LINEAR TIME-INVARIANT (LTI) SYSTEMS (CONTD.)

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TODAY

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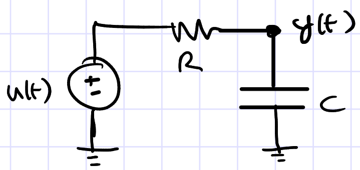
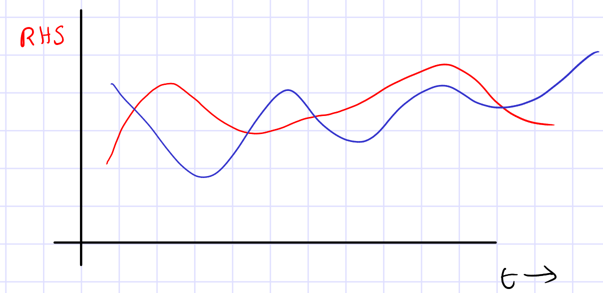
4. EXAMPLES (CONTD.):



$$\underbrace{C \cdot \frac{dy(t)}{dt}}_{LHS} = \underbrace{\frac{u(t) - y(t)}{R}}_{RHS}$$

→ **ASIDE:** A WAY TO LOOK AT EQNS LIKE THESE

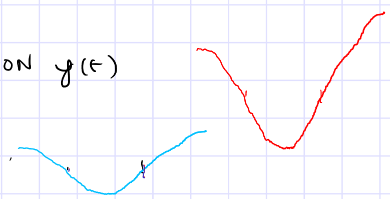
1. Pick $y(t)$ ($u(t)$, the input, is given + fixed)
2. Calculate:
   → $RHS = \frac{u(t) - y(t)}{R}$
   → $LHS = C \frac{d}{dt} y(t)$



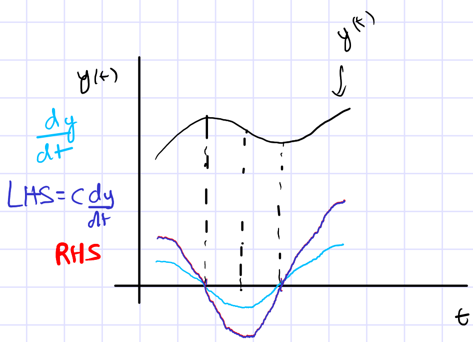
3. CHECK IF THEY OVERLAP
   → IF YES: $y(t)$ is a solution
   → NO? TRY AGAIN.

— SHOWING THAT THE ABOVE EQNS ARE TIME INVARIANT

1) START WITH A SOLUTION $y(t)$



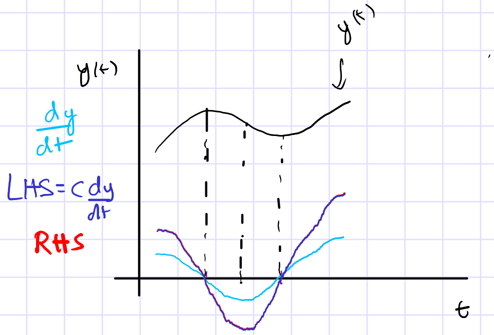
$$\underbrace{C \cdot \frac{dy(t)}{dt}}_{LHS} = \underbrace{\frac{u(t) - y(t)}{R}}_{RHS}$$

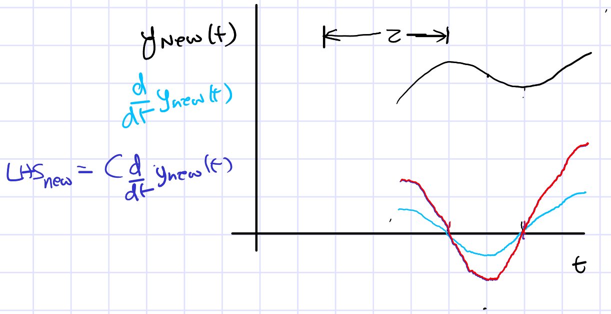


If I try: $u_{new}(t) = u(t-\tau)$, $y_{new}(t) = y(t-\tau)$

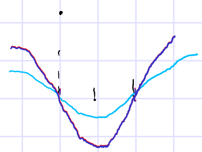
$\rightarrow$ then $u_{new}(t)$ & $y_{new}(t)$ SHOULD ALSO SOLVE

$$\underbrace{C\frac{d\overset{new}{y}(t)}{dt}}_{LHS} = \underbrace{\frac{\overset{new}{u}(t) - \overset{new}{y}(t)}{R}}_{RHS}$$





$$RHS_{new} = \frac{u_{new}(t) - y_{new}(t)}{R}$$



$$= \frac{u(t-\tau) - y(t-\tau)}{R} = RHS(t-\tau)$$

$\rightarrow$ STANDARD STATE SPACE ("LINEAR") SYSTEM:

$\rightarrow$ $\frac{d\vec{x}}{dt} = A\vec{x}(t) + B\vec{u}(t)$, $\quad \vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$

$\rightarrow$ THIS IS LINEAR and TI

$\rightarrow$ PROOF: STEPS IDENTICAL TO THE CKT. EXAMPLE ABOVE

— POINT TO PONDER: HOW DO I.C.s OF STATE-SPACE SYSTEMS

$\rightarrow$ FIGURE IN LINEARITY AND TIME INVARIANCE?

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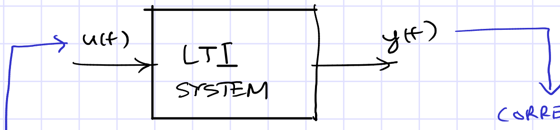
5. IMPULSE RESPONSES OF LTI SYSTEMS

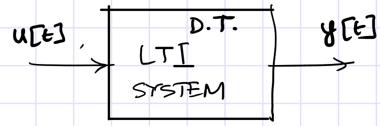


SPECIAL INPUT
$\delta(t)$
"IMPULSE"
or "DELTA FUNCTION"

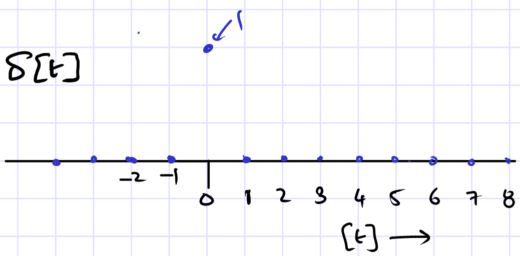
CORRESPONDING OUTPUT
CALLED "IMPULSE RESPONSE"
$h(t)$

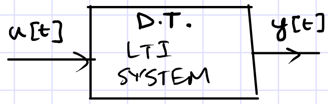
$\rightarrow$ AMAZING FACT: IF YOU KNOW $h(t)$, you can calculate $y(t)$ for ANY input $u(t)$

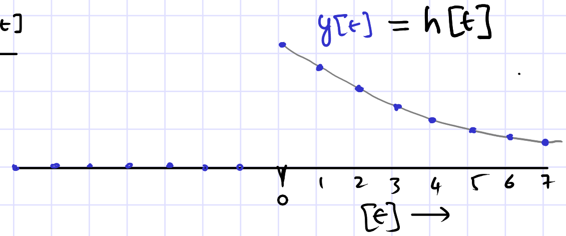
→ (MOVING TO D.T. FOR EASE)



D.T. δ function or impulse







$$\delta[t] = \begin{cases} 1, & \text{if } t=0 \\ 0, & \text{otherwise} \end{cases}$$

CAUSALITY : (A PROPERTY OF SYSTEMS)

→ **WORDS**: THE SYSTEM WON'T RESPOND TO AN INPUT BEFORE THE INPUT IS APPLIED

→ **IN EQNS**:

1) TAKE ANY INPUT $\hat{u}[t]$, apply it: $\hat{u}[t] \mapsto \hat{y}[t]$

2) PICK ANY NUMBER τ

3) DEVISE A NEW INPUT $u[t]$:

   → $u[t] = \hat{u}[t]$ for $t < \tau$
   → AFTER THAT $(t \geq \tau)$, $u[t]$ can be different from $\hat{u}[t]$

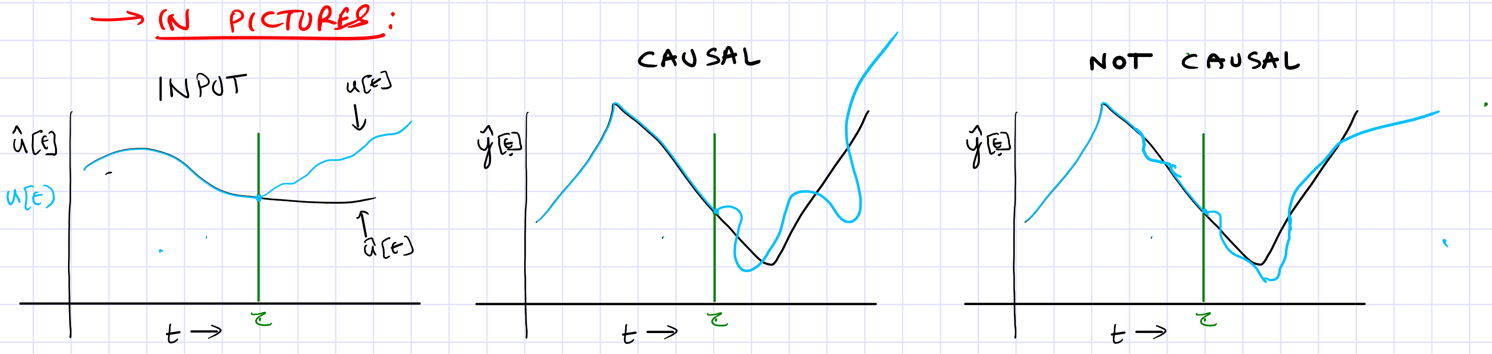
4) Apply $u[t]$ to the system: $u[t] \mapsto y[t]$

5) CHECK: Is $y[t] = \hat{y}[t]$ for $t < \tau$? (FOR ALL CHOICES of $\hat{u}$, u and τ)

   → YES: CAUSAL

   → ELSE : NOT CAUSAL

→ IN PICTURES :



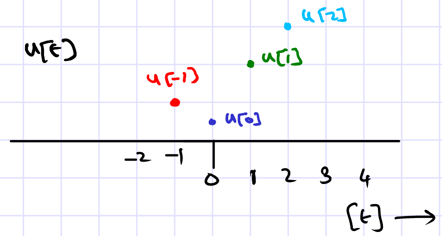
→ CAUSALITY IN LTI SYSTEMS.

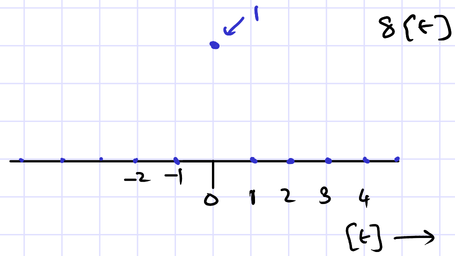
→ IF $h[t] = 0 \;\; \forall t < 0 \iff$ CAUSAL

→ PROOF: try it yourself.

→ CLAIM: FOR LTI SYSTEMS, IF YOU KNOW $h[t]$, and are given ANY input $u[t]$, you can calculate $y[t]$ (where $u[t] \mapsto y[t]$)

→ PROOF:





$$u[t] = \underbrace{u[0]}_{\text{JUST A NUMBER}} \; \underbrace{\delta[t]}_{\text{IMPULSE FUNCTION}} + u[1]\,\delta[t-1] + u[2]\,\delta[t-2] + u[-1]\,\delta[t+1] + \cdots$$

superposition, scaling, shifted $\delta[t]$ functions

$$u[t] = \sum_{i=-\infty}^{+\infty} u[i]\,\delta[t-i]$$

→ CONSIDER THE SYSTEM W/ INPUT $\delta[t-i]$

$$\delta[t-i] \mapsto h[t-i] \quad \text{(TIME INVARIANCE)}$$

→ Try input $u[i]\,\delta[t-i] \mapsto u[i]\,h[t-i]$ (SCALING)

→ Try $u[t] = \sum_{i=-\infty}^{+\infty} \left(u[i]\,\delta[t-i]\right) \mapsto \sum_{i=-\infty}^{+\infty} u[i]\,h[t-i]$

$t - i = j \Rightarrow i = t - j$

→ $y[t] = \sum_{i=-\infty}^{+\infty} u[i]\,h[t-i]$ ← CONVOLUTION $\Rightarrow y[t] = \sum_{j=-\infty}^{\infty} u[t-j]\,h[j]$

→ WRITTEN AS $y[t] = u[t] \circledast h[t] = h[t] \circledast u[t]$

→ SUPPOSE CAUSAL: $\Rightarrow h[t] = 0 \;\; \forall t < 0$

$j = t - i \Rightarrow j = 0 \ldots +\infty$

$$y[t] = \sum_{i=-\infty}^{t} u[i]\,h[t-i] = \sum_{j=0}^{+\infty} u[t-j]\,h[j]$$

→ SUPPOSE (FOR CONVENIENCE IN PROBLEMS) $u[t] = 0 \quad \forall t < 0$

$$y[t] = \sum_{i=0}^{t} u[i]\,h[t-i] = \sum_{j=0}^{t} u[t-j]\,h[j]$$