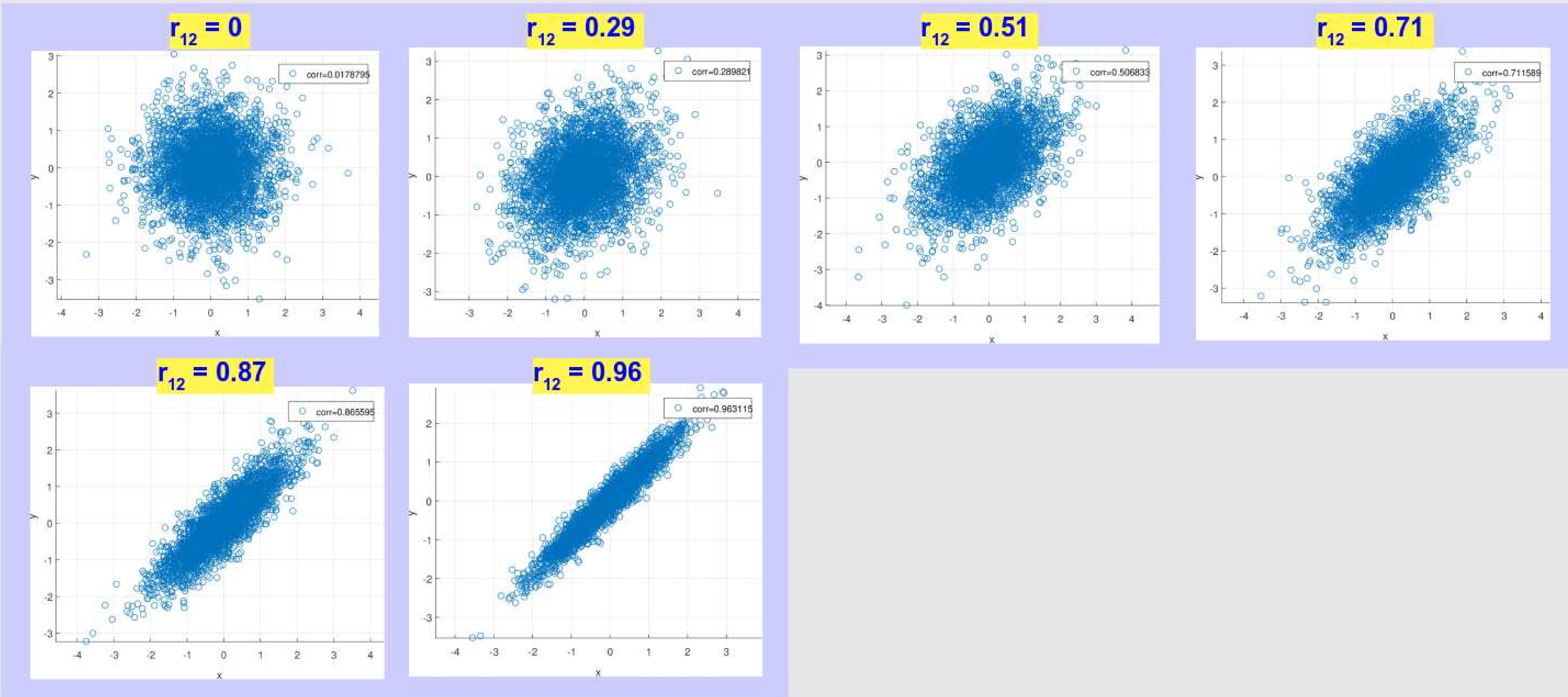
- $r_{ij} \triangleq \dfrac{s_{ij}}{s_i s_j}$; $r_{ij} = r_{ji}$ (symmetry); $\Rightarrow r_{ii} = 1$ why? $\Rightarrow |r_{ij}| <= 1$

  correlation

- $$R = \begin{bmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1m} \\ r_{21} & 1 & r_{23} & \cdots & r_{2m} \\ r_{31} & r_{32} & 1 & \cdots & r_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & r_{m3} & \cdots & 1 \end{bmatrix}$$

  correlation matrix

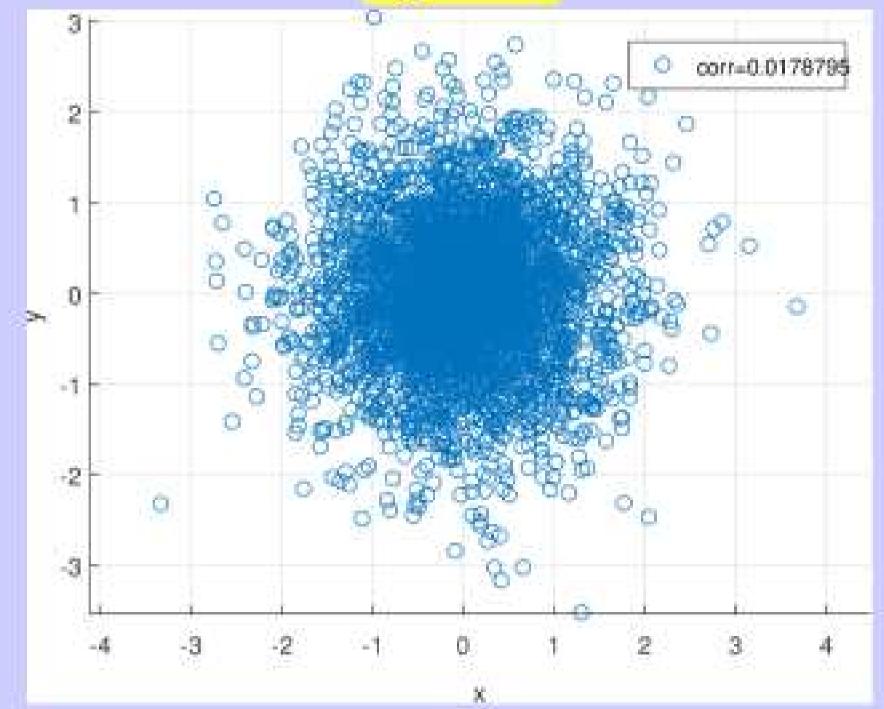
# Correlation: Geometric Intuition



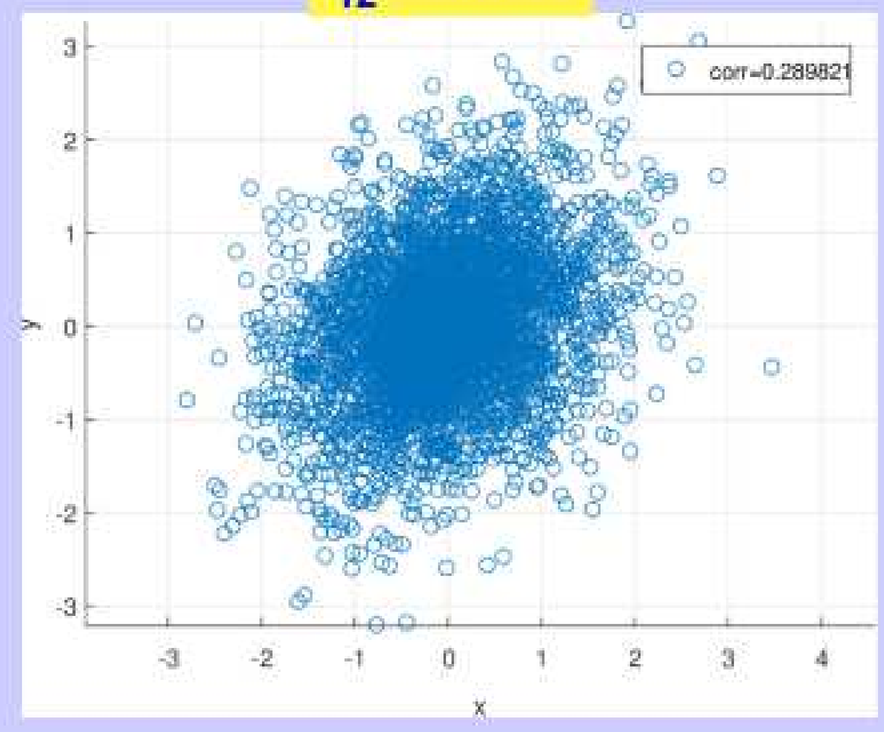

**5000 x 2 matrices**
**(each point is a row)**

# Correlation: Geometric Intuition

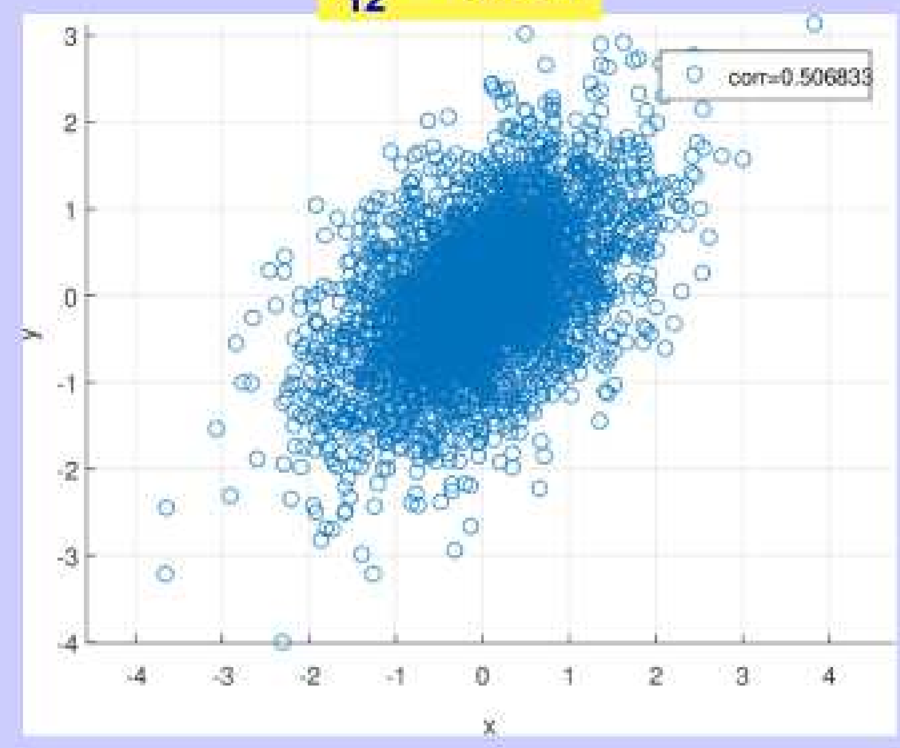
$r_{12} = 0$



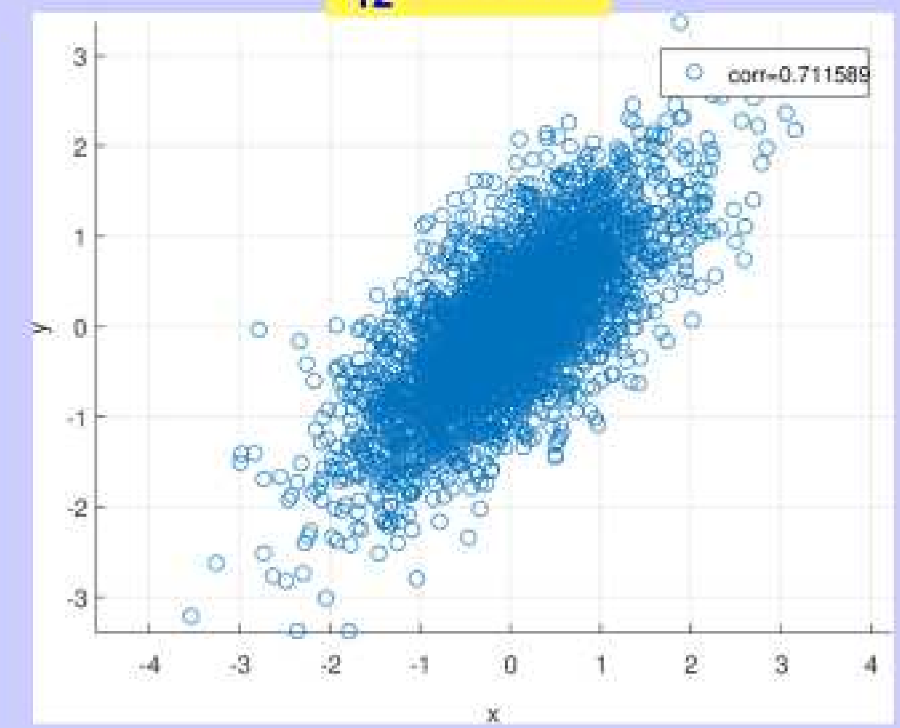

$r_{12} = 0.29$



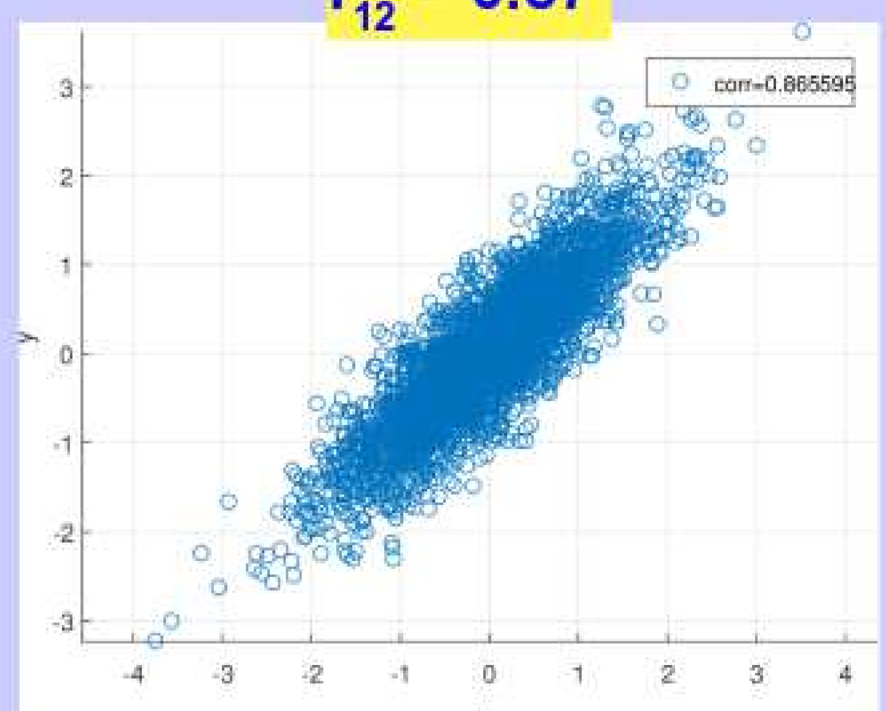

$r_{12} = 0.51$



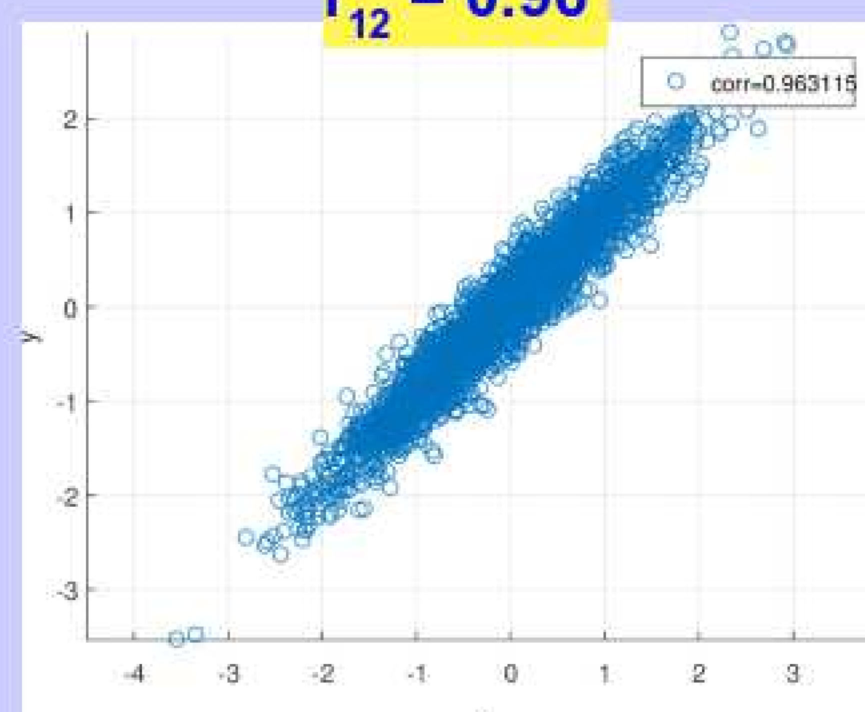

$r_{12} = 0.71$



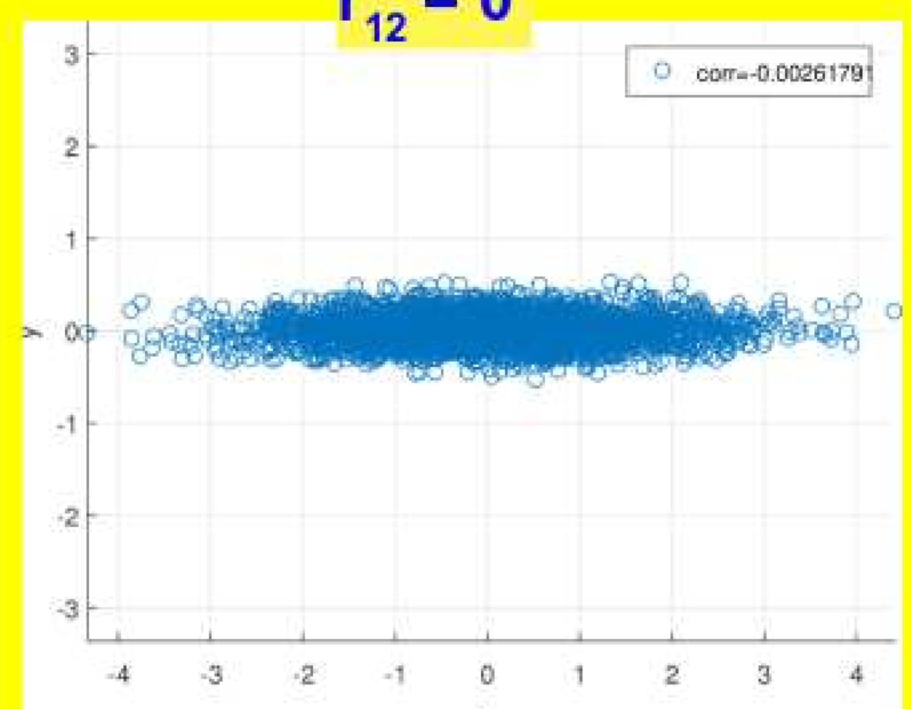

$r_{12} = 0.87$



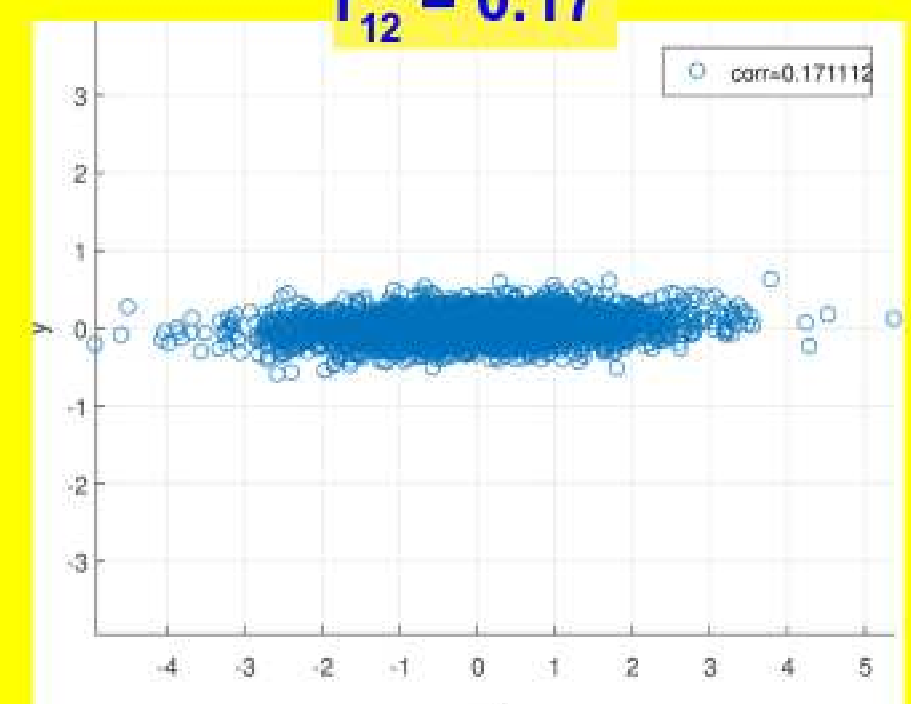

$r_{12} = 0.96$



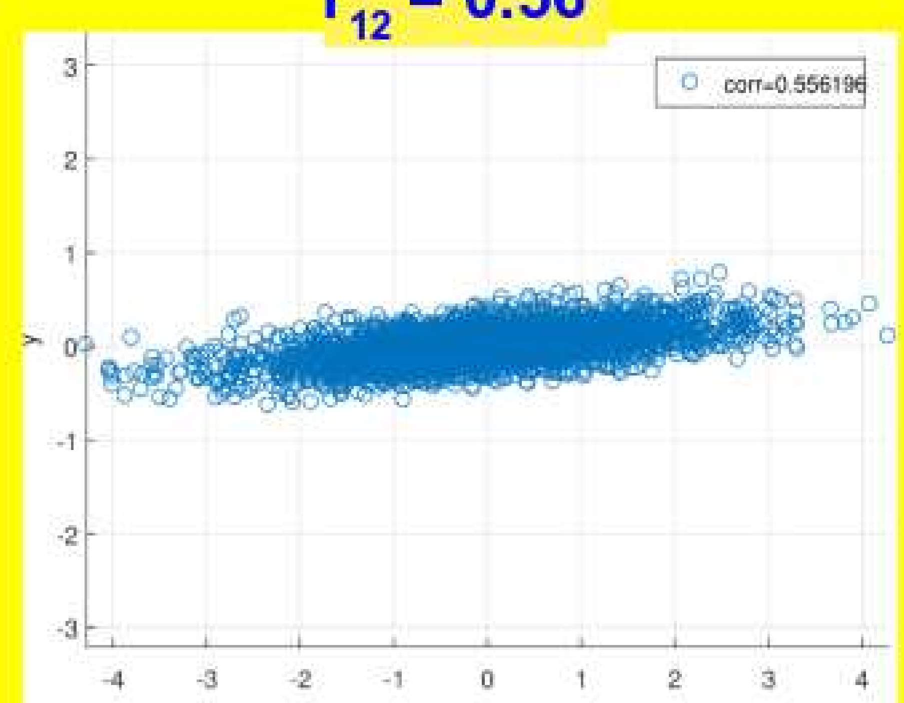

$r_{12} = 0$



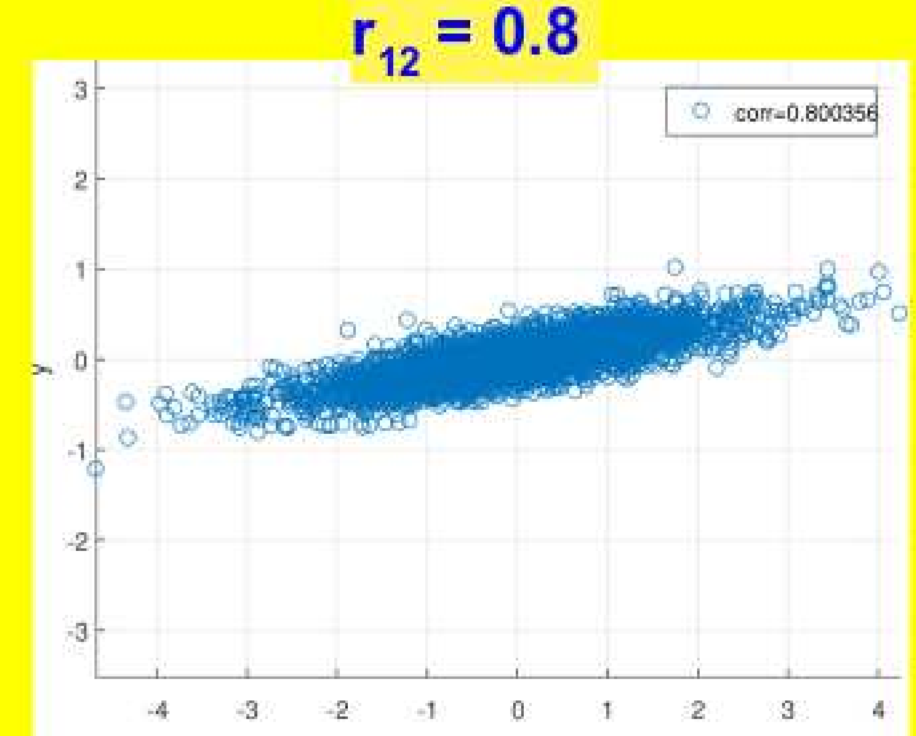

$r_{12} = 0.17$




$r_{12} = 0.56$

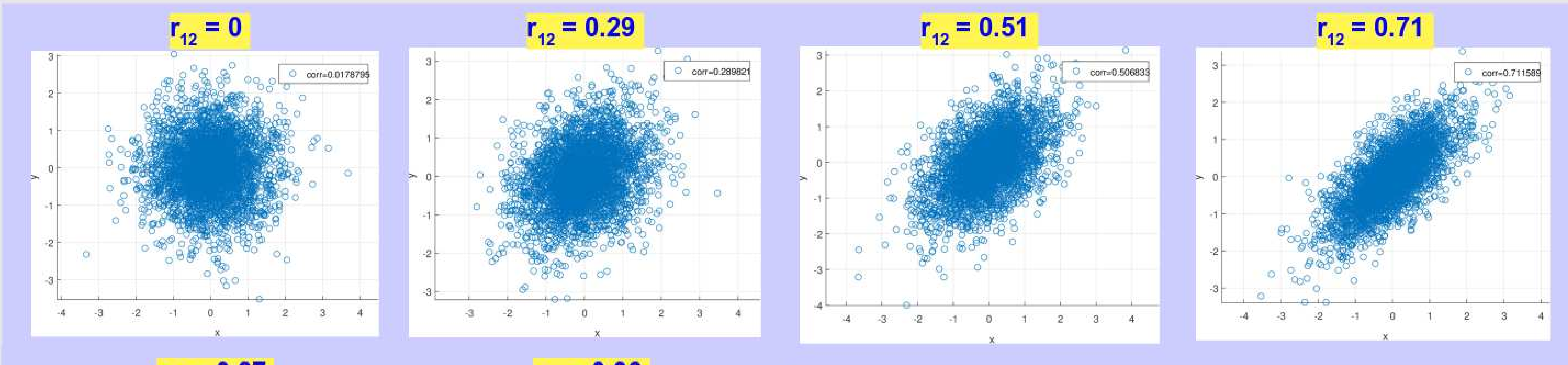


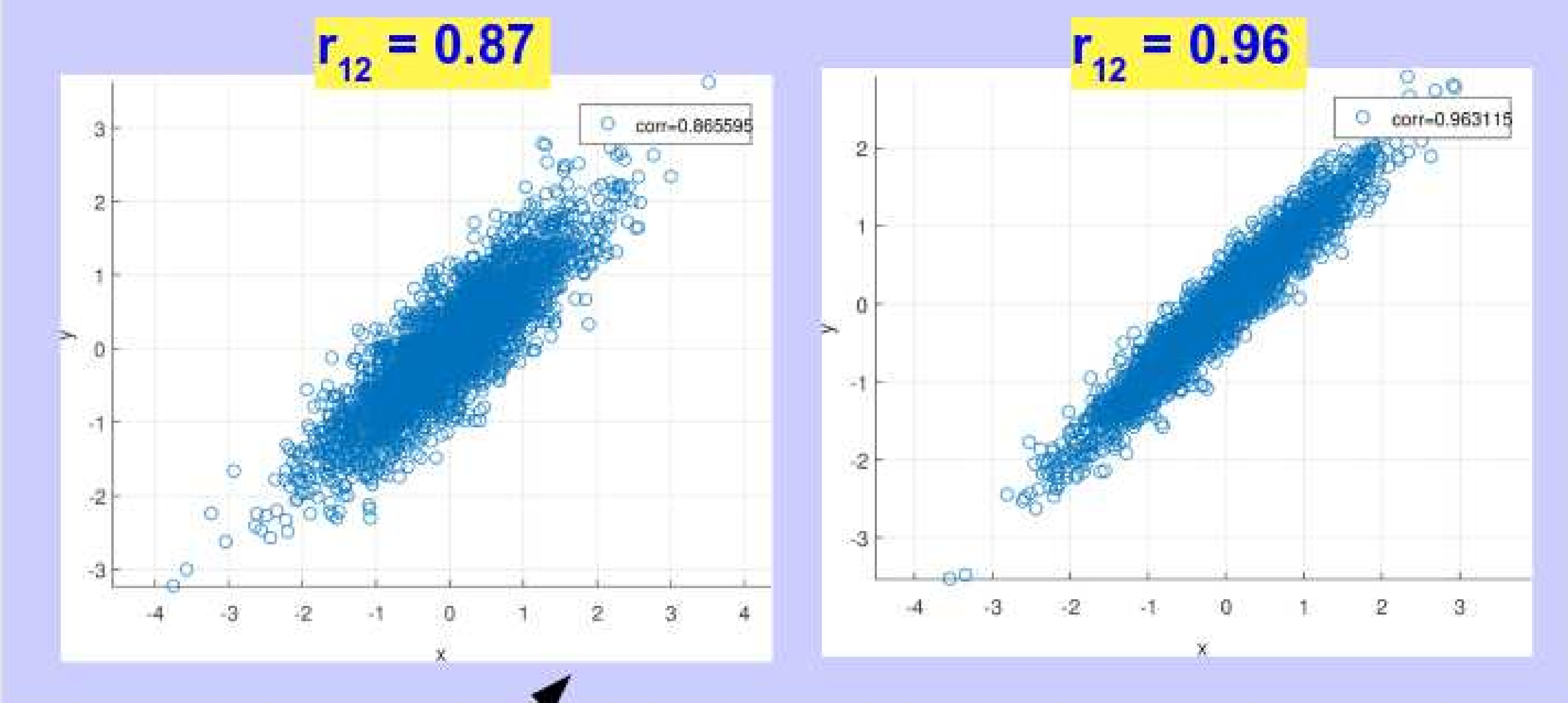

$r_{12} = 0.8$

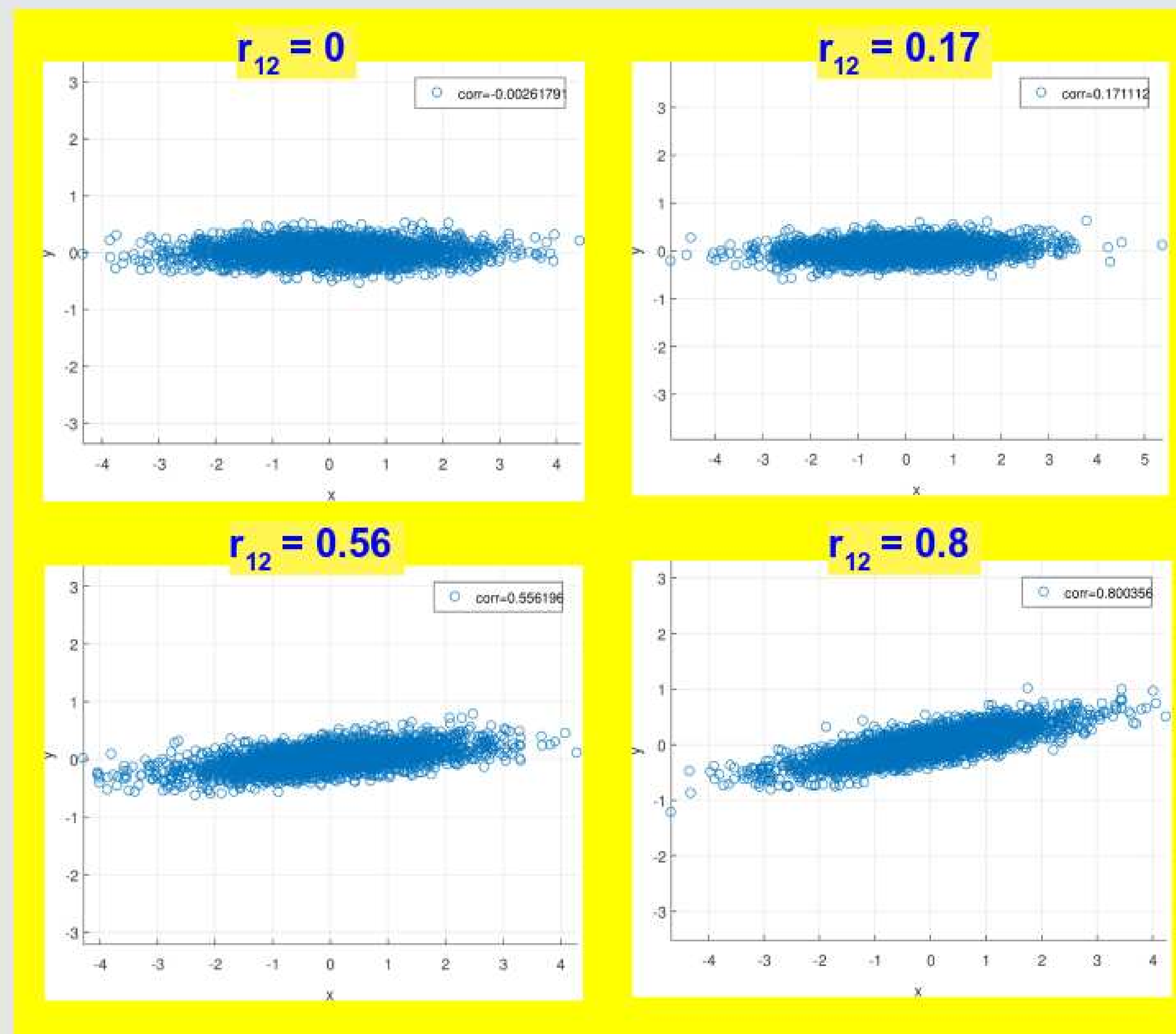



**5000 x 2 matrices**
**(each point is a row)**

# Correlation: Geometric Intuition



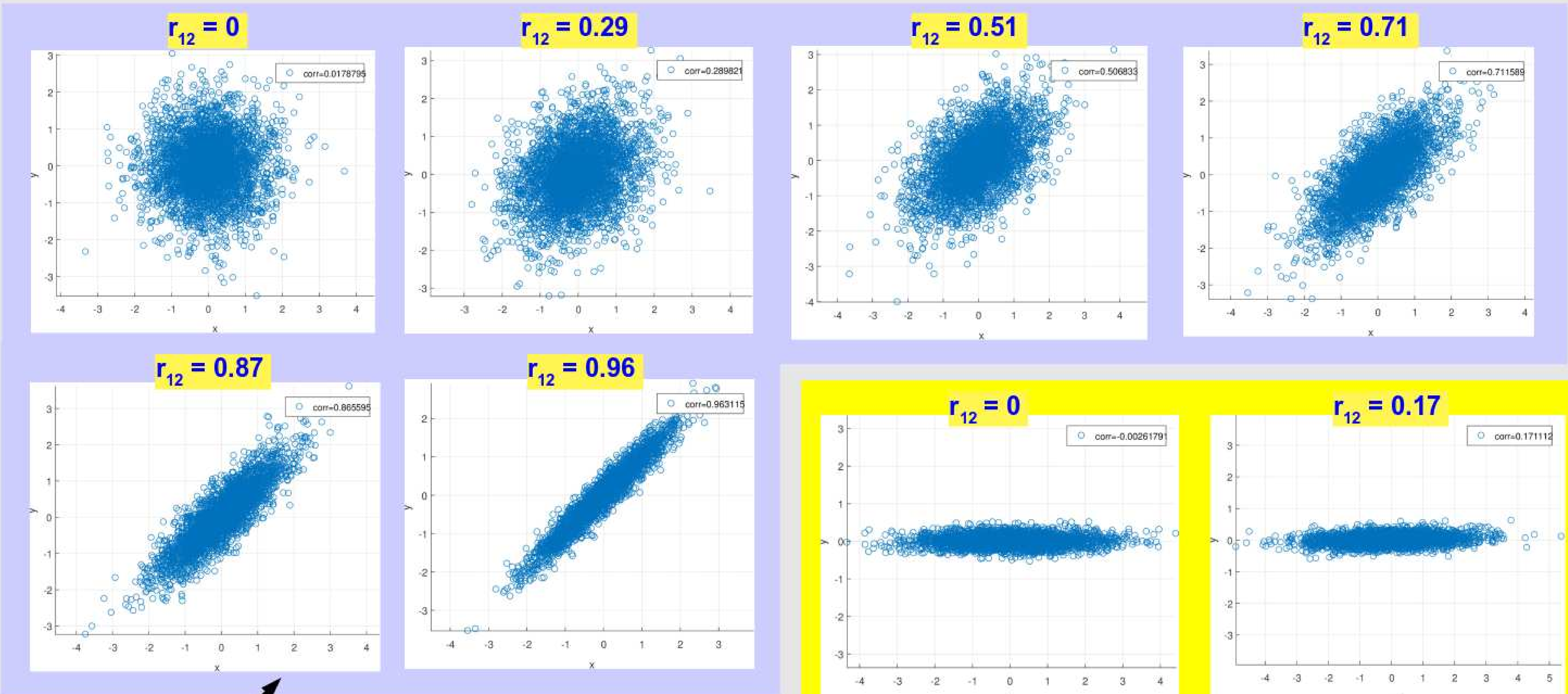



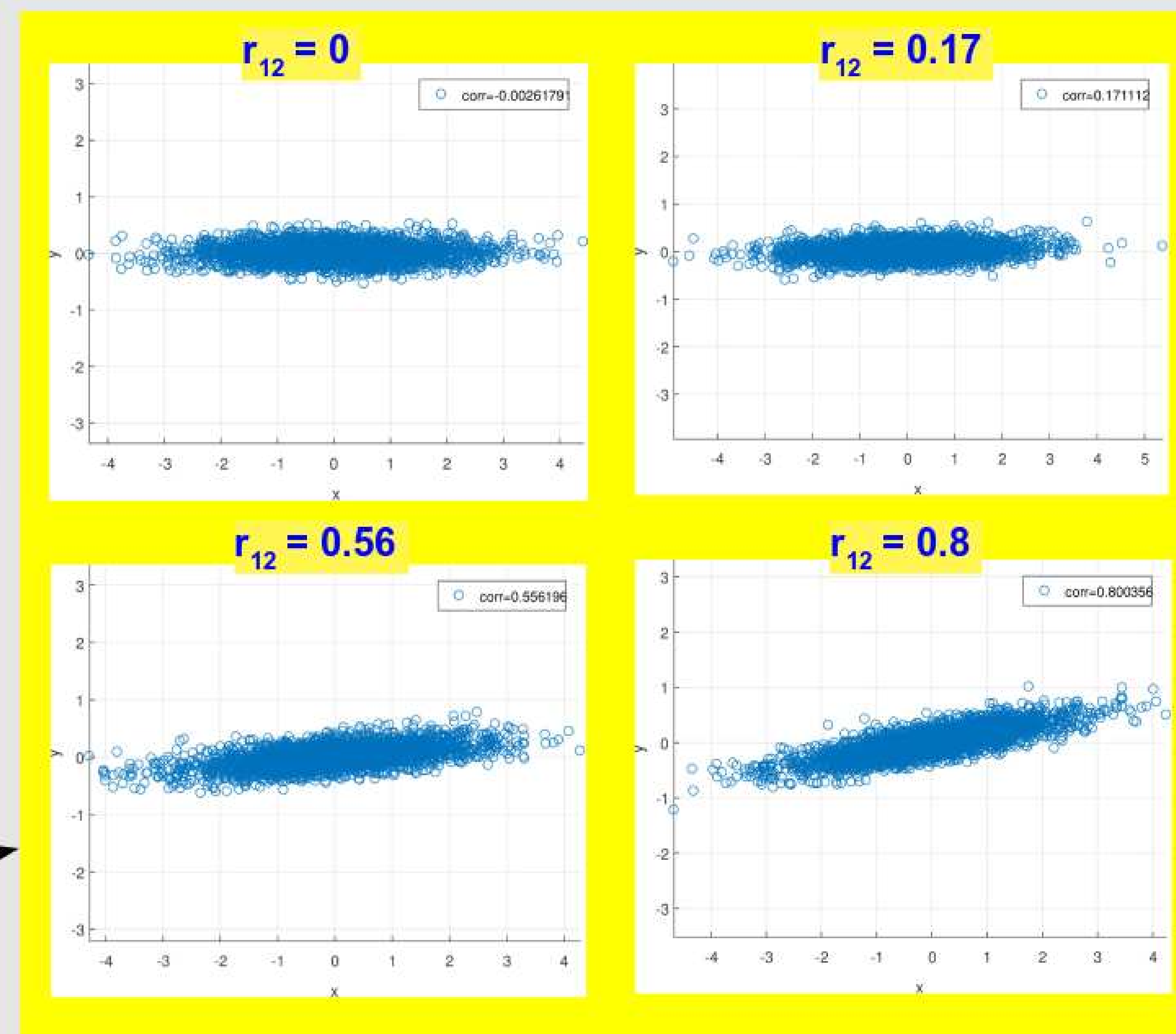




**correlation provides some insight ...**

**5000 x 2 matrices**
**(each point is a row)**

# Correlation: Geometric Intuition




**5000 x 2 matrices**
(each point is a row)

**correlation provides some insight ...**

**... but it leaves a lot out**

# The Intuition behind PCA

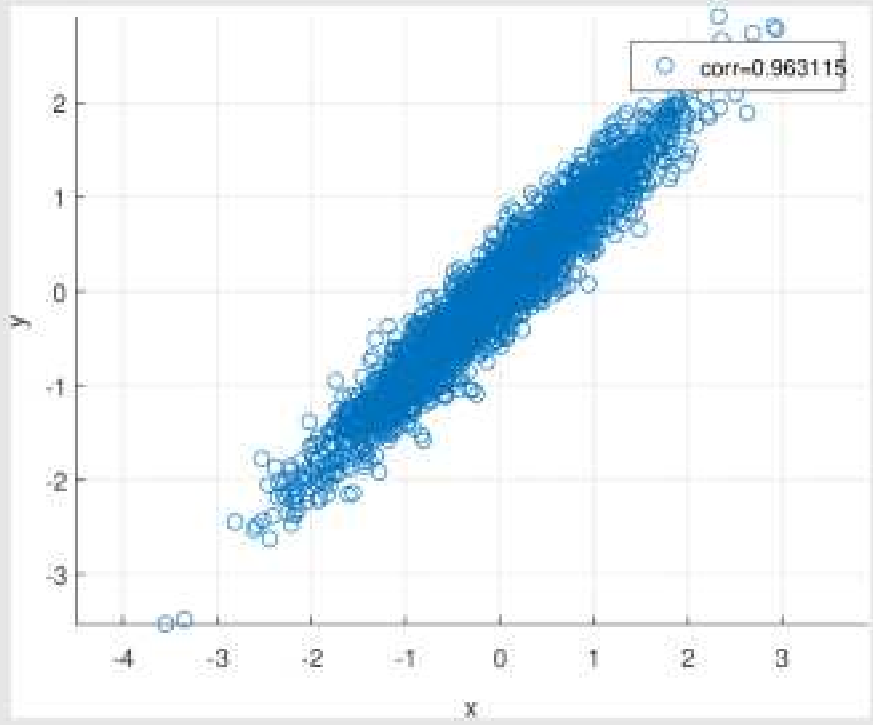
- PCA: finds (orthogonal) “**main axes** along which the data lie”: the **principal components**

# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis

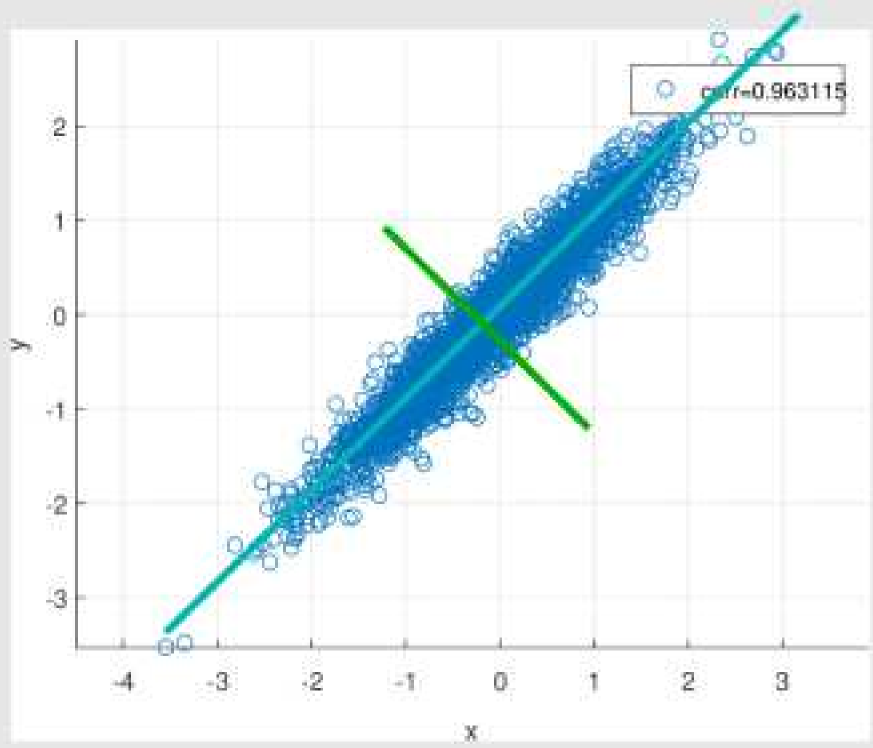
# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis



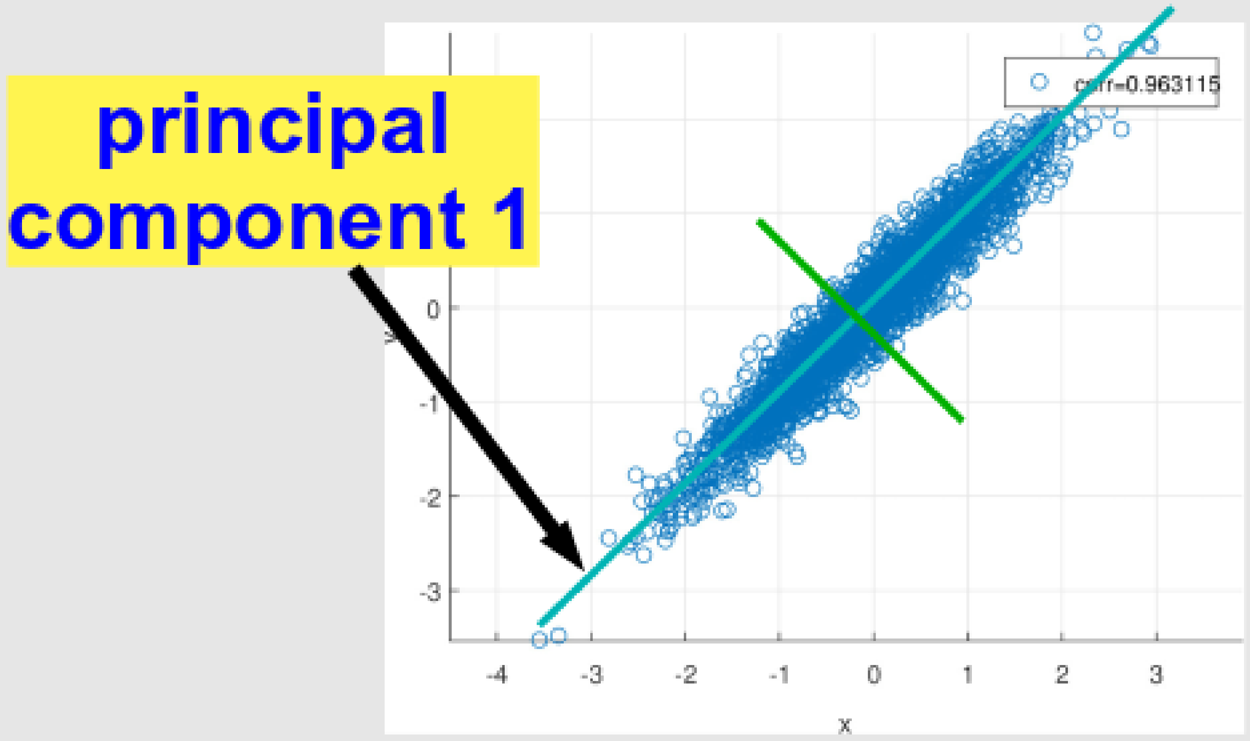
# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis



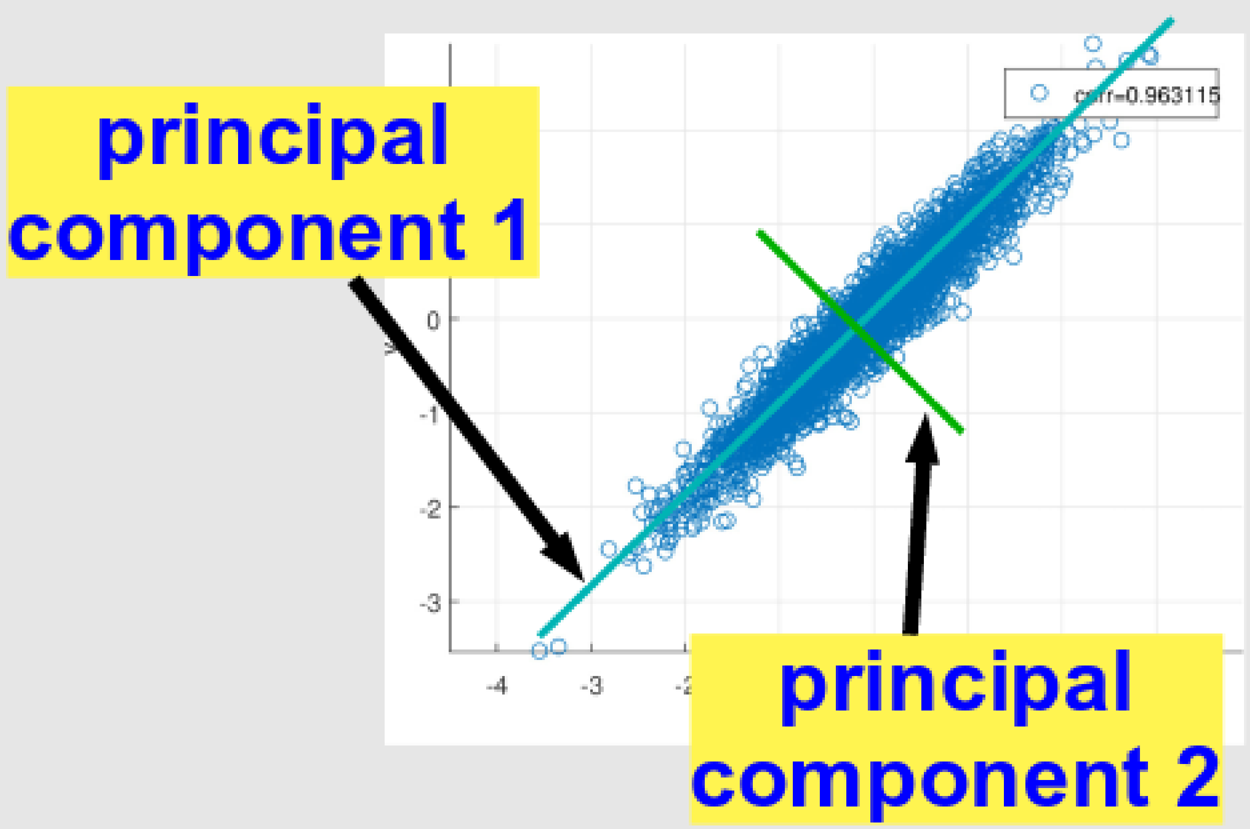
# The Intuition behind PCA

- PCA: finds (orthogonal) “**main axes** along which the data lie”: the **principal components**
  - provides weights indicating “strength” of each axis



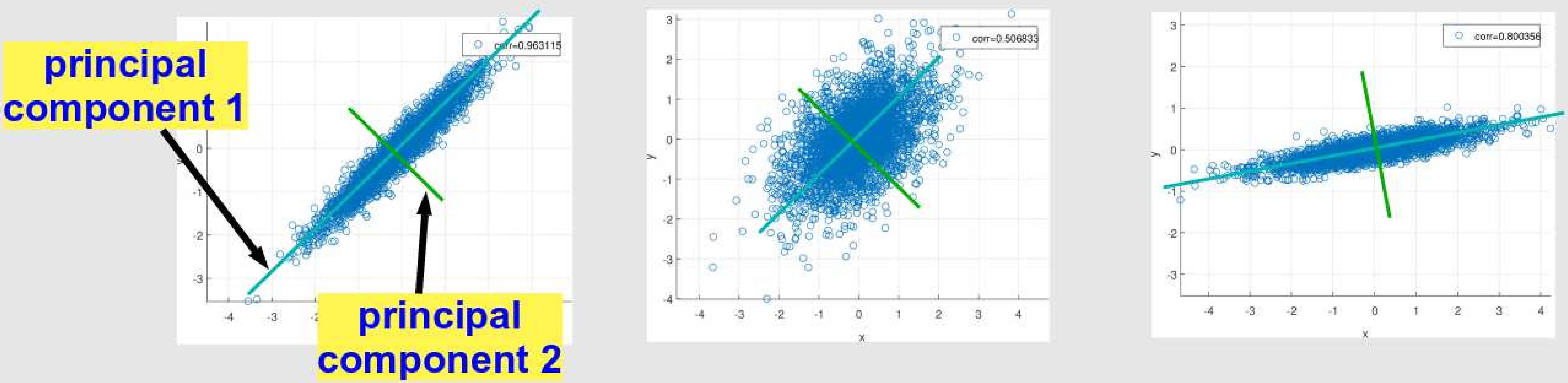
# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis



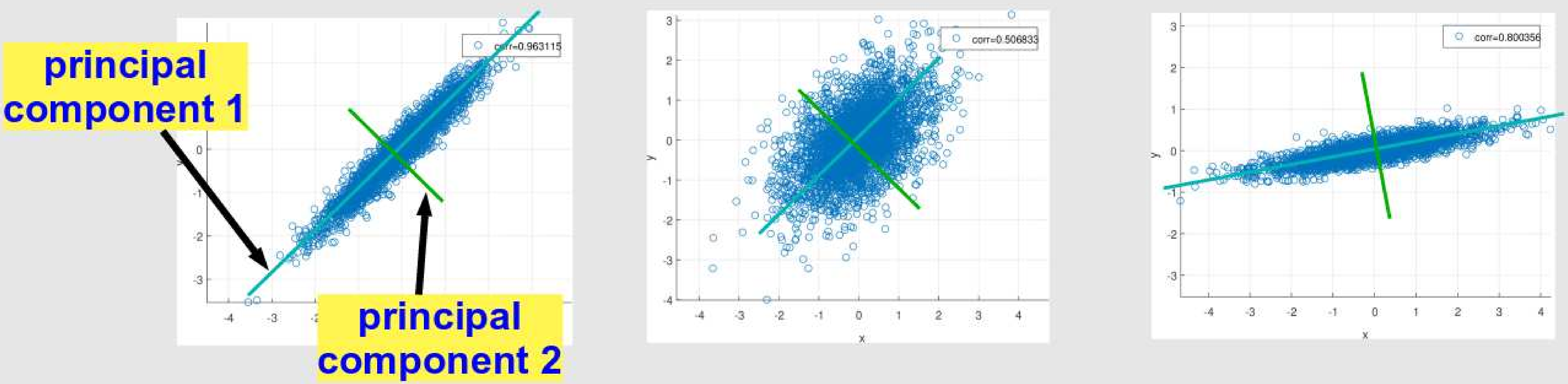
# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis



# The Intuition behind PCA

- PCA: finds (orthogonal) "**main axes** along which the data lie": the **principal components**
  - provides weights indicating "strength" of each axis




  - starting point for PCA: the covariance matrix S

# PCA: The Procedure

- **Eigendecompose** the covariance matrix
- $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P \Lambda P^{-1}$$

# PCA: The Procedure

- **Eigendecompose** the covariance matrix

- $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P\Lambda P^{-1}$$

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

# PCA: The Procedure

- **Eigendecompose** the covariance matrix

- $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P\Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

# PCA: The Procedure

- **Eigendecompose** the covariance matrix

- $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P\Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

# PCA: The Procedure

- **Eigendecompose** the covariance matrix
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P \Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

will be real and ≥0

# PCA: The Procedure

- **Eigendecompose** the covariance matrix
  - $S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P\Lambda P^{-1}$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

will be real and ≥0

- $\sqrt{\lambda_i}$ = the **weights**
  - (i.e., the variances)

# PCA: The Procedure

- **Eigendecompose the covariance matrix**
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P \Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

will be real and ≥0

- $\sqrt{\lambda_i}$ = the **weights**
  - **(i.e., the variances)**
  - **with** $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_m \geq 0$

# PCA: The Procedure

- **Eigendecompose** **the covariance matrix**
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P\Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

will be real and ≥0

- $\sqrt{\lambda_i}$ = the **weights**
  - **(i.e., the variances)**
  - **with** $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_m \geq 0$
- eigenvectors $\vec{p}_i$ = the **principal components**

# PCA: The Procedure

- **Eigendecompose** the covariance matrix
  - $$S = \begin{bmatrix} s_1^2 & s_{12} & s_{13} & \cdots & s_{1m} \\ s_{21} & s_2^2 & s_{23} & \cdots & s_{2m} \\ s_{31} & s_{32} & s_3^2 & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_m^2 \end{bmatrix} = P \Lambda P^{-1}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

cols will be orthonormal

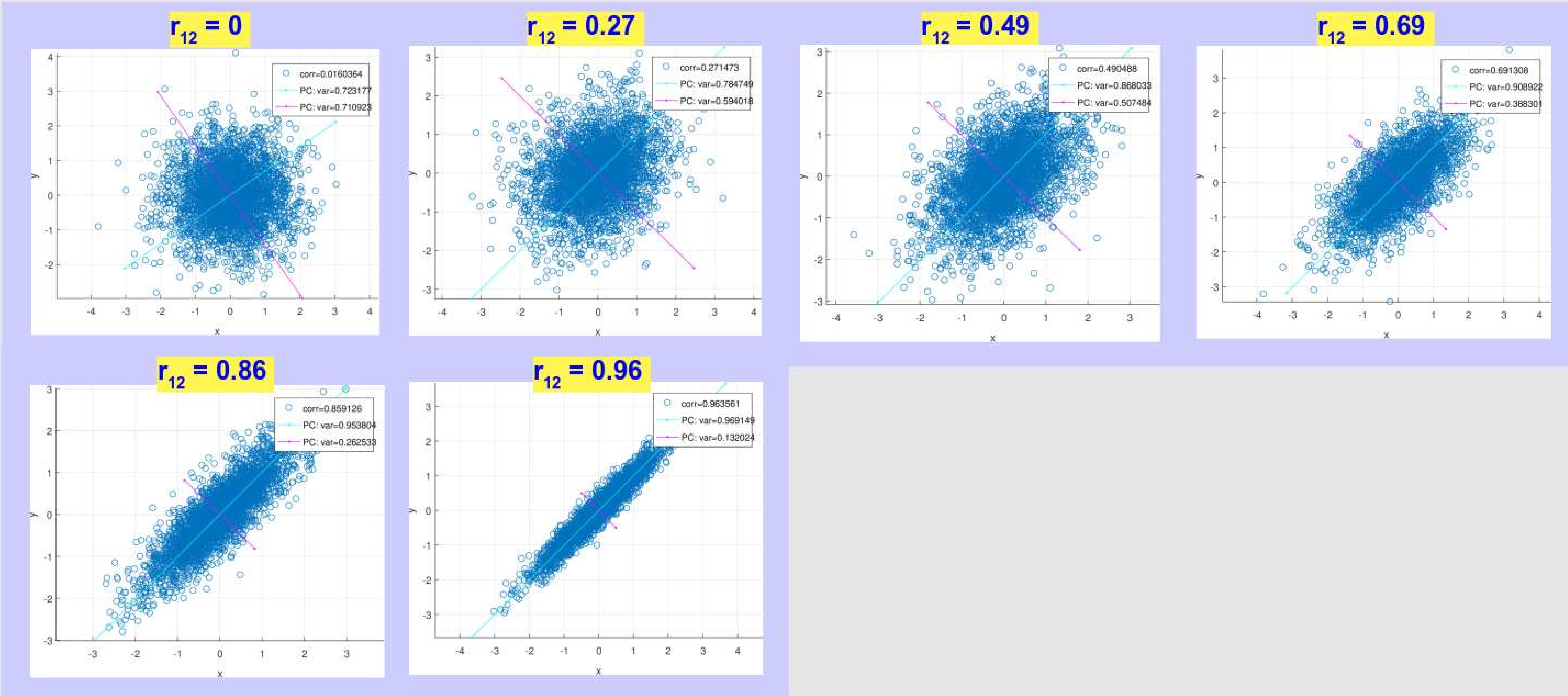
these are the PCs

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$$

will be real and ≥0

- $\sqrt{\lambda_i}$ = the **weights**
  - (i.e., the variances)
  - with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_m \geq 0$
- eigenvectors $\vec{p}_i$ = the **principal components**

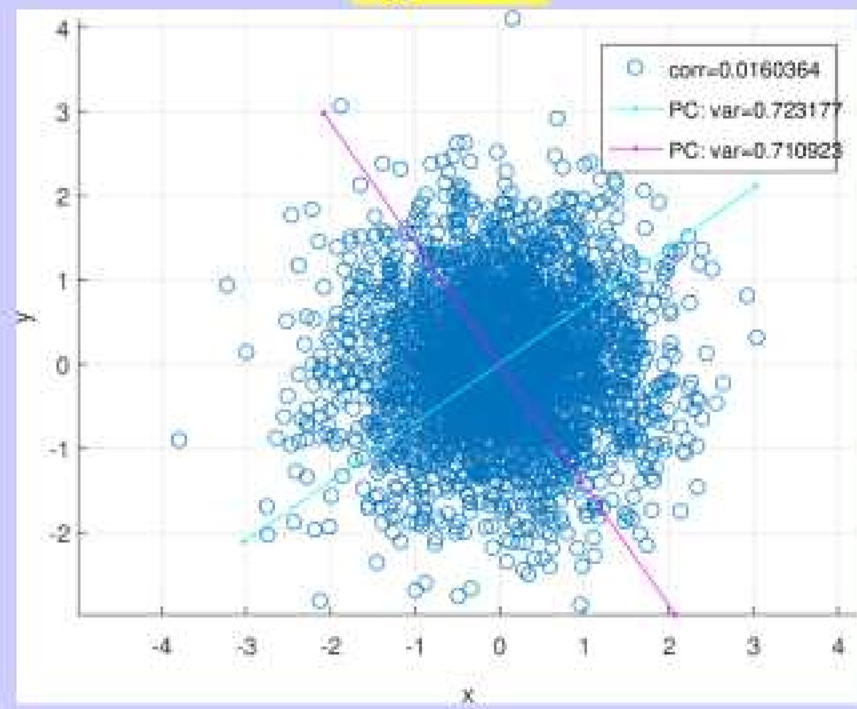
# Principal Components of the Data



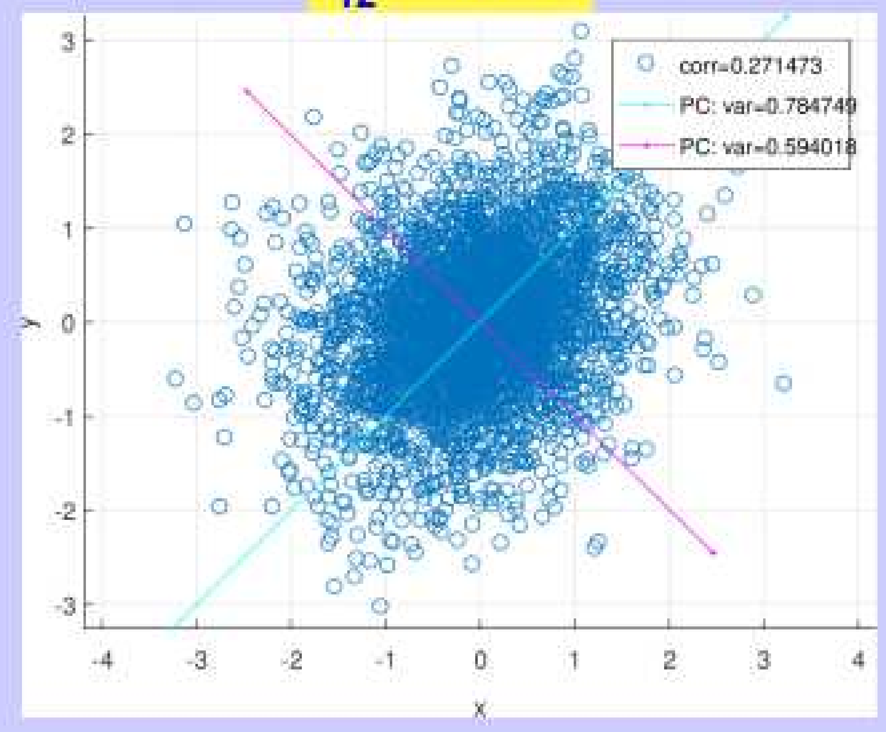

**5000 x 2 matrices**
**(each point is a row)**

# Principal Components of the Data

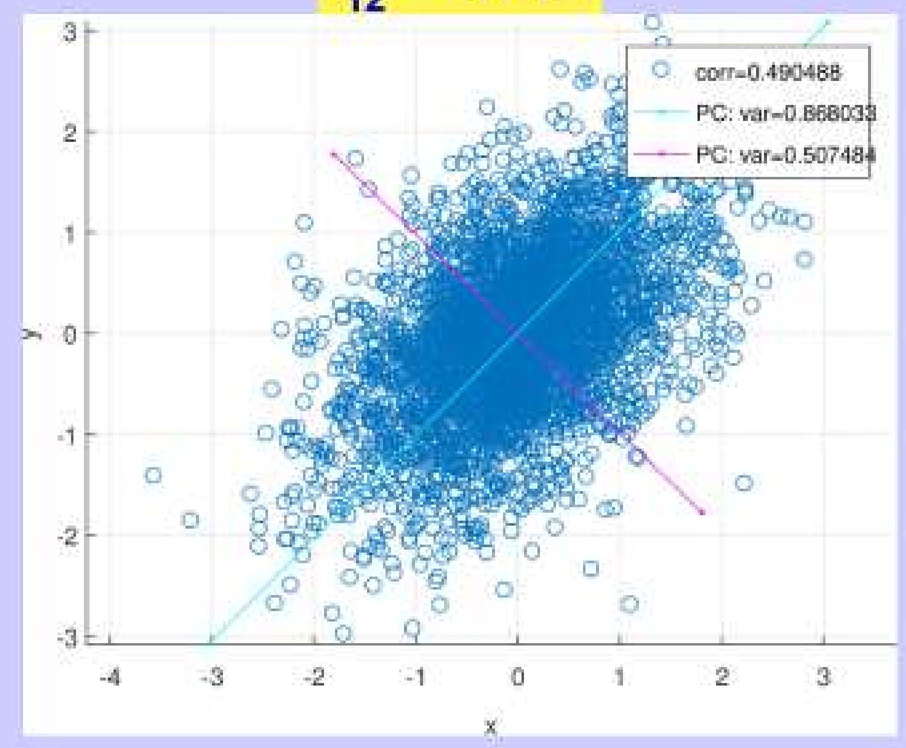
$r_{12} = 0$



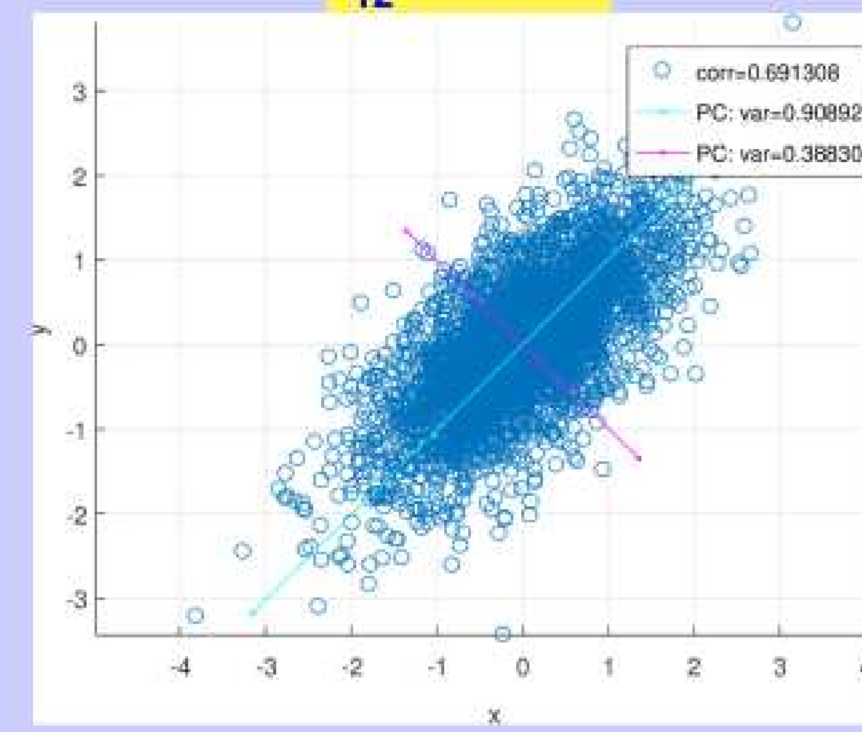

$r_{12} = 0.27$



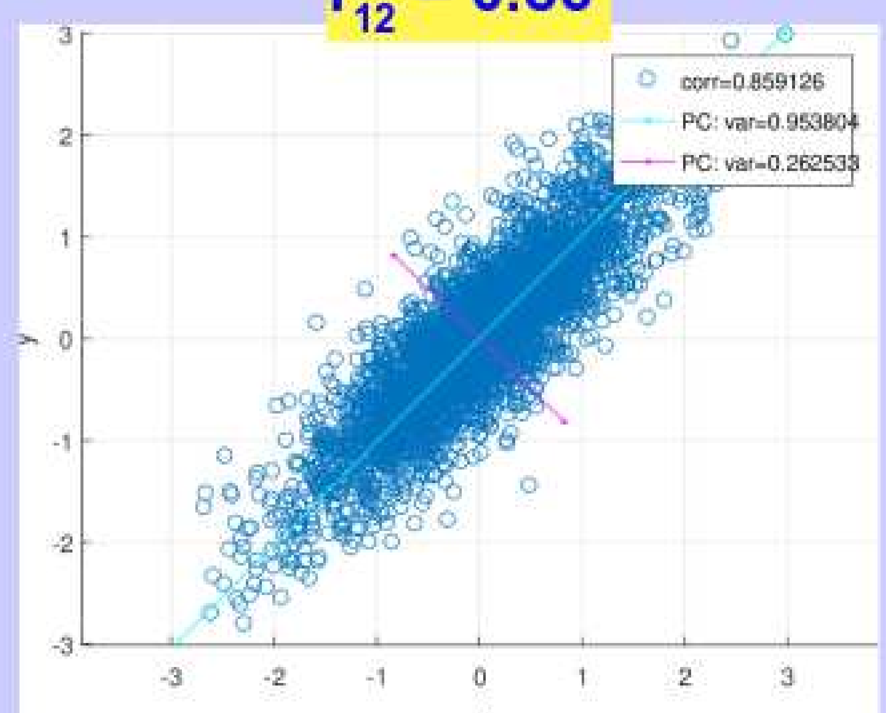

$r_{12} = 0.49$



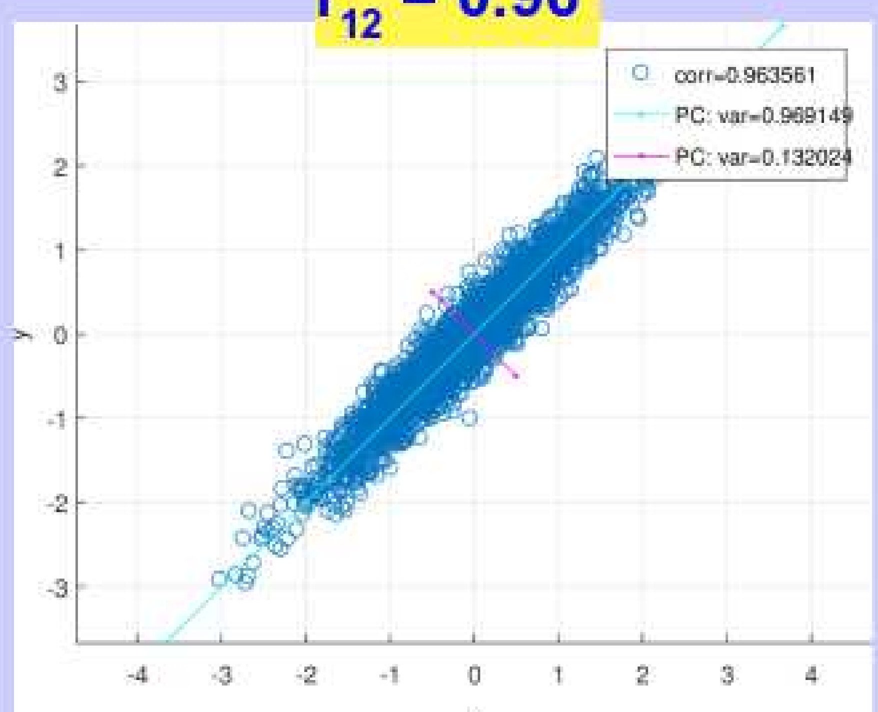

$r_{12} = 0.69$



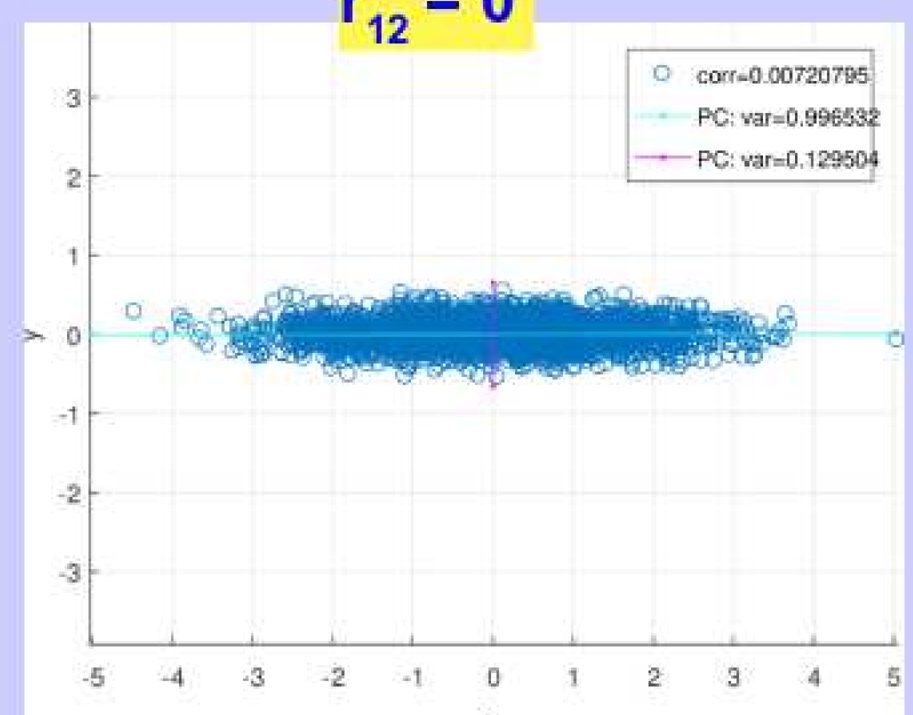

$r_{12} = 0.86$



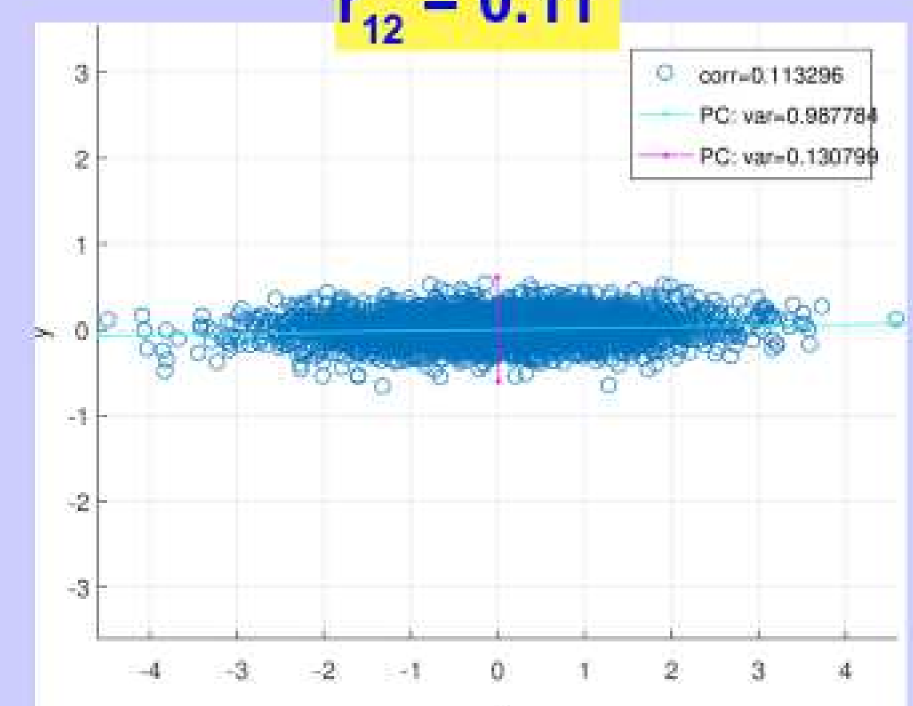

$r_{12} = 0.96$




$r_{12} = 0$

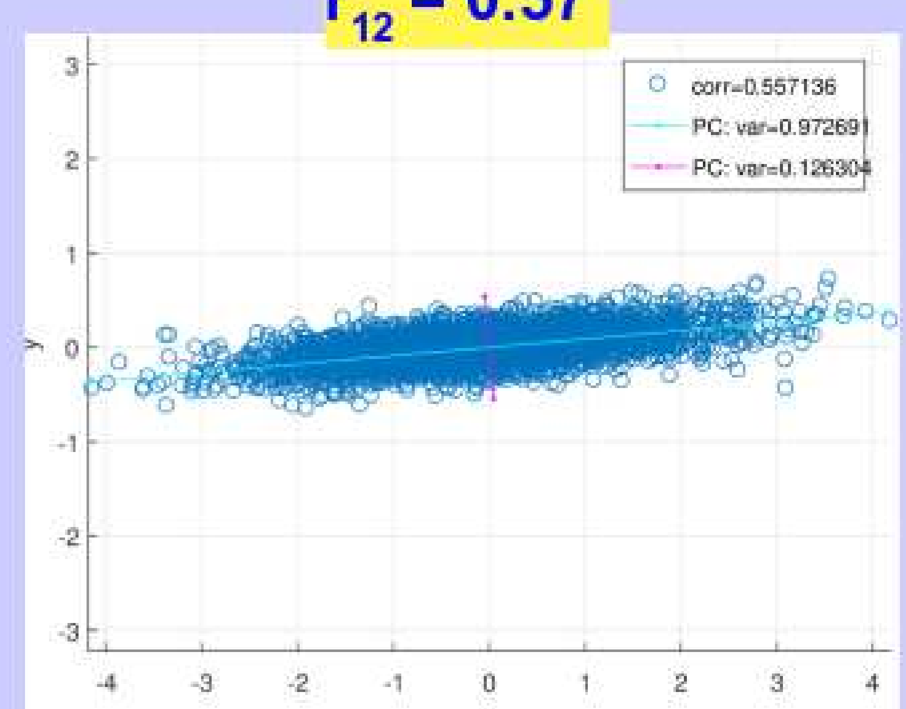



$r_{12} = 0.11$

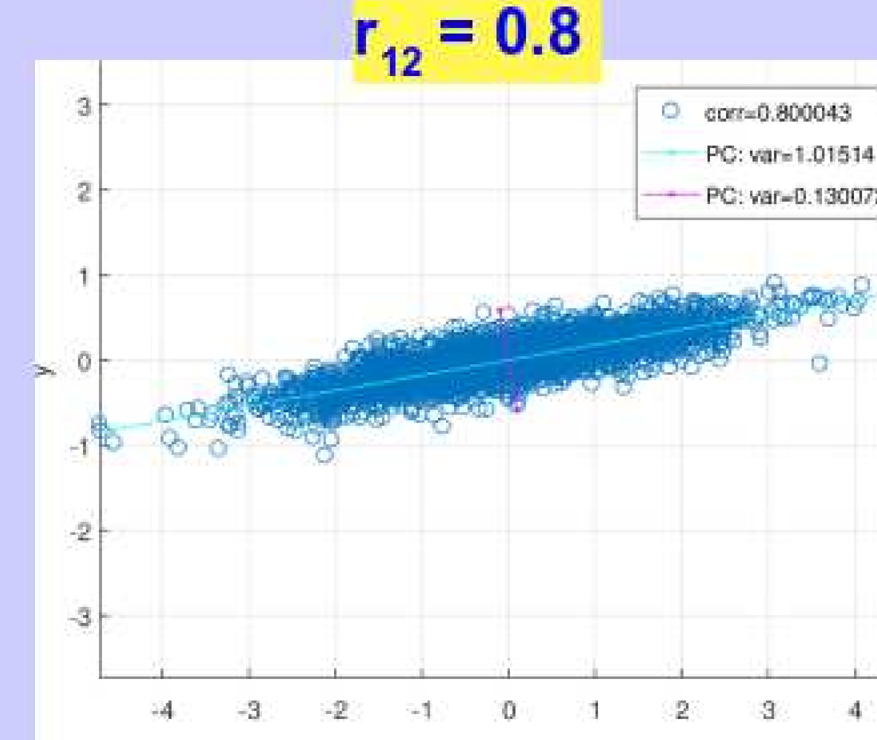



**5000 x 2 matrices**
**(each point is a row)**

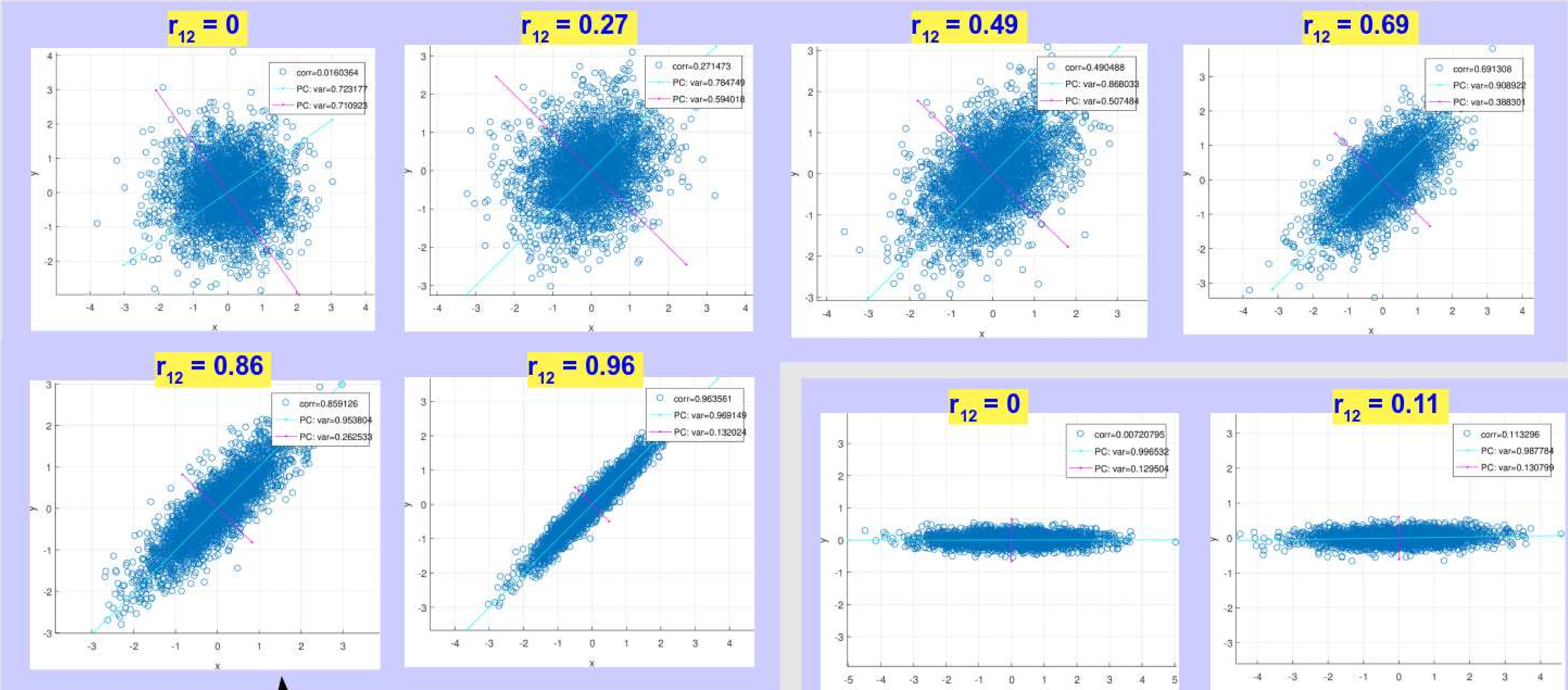
$r_{12} = 0.57$



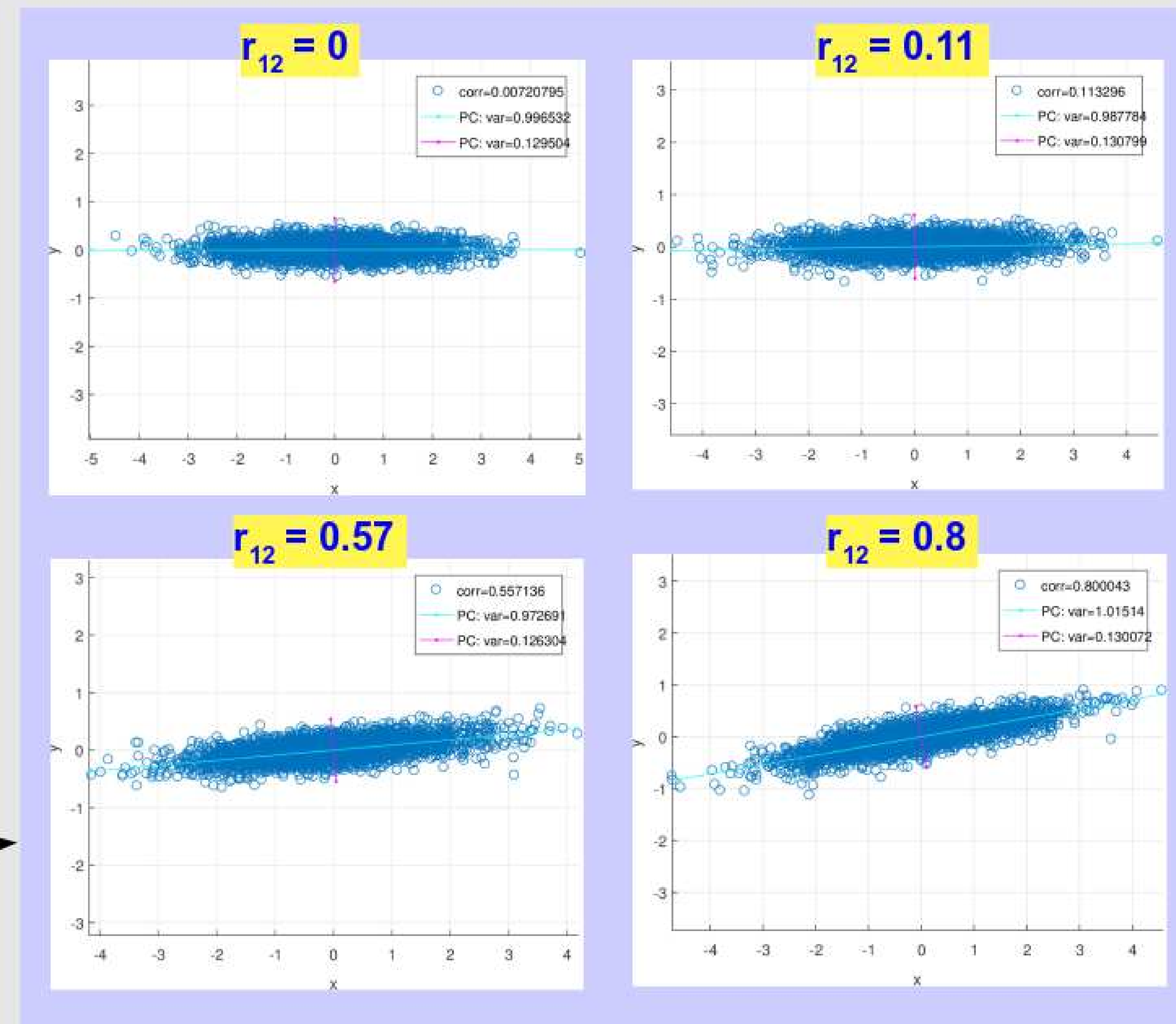

$r_{12} = 0.8$



# Principal Components of the Data




**5000 x 2 matrices**
(each point is a row)




**First PC always captures direction of maximum data spread**
**(2$^{nd}$ PC: max spread in orthogonal direction)**

# PCA: Why it Works: The Flow

- First: establish some properties of P and $\Lambda$

# PCA: Why it Works: The Flow

- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors

# PCA: Why it Works: The Flow

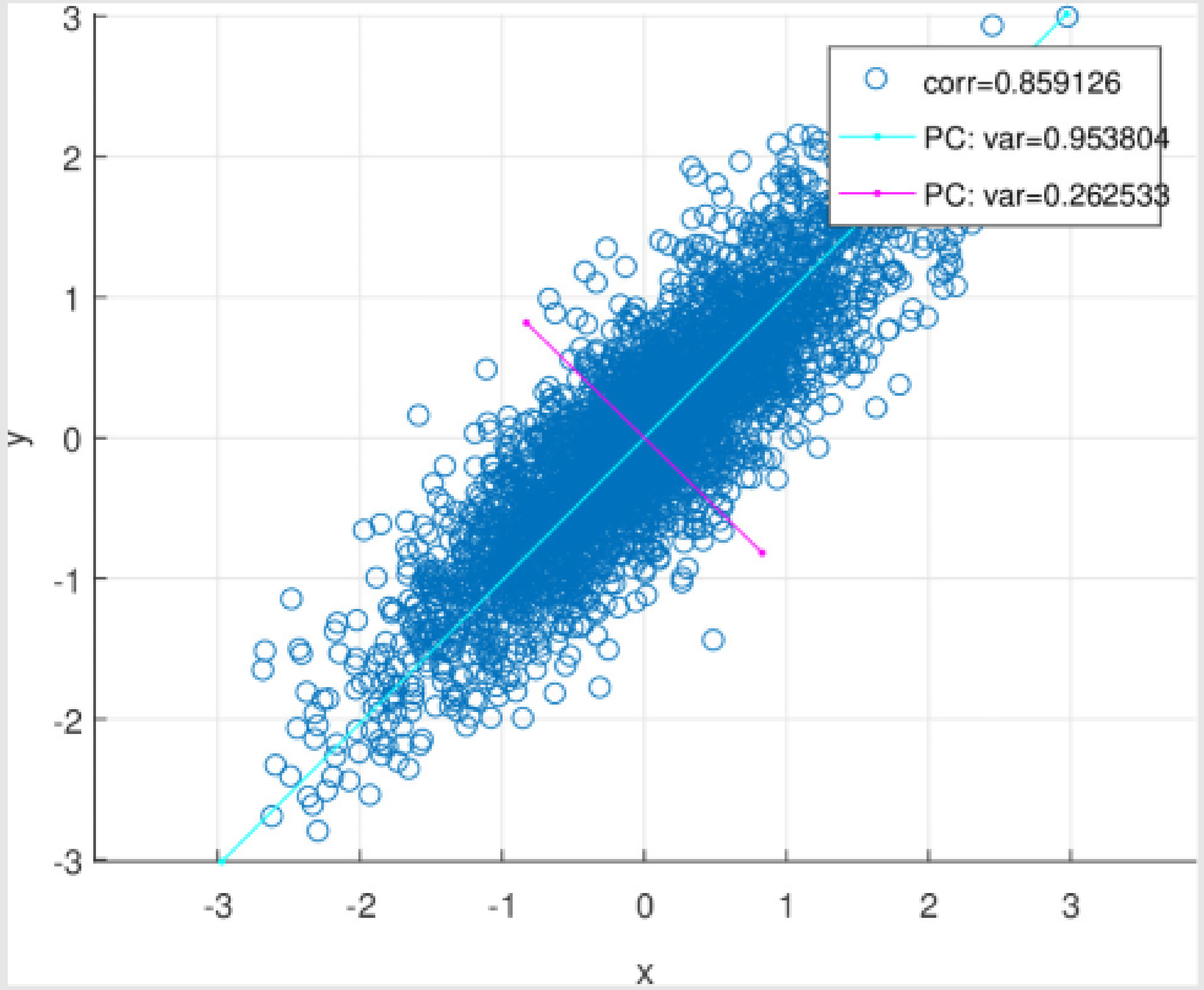
- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - eigenvalues $\geq 0$

# PCA: Why it Works: The Flow

- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - eigenvalues $\geq 0$
- Express data in eigenvector basis
  - project each data point onto eigenvectors

# PCA: Why it Works: The Flow

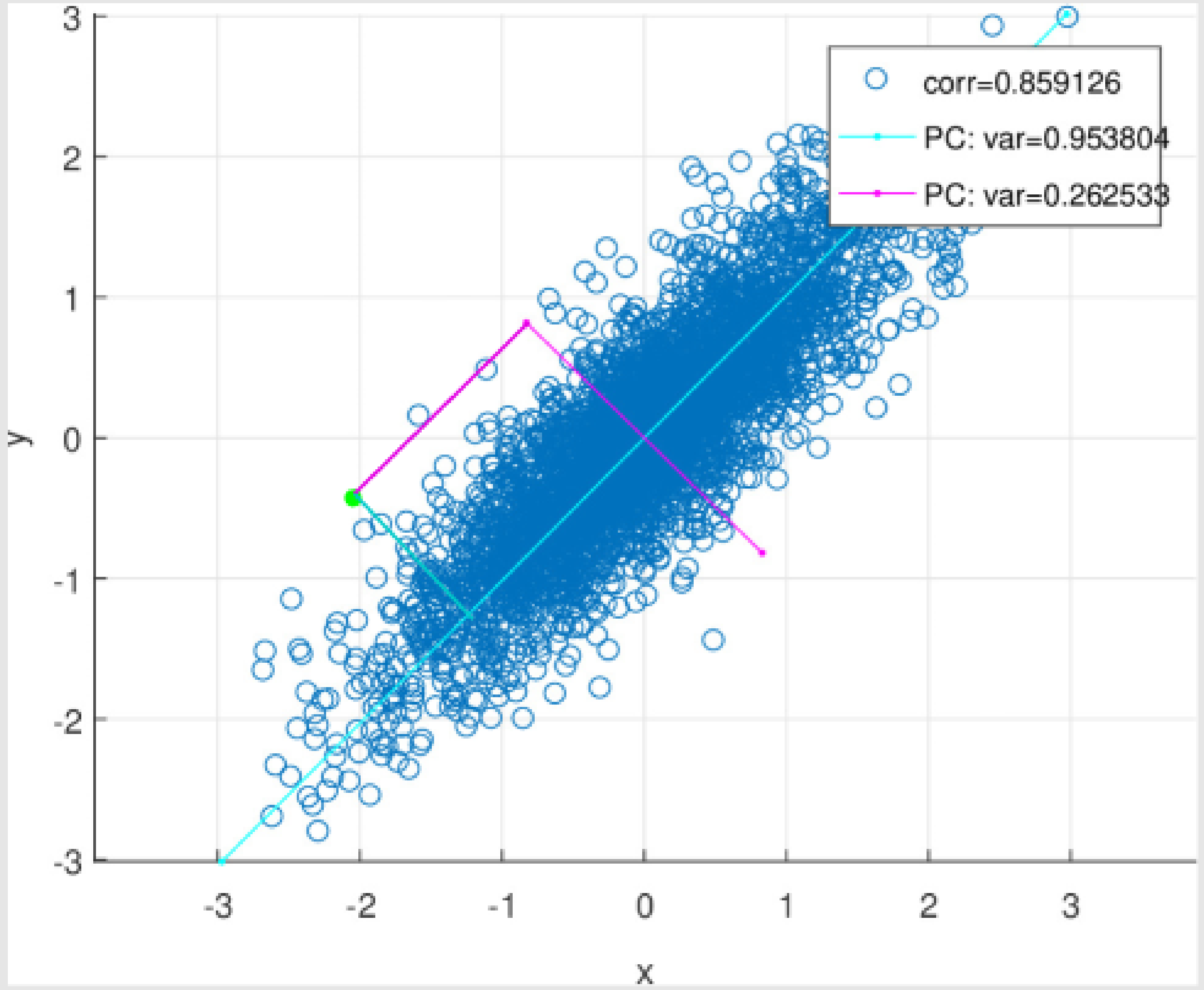
- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - eigenvalues $\geq 0$




- Express data in eigenvector basis
  - project each data point onto eigenvectors

# PCA: Why it Works: The Flow

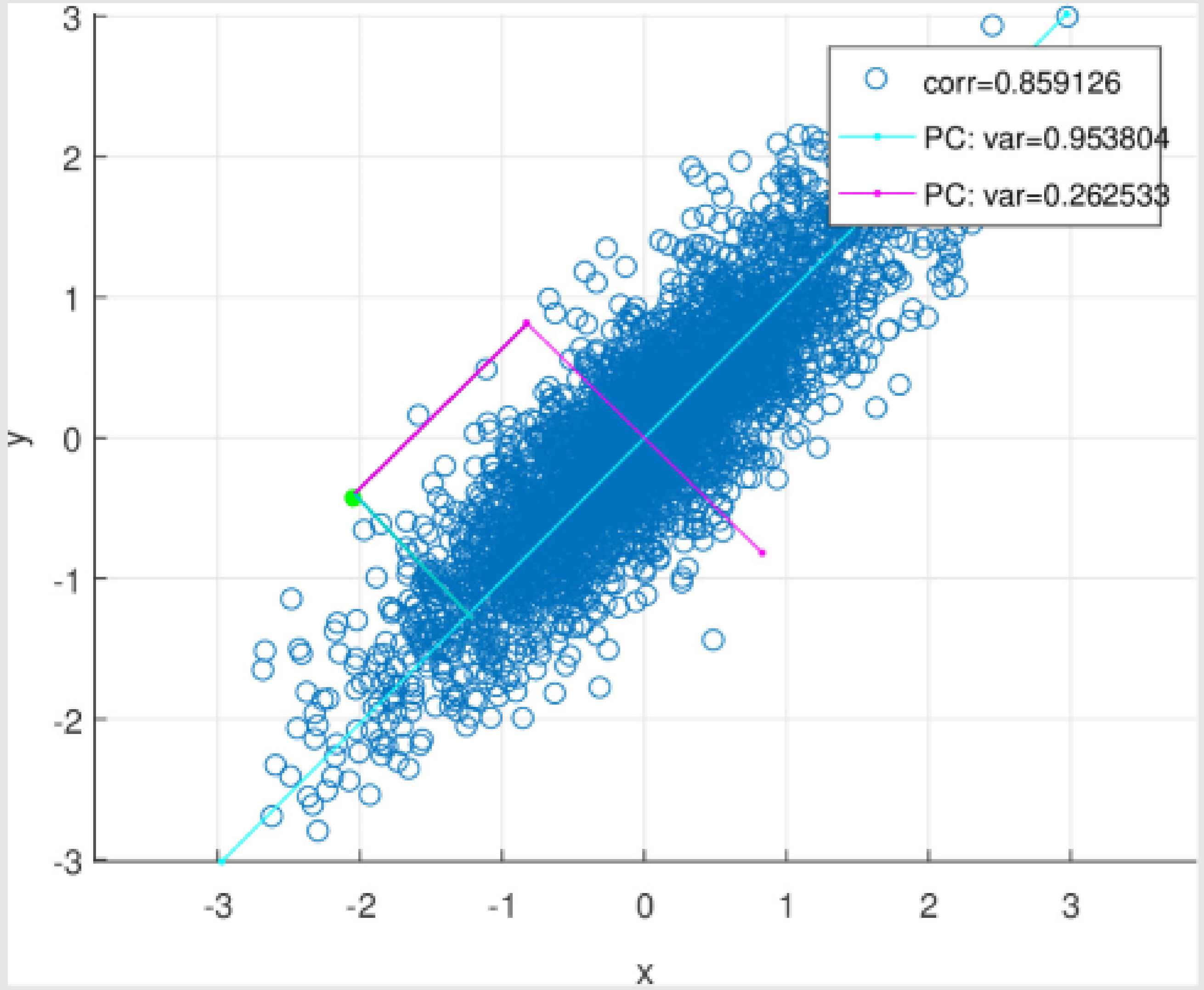
- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - eigenvalues ≥ 0




- Express data in eigenvector basis
  - project each data point onto eigenvectors

# PCA: Why it Works: The Flow

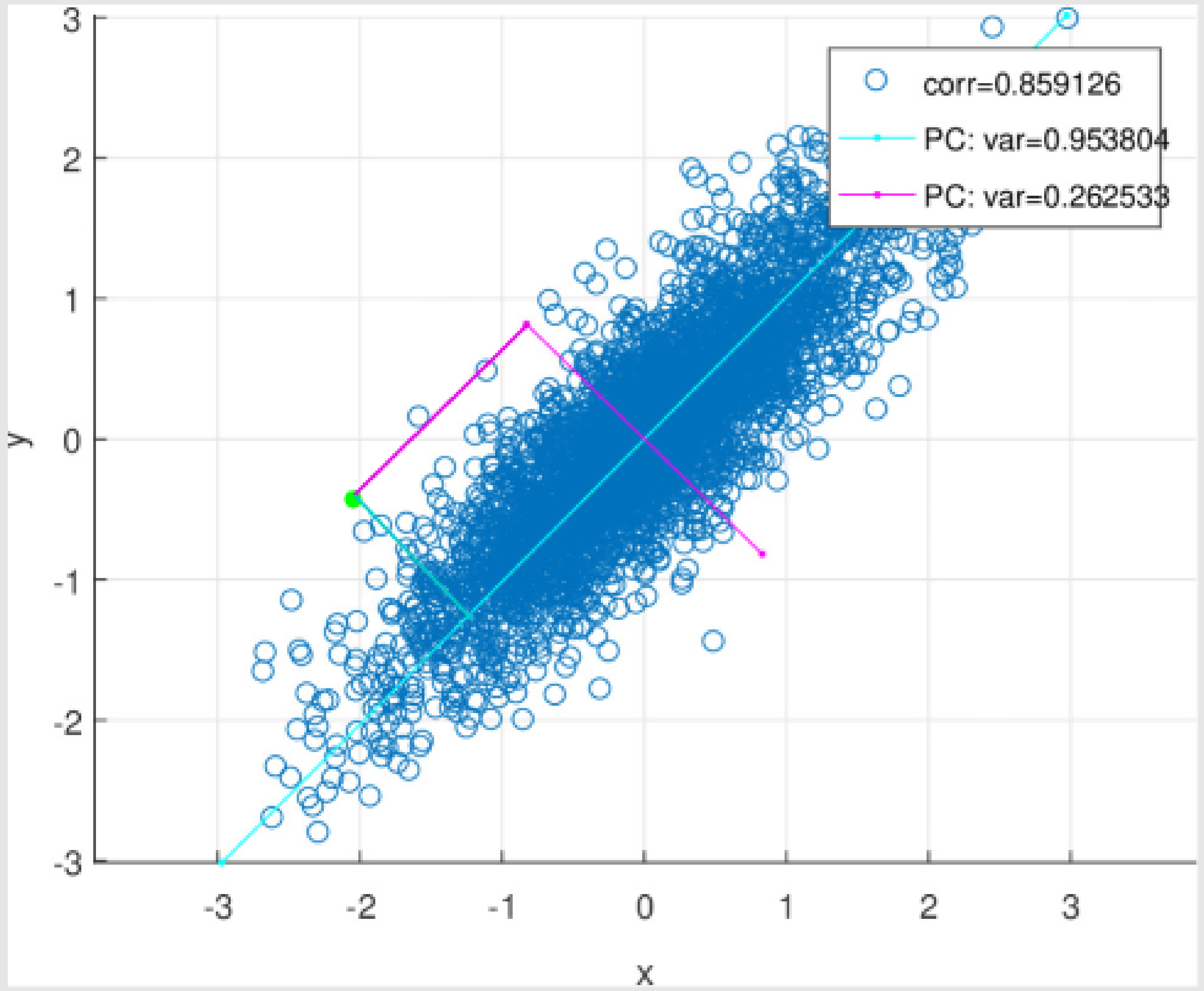
- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - real eigenvalues
    - real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - eigenvalues ≥ 0




- Express data in eigenvector basis
  - project each data point onto eigenvectors
- Show that the covariance matrix of the projected data is diagonal
  - the variances of the projections along each axis/PC

# PCA: Why it Works: The Flow

- First: establish some properties of P and $\Lambda$
  - properties of real symmetric matrices
    - ➔ real eigenvalues
    - ➔ real set of orthonormal eigenvectors
  - properties of real $A^T A$
    - ➔ eigenvalues ≥ 0




- Express data in eigenvector basis
  - project each data point onto eigenvectors
- Show that the covariance matrix of the projected data is diagonal
  - the variances of the projections along each axis/PC
- First PC maximizes variance along any 1D projection
  - $2^{nd}$ PC maximizes remaining variance; and so on

# Properties of Covariance Matrices

- If S is a real m x m symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real

# Properties of Covariance Matrices

- If S is a real mxm symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real
    - $S\vec{p} = \lambda\vec{p}$. S symmetric $\rightarrow$ $\vec{p}^T S = \lambda\vec{p}^T$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \lambda\vec{p}^T\overline{\vec{p}} = \lambda||\vec{p}||^2$
    - S real $\rightarrow$ $S\overline{\vec{p}} = \overline{\lambda}\,\overline{\vec{p}}$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \overline{\lambda}\vec{p}^T\overline{\vec{p}} = \overline{\lambda}||\vec{p}||^2$.
    - hence $\lambda||\vec{p}||^2 = \overline{\lambda}||\vec{p}||^2$ $\rightarrow$ $\lambda = \overline{\lambda}$ $\rightarrow$ λ is real.

# Properties of Covariance Matrices

- If S is a real mxm symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real
    - $S\vec{p} = \lambda\vec{p}$. S symmetric $\rightarrow$ $\vec{p}^T S = \lambda\vec{p}^T$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \lambda\vec{p}^T\overline{\vec{p}} = \lambda\|\vec{p}\|^2$
    - S real $\rightarrow$ $S\overline{\vec{p}} = \overline{\lambda}\,\overline{\vec{p}}$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \overline{\lambda}\vec{p}^T\overline{\vec{p}} = \overline{\lambda}\|\vec{p}\|^2$.
    - hence $\lambda\|\vec{p}\|^2 = \overline{\lambda}\|\vec{p}\|^2$ $\rightarrow$ $\lambda = \overline{\lambda}$ $\rightarrow$ $\lambda$ is real.
  - 2. A set of real eigenvectors can be found (see the notes)

# Properties of Covariance Matrices

- If S is a real mxm symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real
    - $S\vec{p} = \lambda\vec{p}$. S symmetric $\rightarrow$ $\vec{p}^T S = \lambda\vec{p}^T$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \lambda\vec{p}^T\overline{\vec{p}} = \lambda\|\vec{p}\|^2$
    - S real $\rightarrow$ $S\overline{\vec{p}} = \overline{\lambda}\,\overline{\vec{p}}$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \overline{\lambda}\vec{p}^T\overline{\vec{p}} = \overline{\lambda}\|\vec{p}\|^2$.
    - hence $\lambda\|\vec{p}\|^2 = \overline{\lambda}\|\vec{p}\|^2$ $\rightarrow$ $\lambda = \overline{\lambda}$ $\rightarrow$ λ is real.
  - 2. A set of real eigenvectors can be found (see the notes)
  - 3. The eigenvectors form an orthonormal set (basis).
    - (see the notes)

# Properties of Covariance Matrices

- If S is a real mxm symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real
    - $S\vec{p} = \lambda\vec{p}$. S symmetric $\rightarrow$ $\vec{p}^T S = \lambda\vec{p}^T$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \lambda\vec{p}^T\overline{\vec{p}} = \lambda\|\vec{p}\|^2$
    - S real $\rightarrow$ $S\overline{\vec{p}} = \overline{\lambda}\,\overline{\vec{p}}$. $\rightarrow$ $\vec{p}^T S\overline{\vec{p}} = \overline{\lambda}\vec{p}^T\overline{\vec{p}} = \overline{\lambda}\|\vec{p}\|^2$.
    - hence $\lambda\|\vec{p}\|^2 = \overline{\lambda}\|\vec{p}\|^2$ $\rightarrow$ $\lambda = \overline{\lambda}$ $\rightarrow$ λ is real.
  - 2. A set of real eigenvectors can be found (see the notes)
  - 3. The eigenvectors form an orthonormal set (basis).
    - (see the notes)
- If S is in the form $A^T A$ (A real)
  - 4. its eigenvalues are all ≥ 0.

# Properties of Covariance Matrices

- If S is a real mxm symmetric matrix ($s_{ij} = s_{ji}$)
  - 1. its eigenvalues are all real
    - $S\vec{p} = \lambda\vec{p}$. S symmetric $\longrightarrow$ $\vec{p}^T S = \lambda\vec{p}^T$. $\longrightarrow$ $\vec{p}^T S\overline{\vec{p}} = \lambda\vec{p}^T\overline{\vec{p}} = \lambda||\vec{p}||^2$
    - S real $\longrightarrow$ $S\overline{\vec{p}} = \overline{\lambda}\,\overline{\vec{p}}$. $\longrightarrow$ $\vec{p}^T S\overline{\vec{p}} = \overline{\lambda}\vec{p}^T\overline{\vec{p}} = \overline{\lambda}||\vec{p}||^2$.
    - hence $\lambda||\vec{p}||^2 = \overline{\lambda}||\vec{p}||^2$ $\longrightarrow$ $\lambda = \overline{\lambda}$ $\longrightarrow$ $\lambda$ is real.
  - 2. A set of real eigenvectors can be found (see the notes)
  - 3. The eigenvectors form an orthonormal set (basis).
    - (see the notes)
- If S is in the form $A^TA$ (A real)
  - 4. its eigenvalues are all $\geq 0$.
    - $A^T A\vec{p} = \lambda\vec{p}$ $\longrightarrow$ $\vec{p}^T A^T A\vec{p} = \lambda\vec{p}^T\vec{p}$ $\longrightarrow$ $(A\vec{p})^T A\vec{p} = \lambda\vec{p}^T\vec{p}$
    - $\longrightarrow$ $||A\vec{p}||^2 = \lambda||\vec{p}||^2$ $\longrightarrow$ $\lambda = \dfrac{||A\vec{p}||^2}{||\vec{p}||^2} \geq 0$.

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\rightarrow PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\rightarrow$ $PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\rightarrow$ $PP^T = I$ $\rightarrow$ $P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\longrightarrow PP^T = I \longrightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\longrightarrow$ $PP^T = I$ $\longrightarrow$ $P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\longrightarrow$ $PP^T = I$ $\longrightarrow$ $P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

**= $\Lambda$ (diagonal)**

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\rightarrow$ $PP^T = I$ $\rightarrow$ $P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

**$= \Lambda$ (diagonal)**

**Data projected on PC basis becomes UNCORRELATED**

# PCA Basis Diagonalises the Data

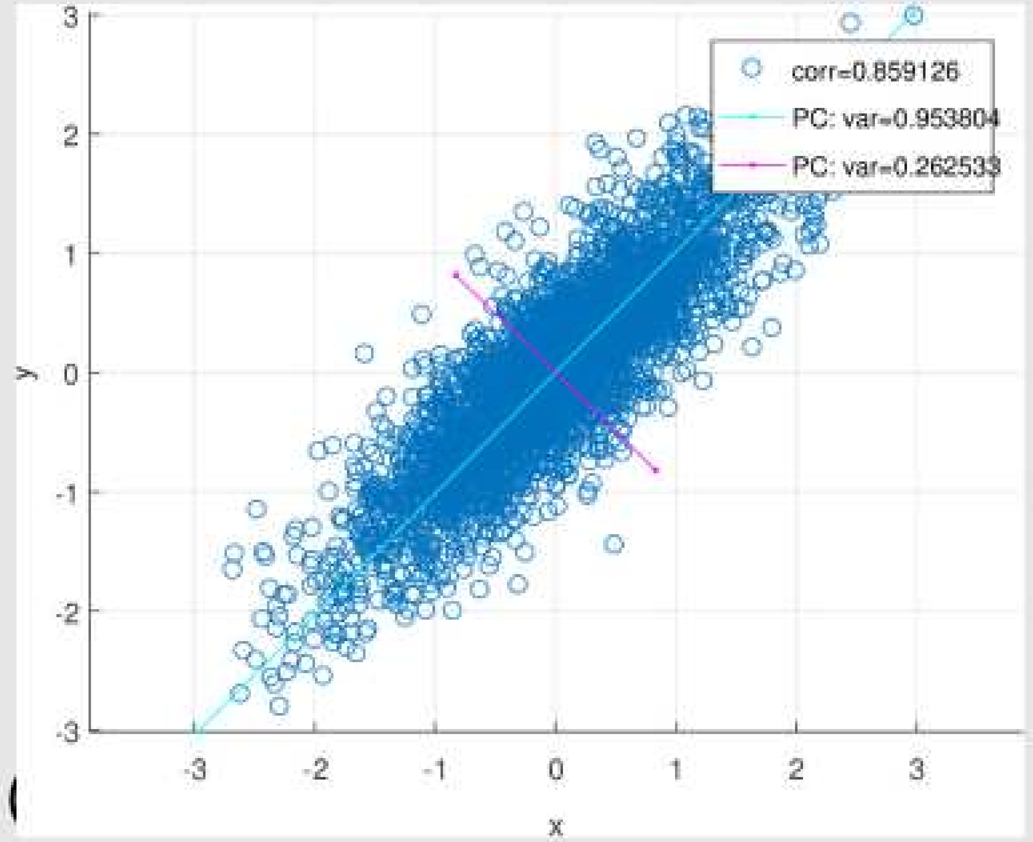
- eigenvectors orthonormal $\rightarrow$ $PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

    **= $\Lambda$ (diagonal)** ← **Data projected on PC basis becomes UNCORRELATED**
  - the diagonal entries are the variances of the data projected along $\vec{p}_i$ (recall: from defn. of covariance matrix)

# PCA Basis Diagonalises the Data

- eigenvectors orthonormal → $PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

    **= $\Lambda$ (diagonal)** ← **Data projected on PC basis becomes UNCORRELATED**
  - the diagonal entries are the variances of the data projected along $\vec{p}_i$ (recall: from defn. of covariance matrix)

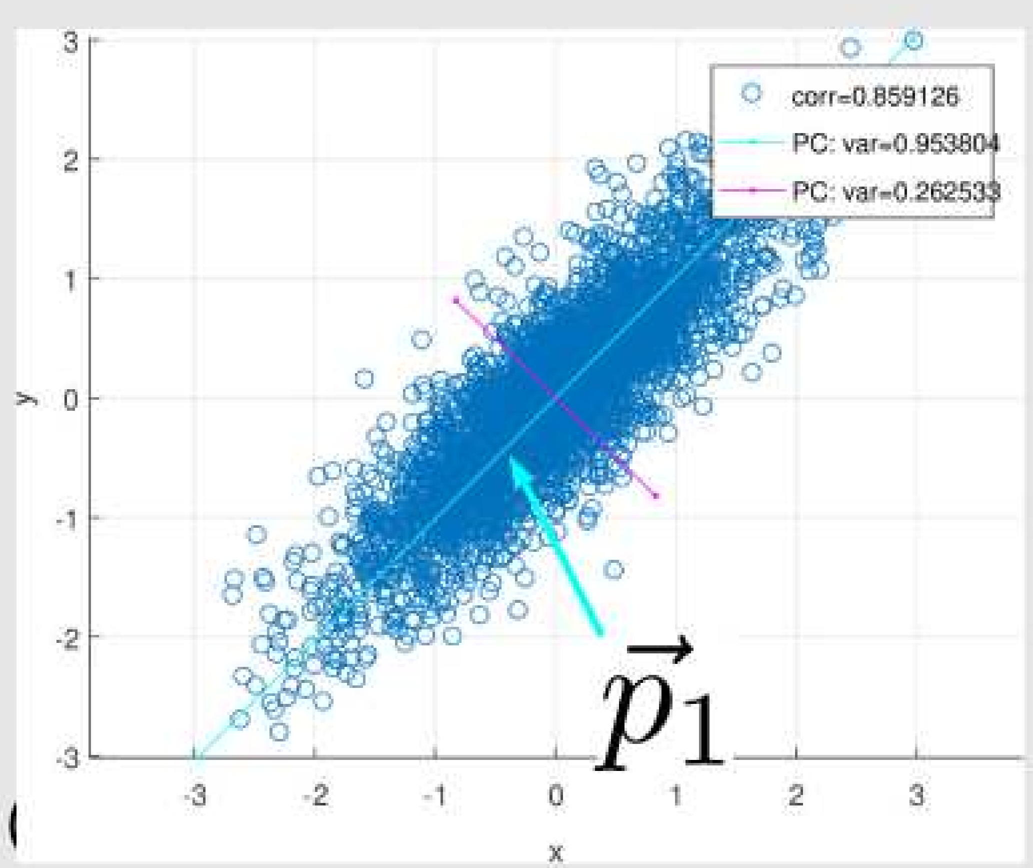


# PCA Basis Diagonalises the Data

- eigenvectors orthonormal → $PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

**= $\Lambda$ (diagonal)** ← **Data projected on PC basis becomes UNCORRELATED**

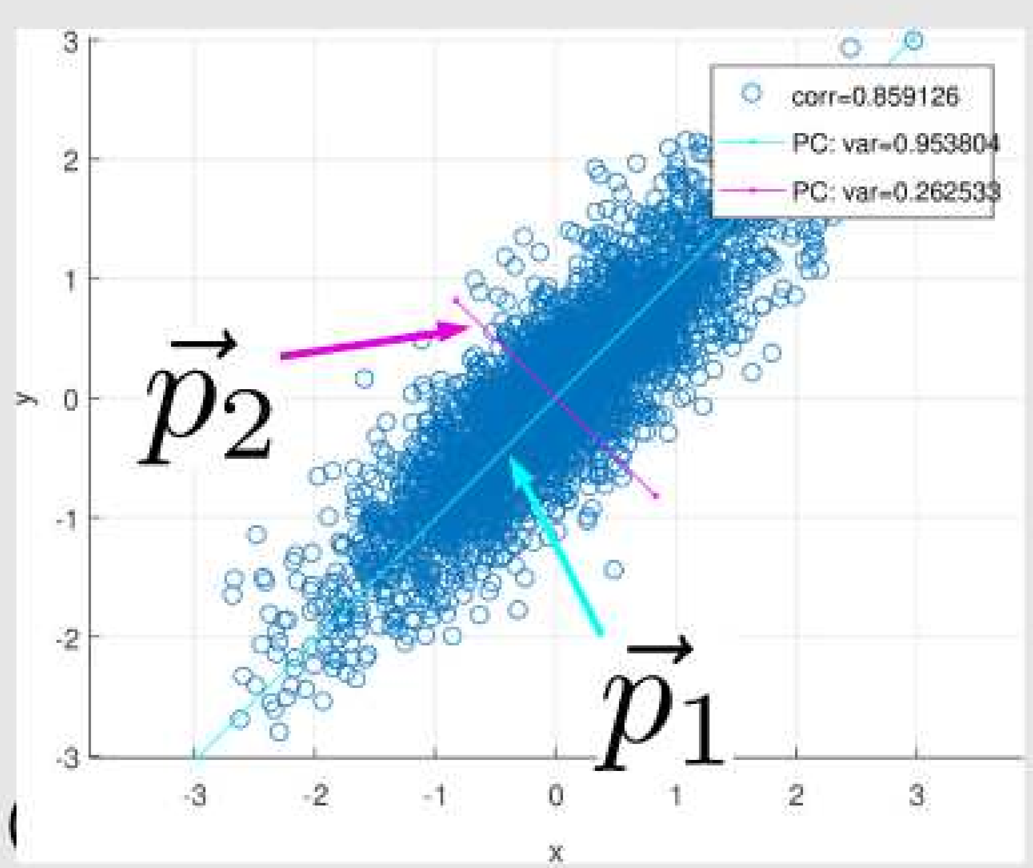
  - the diagonal entries are the variances of the data projected along $\vec{p}_i$ (recall: from defn. of covariance matrix)



# PCA Basis Diagonalises the Data

- eigenvectors orthonormal $\rightarrow$ $PP^T = I \rightarrow P^T = P^{-1}$
- eigendecomposition of S: $S = P\Lambda P^T$
- project rows of (zero-mean) A in basis P: $F = \tilde{A}P$
  - columns of F are the projections along $\vec{p}_i$
- Let G be the co-variance matrix of F: $G \triangleq (F^T F)/n$
  - $nG = F^T F = P^T \tilde{A}^T \tilde{A} P = nP^T S P = nP^T P \Lambda P^T P = n\Lambda$

    **= $\Lambda$ (diagonal)** ← **Data projected on PC basis becomes UNCORRELATED**
  - the diagonal entries are the variances of the data projected along $\vec{p}_i$ (recall: from defn. of covariance matrix)



# Why do PCs Align with Visual Axes?

- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated

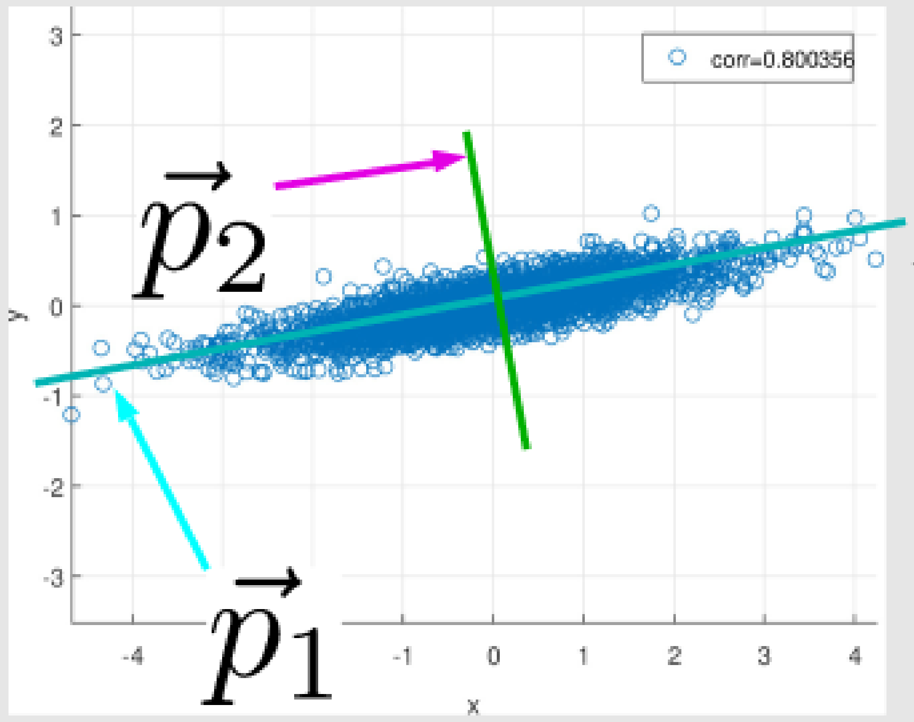
# Why do PCs Align with Visual Axes?


- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?

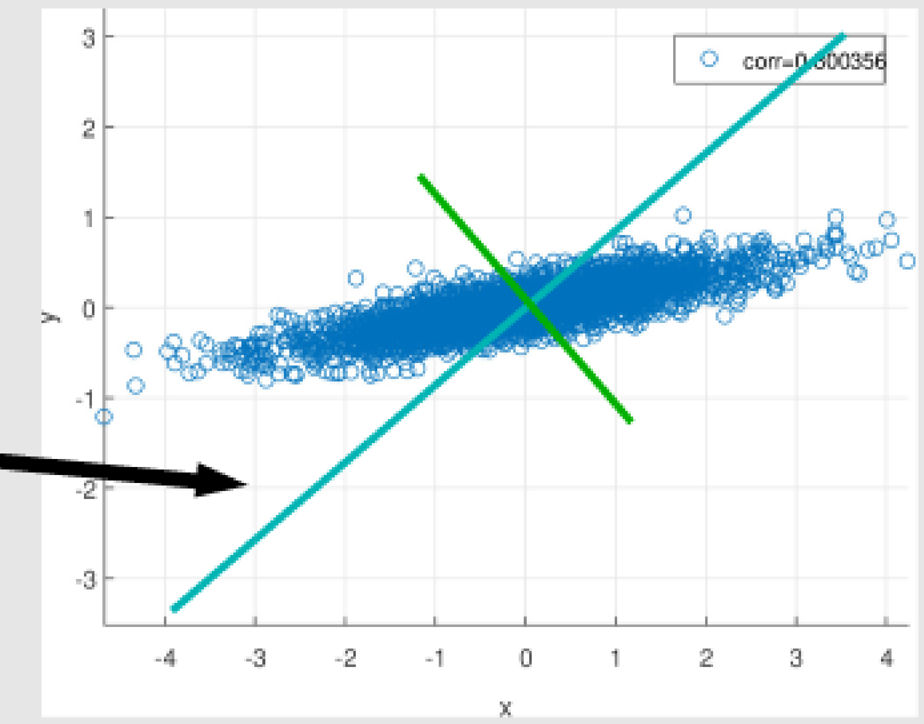
# Why do PCs Align with Visual Axes?


- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?
- Key property of PCA
  - consider **any** norm-1 vector ("direction") $\vec{p}$

# Why do PCs Align with Visual Axes?

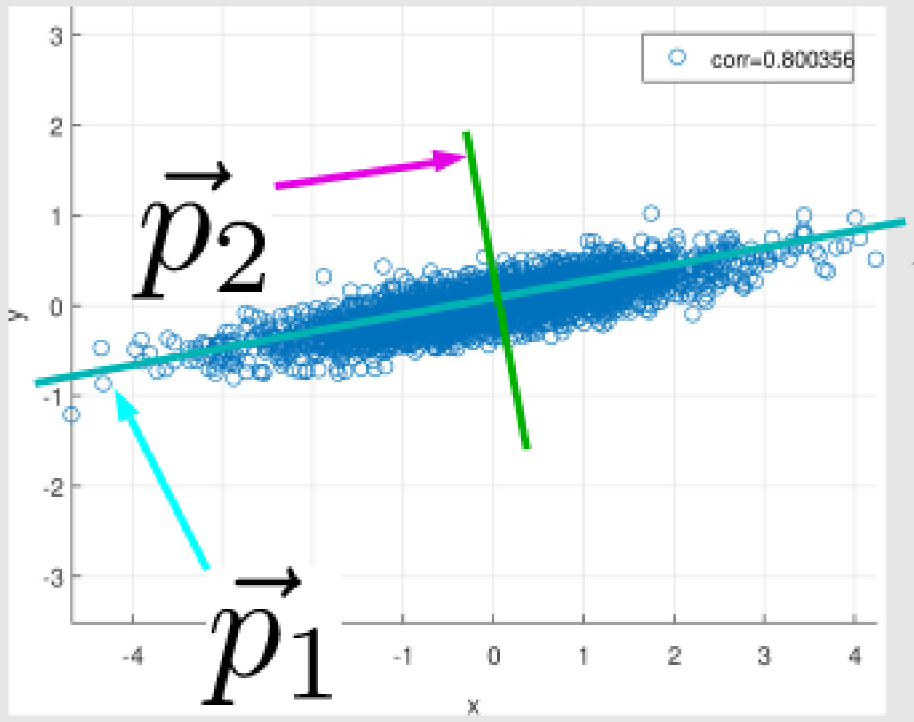



**Why this?**
**and not this?**

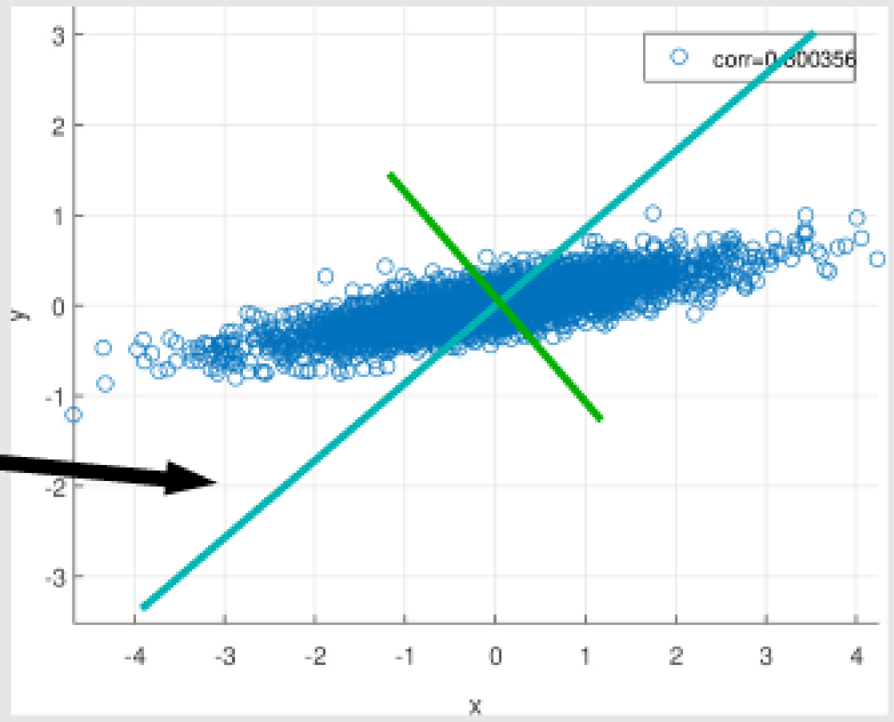



- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?
- Key property of PCA
  - consider **any** norm-1 vector ("direction") $\vec{p}$
  - project the data along it: $\tilde{A}\vec{p}$

# Why do PCs Align with Visual Axes?

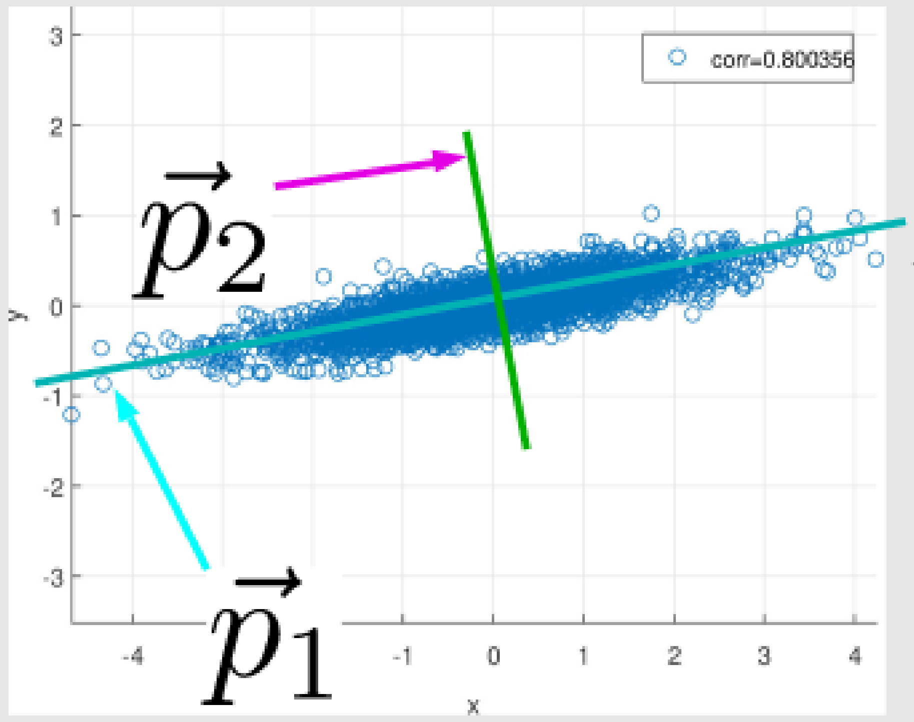



Why this?
and not this?

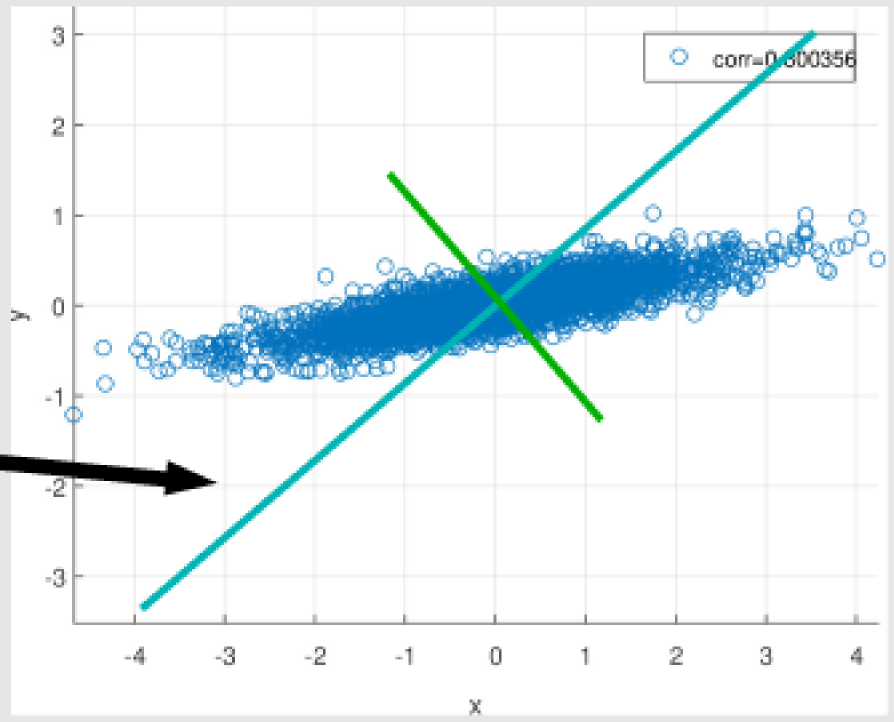



- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?
- Key property of PCA
  - consider **any** norm-1 vector (“direction”) $\vec{p}$
  - project the data along it: $\tilde{A}\vec{p}$
  - find the variance of the projected data: $\frac{1}{n}(\tilde{A}\vec{p})^T(\tilde{A}\vec{p})$

# Why do PCs Align with Visual Axes?

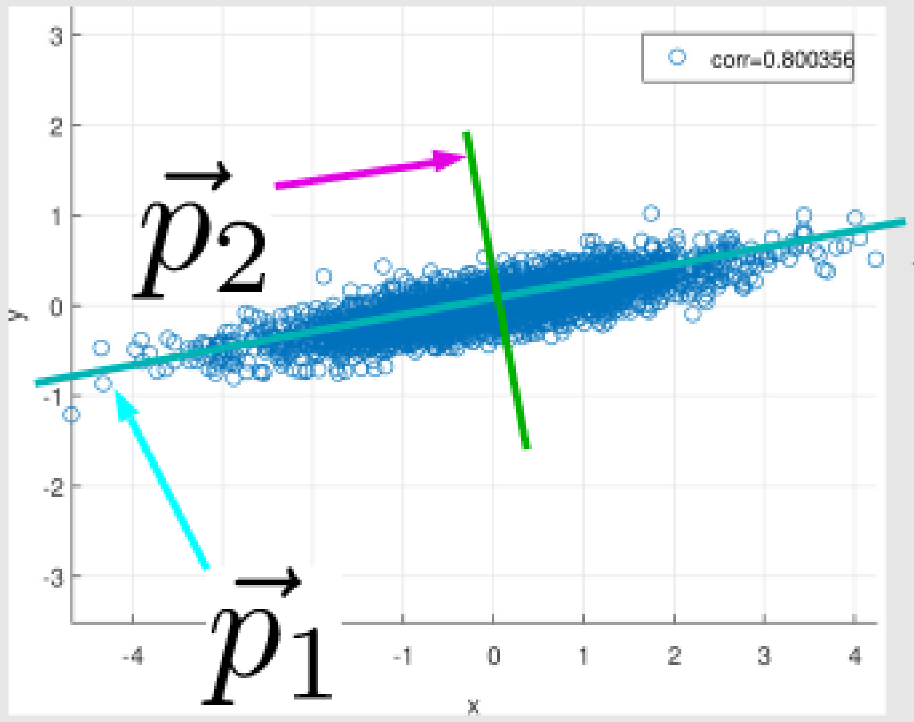



**Why this?**
**and not this?**

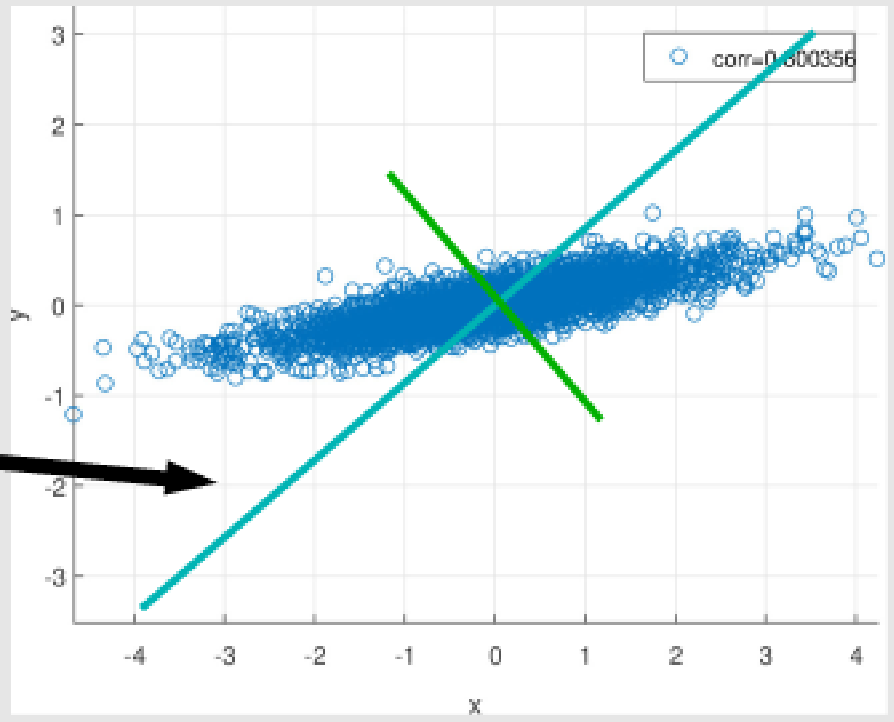



- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?
- Key property of PCA
  - consider **any** norm-1 vector (“direction”) $\vec{p}$
  - project the data along it: $\tilde{A}\vec{p}$
  - find the variance of the projected data: $\frac{1}{n}(\tilde{A}\vec{p})^T(\tilde{A}\vec{p})$
  - **the first PC $\vec{p}_1$ maximizes this variance** (the max is $\vec{\lambda}_1$)

# Why do PCs Align with Visual Axes?




**Why this?**
**and not this?**




- So far: have shown that PCs are orthonormal
  - data projected onto them becomes uncorrelated
- but why is the first **PC aligned with the direction of maximum spread**?
- Key property of PCA ← **proof → notes**
  - consider **any** norm-1 vector (“direction”) $\vec{p}$
  - project the data along it: $\tilde{A}\vec{p}$
  - find the variance of the projected data: $\frac{1}{n}(\tilde{A}\vec{p})^T(\tilde{A}\vec{p})$
  - **the first PC $\vec{p}_1$ maximizes this variance** (the max is $\vec{\lambda}_1$ )
    - → 2nd PC: maximizes variance along directions orthogonal to $\vec{p}_1$
    - $\longrightarrow$ 3rd PC: maximizes var. along dirs. orthogonal to $\vec{p}_1$ and $\vec{p}_2$ ; and so on

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U\Sigma V^T$

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U \Sigma V^T$
  - the covariance matrix is:
    - $S \triangleq \frac{1}{n} \tilde{A}^T \tilde{A} = \frac{1}{n} V \Sigma^T U^T U \Sigma V^T = V \, \frac{\Sigma^T \Sigma}{n} \, V^T$

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U\Sigma V^T$
  - the covariance matrix is:
    - $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A} = \frac{1}{n}V\Sigma^T U^T U\Sigma V^T = V\boxed{\frac{\Sigma^T\Sigma}{n}}V^T$

$$\frac{1}{n}\begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$$

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U \Sigma V^T$
  - the covariance matrix is:
    - ➔ $S \triangleq \frac{1}{n} \tilde{A}^T \tilde{A} = \frac{1}{n} V \Sigma^T U^T U \Sigma V^T = V \boxed{\frac{\Sigma^T \Sigma}{n}} V^T$

$$\frac{1}{n} \begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$$

    - ➔ recall PCA: $S = P \Lambda P^T$

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U\Sigma V^T$
  - the covariance matrix is:
    - $$S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A} = \frac{1}{n}V\Sigma^T U^T U \Sigma V^T = V \boxed{\frac{\Sigma^T\Sigma}{n}} V^T$$ $$\frac{1}{n}\begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$$
    - recall PCA: $S = P \boxed{\Lambda} P^T$ **diagonal and ≥ 0**

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U\Sigma V^T$
  - the covariance matrix is:
    - ➔ $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A} = \frac{1}{n}V\Sigma^T U^T U\Sigma V^T = \boxed{V \boxed{\frac{\Sigma^T\Sigma}{n}} V^T}$ $\qquad \frac{1}{n}\begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$
    - ➔ recall PCA: $S = \boxed{P\boxed{\Lambda}P^T}$ — **diagonal and ≥ 0**

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U \Sigma V^T$
  - the covariance matrix is:
    - $S \triangleq \frac{1}{n} \tilde{A}^T \tilde{A} = \frac{1}{n} V \Sigma^T U^T U \Sigma V^T = \boxed{V \boxed{\frac{\Sigma^T \Sigma}{n}} V^T}$ $\quad \frac{1}{n} \begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$
    - recall PCA: $S = \boxed{P \boxed{\Lambda} P^T}$

IDENTICAL FORM

diagonal and ≥ 0

# PCA: the Connection with the SVD

- Suppose you run an SVD on the data: $\tilde{A} = U\Sigma V^T$
  - the covariance matrix is:
    - $S \triangleq \frac{1}{n}\tilde{A}^T\tilde{A} = \frac{1}{n}V\Sigma^T U^T U \Sigma V^T = \boxed{V \boxed{\frac{\Sigma^T\Sigma}{n}} V^T}$ $\quad \frac{1}{n}\begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & \sigma_3^2 & & \\ & & & \ddots & \\ & & & & \sigma_m^2 \end{bmatrix}$
    - recall PCA: $S = \boxed{P \boxed{\Lambda} P^T}$ IDENTICAL FORM

      diagonal and ≥ 0
  - i.e., can use the SVD of Ã for PCA:
    - **just set** $\lambda_i \triangleq \frac{\sigma_i^2}{n}$ **and** $P \triangleq V$ **(no need to even form S)!**

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \, \frac{\Sigma^T \Sigma}{n} \, V^T$ ; PCA: $S = P \, \Lambda \, P^T$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \dfrac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \, \Lambda \, P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \dfrac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \dfrac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \, \Lambda \, P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$
  - what about U?

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \dfrac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P\,\Lambda\,P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$ , $V \triangleq P$
  - what about U?
    - just eigendecompose $\hat{S} \triangleq AA^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \dfrac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \, \Lambda \, P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$ , $V \triangleq P$
  - what about U?
    - just eigendecompose $\underbrace{\hat{S}}_{n \times n} \triangleq A A^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$ , $V \triangleq P$

    more work, because (we had assumed) n ≥ m

    nxn
  - what about U?
    - just eigendecompose $\hat{S} \triangleq A A^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$, eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$

    more work, because (we had assumed) n ≥ m

    nxn
  - what about U?
    - just eigendecompose $\hat{S} \triangleq A A^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$
    - can also get V from the **same** eigendecomposition
      - $A = U \Sigma V^T \rightarrow U^T A = \Sigma V^T \rightarrow A^T U = V \Sigma^T \rightarrow$ $\vec{v}_i = \frac{A^T \vec{u}_i}{\sigma_i}$, $i = 1, \cdots, m$

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$, eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$

    more work, because (we had assumed) n ≥ m

    nxn
  - what about U?
    - just eigendecompose $\hat{S} \triangleq AA^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$
    - can also get V from the **same** eigendecomposition
      - $A = U \Sigma V^T \to U^T A = \Sigma V^T \to A^T U = V \Sigma^T \to$ $\vec{v}_i = \frac{A^T \vec{u}_i}{\sigma_i}$ $i = 1, \cdots, m$

        if $\sigma_i$ = 0, choose $v_i$ arbitrarily to complete orthonormal basis for V

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$, eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$

    more work, because (we had assumed) n ≥ m

    nxn
  - what about U?
    - just eigendecompose $\hat{S} \triangleq A A^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$
    - can also get V from the **same** eigendecomposition
      - $A = U \Sigma V^T \rightarrow U^T A = \Sigma V^T \rightarrow A^T U = V \Sigma^T \rightarrow$ $\vec{v}_i = \frac{A^T \vec{u}_i}{\sigma_i}$, $i = 1, \cdots, m$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$

    if $\sigma_i$ = 0, choose $v_i$ arbitrarily to complete orthonormal basis for V

# Computing SVDs via Eigendecomposition

- Prev. slide: SVD: $S = V \frac{\Sigma^T \Sigma}{n} V^T$ ; PCA: $S = P \Lambda P^T$
- Q: how to calculate an SVD of a matrix A?
  - using eigendecomposition
- A: just use the above insight (PCA/eigendecomposition)!
  - form $S \triangleq A^T A$ , eigendecompose $S = P \Lambda P^T$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$, $V \triangleq P$

    more work, because (we had assumed) n ≥ m

    nxn
  - what about U?
    - just eigendecompose $\hat{S} \triangleq A A^T = Q \hat{\Lambda} Q^T$ ; then $U \triangleq Q$
    - can also get V from the **same** eigendecomposition
      - $A = U \Sigma V^T \rightarrow U^T A = \Sigma V^T \rightarrow A^T U = V \Sigma^T \rightarrow$ $\vec{v}_i = \frac{A^T \vec{u}_i}{\sigma_i}$ $i = 1, \cdots, m$
  - set $\sigma_i \triangleq \sqrt{\lambda_i}$

    if $\sigma_i = 0$, choose $v_i$ arbitrarily to complete orthonormal basis for V

* why didn't we subtract means from A and normalize by n?

# Who Invented the SVD?

- SVD: "**Swiss Army Knife**" of numerical analysis

# Who Invented the SVD?

- SVD: "**Swiss Army Knife**" of numerical analysis

Eugenio Beltrami
1835-1900

# Who Invented the SVD?

- SVD: "**Swiss Army Knife**" of numerical analysis

Eugenio Beltrami
1835-1900

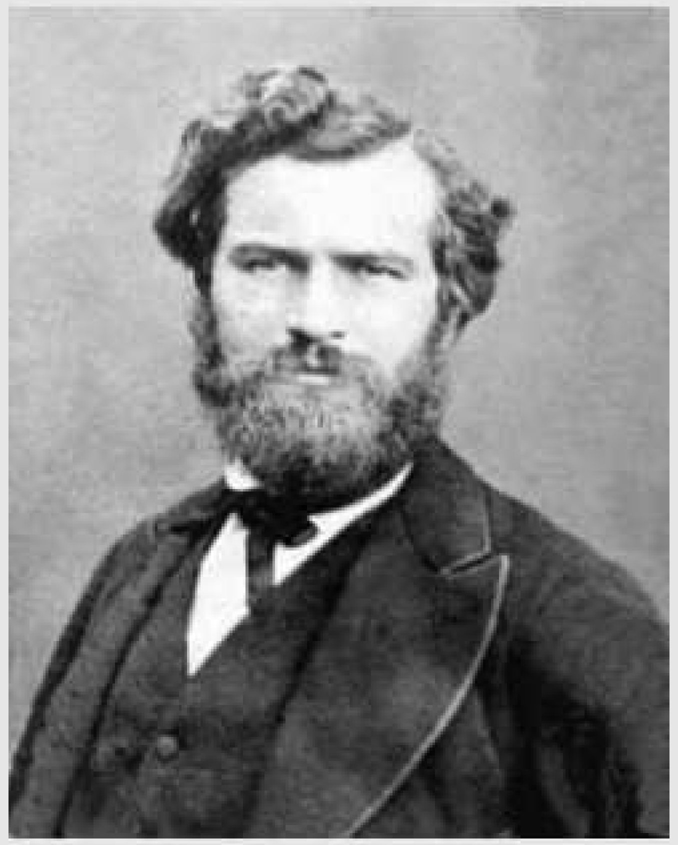
**proposed the SVD**
**via eigendecomposition of $A^T A$ or $A A^T$**

# Who Invented the SVD?

- SVD: “**Swiss Army Knife**” of numerical analysis

Eugenio Beltrami
1835-1900

**proposed the SVD**
**via eigendecomposition of $A^T A$ or $A A^T$**



Camille Jordan
1838-1922

Erhardt Schmidt
1878-1959

James Joseph Sylvester 1814-97

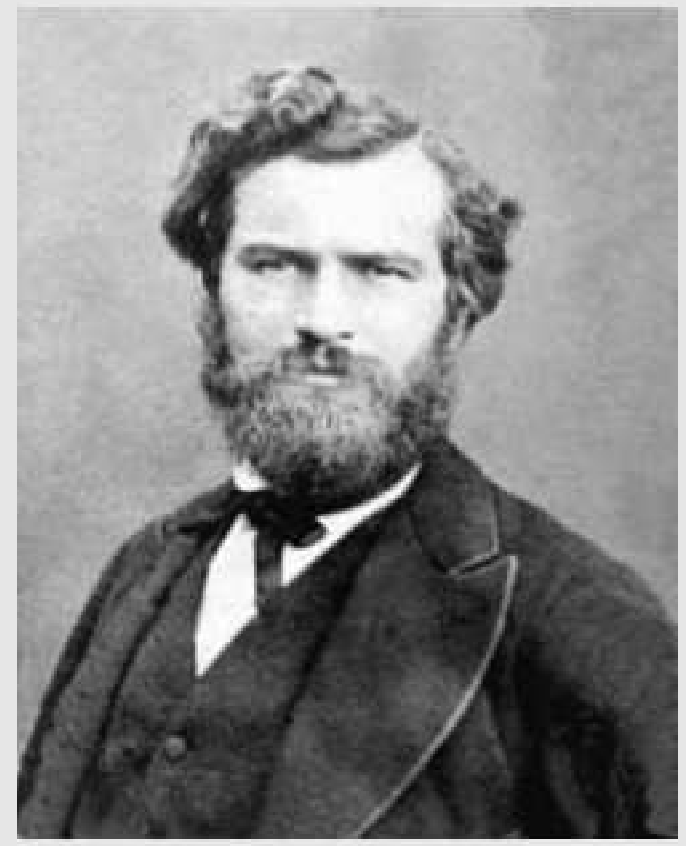
Hermann Weyl
1885-1955

# Who Invented the SVD?

- SVD: "**Swiss Army Knife**" of numerical analysis

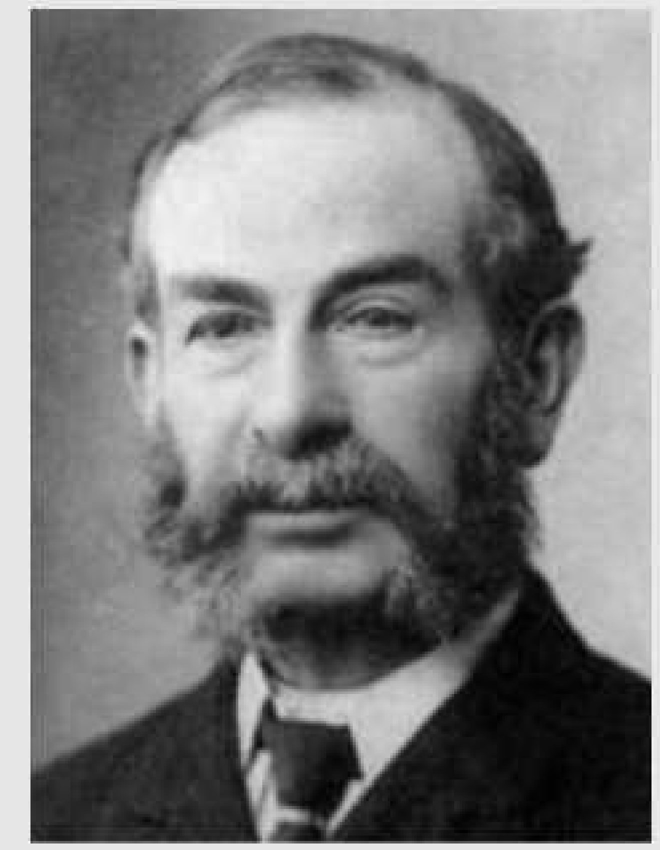
Eugenio Beltrami
1835-1900

**proposed the SVD**
**via eigendecomposition of $A^T A$ or $A A^T$**



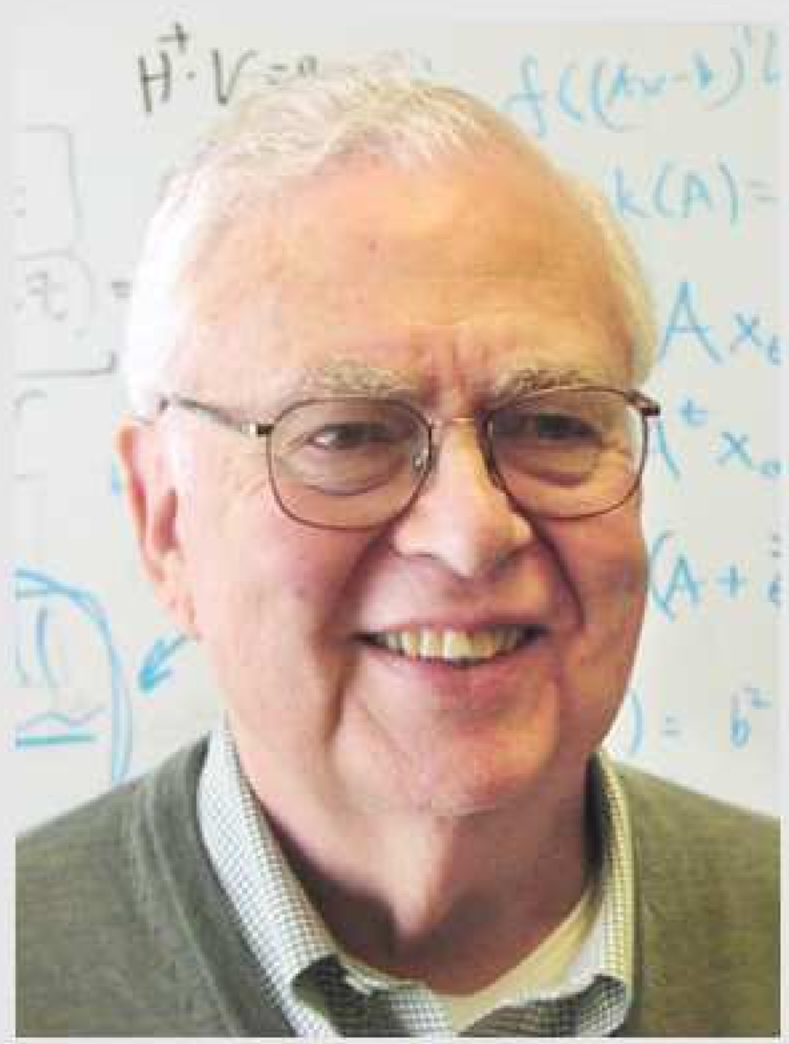
Camille Jordan
1838-1922

Erhardt Schmidt
1878-1959



James Joseph Sylvester 1814-97

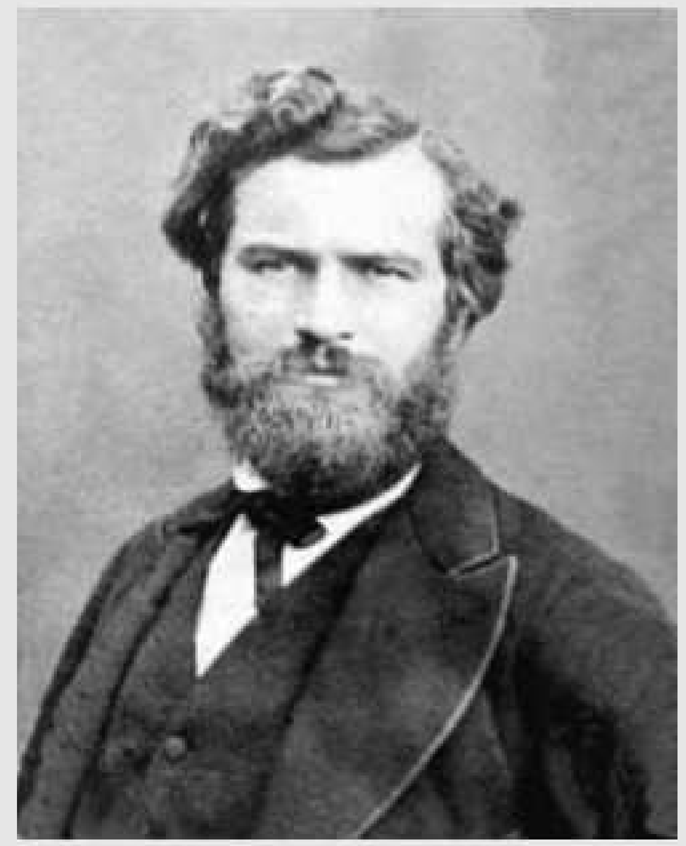
Hermann Weyl
1885-1955



**Gene Golub**
**1932-2007**

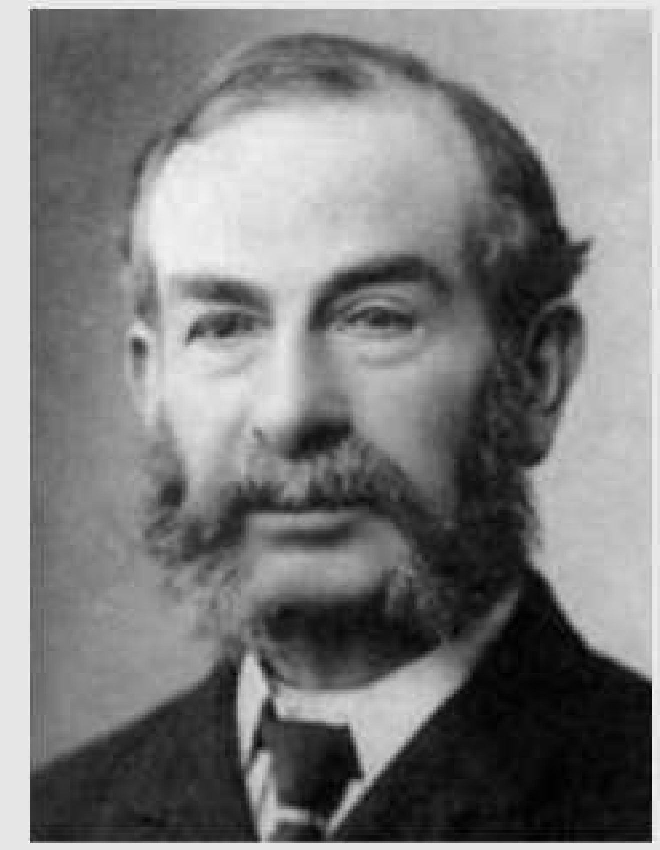
# Who Invented the SVD?

- SVD: “**Swiss Army Knife**” of numerical analysis



Camille Jordan
1838-1922

Erhardt Schmidt
1878-1959

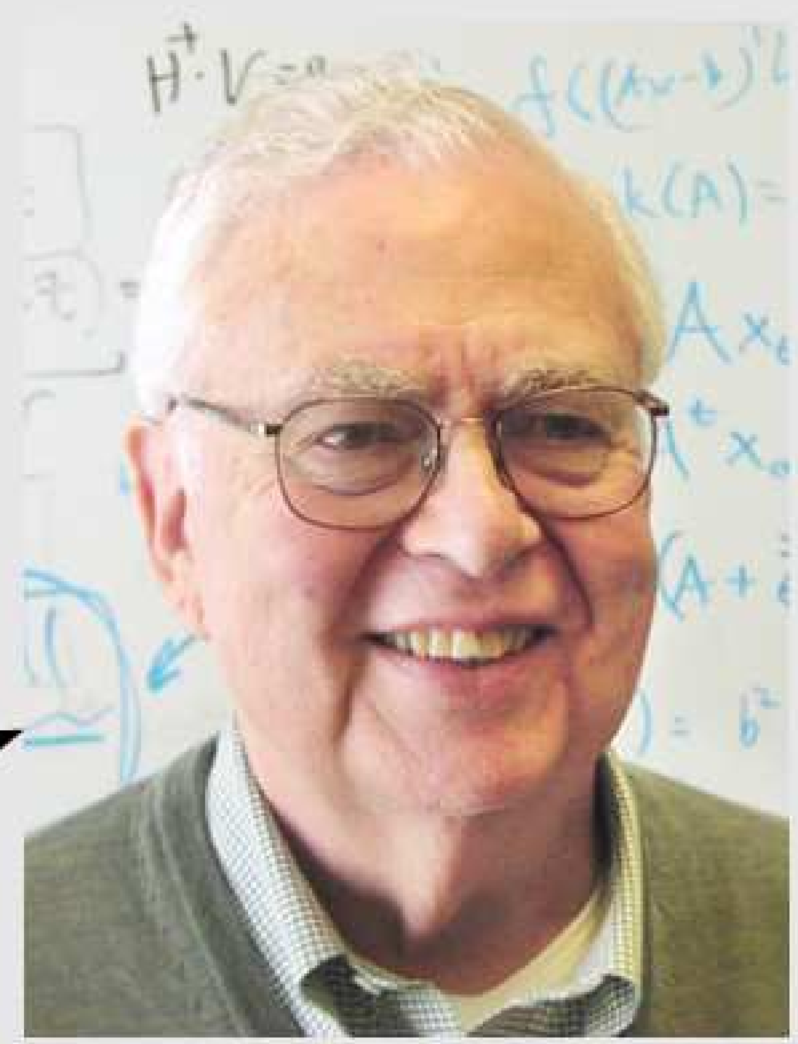


James Joseph
Sylvester 1814-97

Hermann Weyl
1885-1955

Eugenio Beltrami
1835-1900

**proposed the SVD**
**via eigendecomposition**
**of $A^T A$ or $A A^T$**

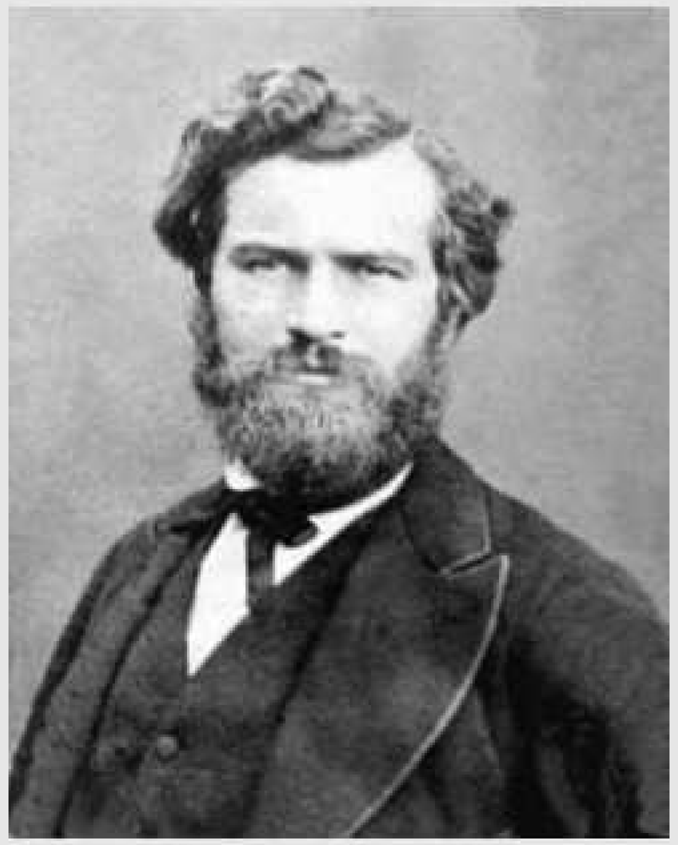
**Gene Golub**
**1932-2007**

# Who Invented the SVD?

- SVD: "**Swiss Army Knife**" of numerical analysis

Eugenio Beltrami
1835-1900

**proposed the SVD**
**via eigendecomposition of $A^T A$ or $A A^T$**

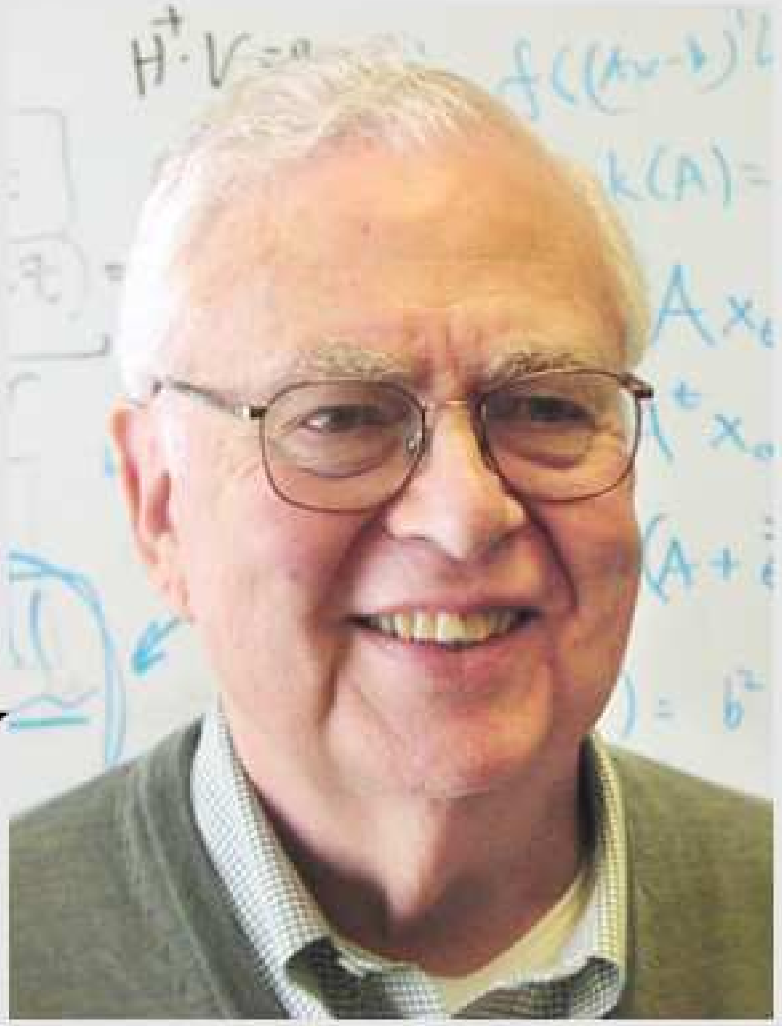


Camille Jordan
1838-1922

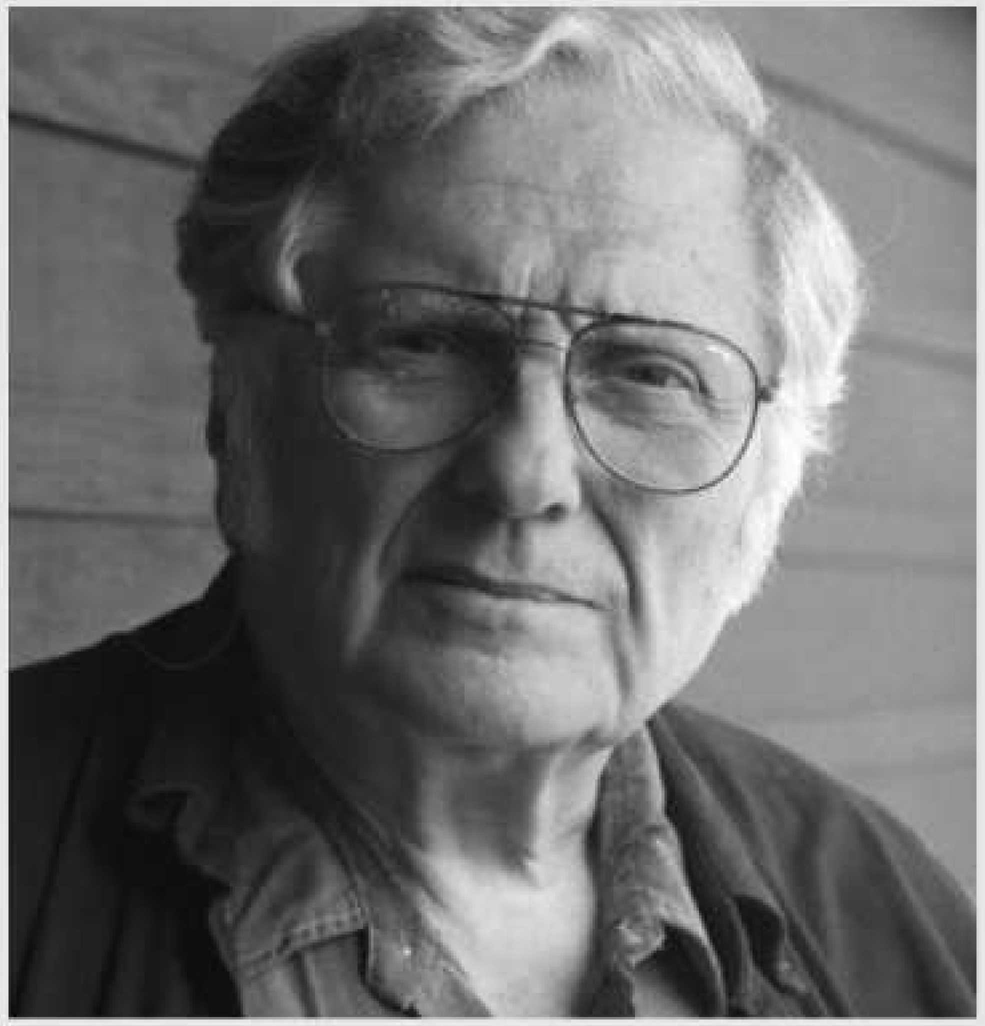
Erhardt Schmidt
1878-1959

James Joseph Sylvester 1814-97

Hermann Weyl
1885-1955

**Gene Golub**
**1932-2007**



**Bill Kahan**
**UCB EECS**

**Jim Demmel**
**UCB EECS**

# Summary: SVD and PCA

- Singular Value Decomposition (SVD)
  - useful for "low-rank approximations" of matrices
    - image analysis and compression
    - general data analysis, finding important features, clustering
- Covariance, Correlation and PCA
  - visualizing data as scatter plots
  - covariance and correlation matrices of data
  - Principal Component Analysis
    - eigenvecs of covariance matrix: principal components
      - directions along which data varies maximally
        - dropping later PCs can, eg, clean out (small) noise
    - eigenvalues correspond to variances along PCs
    - SVD can be used instead of eigendecomposition
      - eigendecomposition of covariance matrix: performs SVD