EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet

Discussion 10A

Properties of Orthonormal Vectors

Please refer to the lecture 10A notes online for proofs of the following properties.

(a) **Definition**: *Orthonormal*

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is **orthonormal** if all the vectors are mutually orthogonal to each other and all are of unit length. That is:

- Orthogonal: For all pairs of vectors $\vec{v}_i, \vec{v}_j$ where $i \neq j$, $\langle \vec{v}_i, \vec{v}_j \rangle = 0$. For real vectors, this means $\vec{v}_i^T \vec{v}_j = 0$.
- Normalized: For all i, $\|\vec{v}_i\| = 1$. (This implies that $\|\vec{v}_i\| = \langle \vec{v}_i, \vec{v}_i \rangle = 1$.)

(b) Any set of orthogonal (and by extension orthonormal) vectors are linearly independent.

(c) Any set of orthogonal (and orthonormal) vectors $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$ form a basis in $\mathbb{R}^n$.

Let us consider the orthonormal set of vectors $\{\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_n\}$, which form a basis in $\mathbb{R}^n$. For any vector $\vec{v}$ represented in this basis, we have

$$\vec{v} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \cdots + \alpha_n \vec{q}_n.$$

Hence, $\alpha_i = \vec{q}_i^{\,T} \vec{v}$. More compactly, we can write

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \longleftarrow & \vec{q}_1^{\,T} & \longrightarrow \\ \longleftarrow & \vec{q}_2^{\,T} & \longrightarrow \\ & \vdots & \\ \longleftarrow & \vec{q}_n^{\,T} & \longrightarrow \end{bmatrix} \vec{v} = Q^T \vec{v}$$

$\vec{v}$, $\vec{\alpha}$, $\{\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_n\}$, and Q have the following properties.

(d) The vector of projections, $\vec{\alpha}$, has the same norm as the original vector $\vec{v}$.

(e) Given that the columns of Q are orthonormal, the rows of Q are also orthonormal.

(f) Given a set of orthonormal vectors $\{\vec{q}_1, \vec{q}_2, \cdots, \vec{q}_n\}$, for any $r \leq n$, we have

$$||\vec{q}_1||^2 + ||\vec{q}_2||^2 + \cdots + ||\vec{q}_r||^2 = \sum_{i=1}^{r} \sum_{j=1}^{n} q_{ji}^2 = r.$$

Here, q_{ji} is the j^{th} element of $\vec{q}_i$.

Properties of Real Symmetric Matrices

Let T be a symmetric matrix on $\mathbb{R}^{n\times n}$. Then,

(a) The eigenvalues of T are real.

(b) A set of real eigenvectors $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$ of T can be found.

(c) The eigenvectors $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$ of T are orthogonal.

(d) A real orthonormal set of eigenvectors of T can be found by normalizing each vector in the set $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n\}$.

That is, there exists n real eigenvalues and n real linearly independent eigenvectors of T that form a basis for $\mathbb{R}^n$. Furthermore, these eigenvectors can be normalized to make an orthonormal basis.

Outer Products

We can define an outer product between two vectors $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$ as follows:

$$\vec{x}\vec{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}_{1\times m} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_m \end{bmatrix}$$

Hence, the outer product gives us an $n \times m$ rank-1 matrix. *Note: Do not confuse the outer product $\vec{x}\vec{y}^T$ with the inner product given by $\vec{x}^T\vec{y}$.*

We can represent the matrix multiplication as a sum of outer products as follows,

$$XY^T = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & \vec{y}_1^{\,T} & — \\ — & \vec{y}_2^{\,T} & — \\ & \vdots & \\ — & \vec{y}_r^{\,T} & — \end{bmatrix} = \vec{x}_1\vec{y}_1^{\,T} + \vec{x}_2\vec{y}_2^{\,T} + \cdots + \vec{x}_r\vec{y}_r^{\,T} = \sum_{i=1}^{r} \vec{x}_i\vec{y}_i^{\,T}.$$

Where X and Y^T are any $n \times r$ and $r \times m$ matrices.

Singular Value Decomposition

The SVD is a useful way to characterize a matrix. Let A be a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or $A \in \mathbb{R}^{m\times n}$) of rank r. It can be decomposed into a sum of r rank-1 matrices:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

where

- $\vec{u}_1, \ldots, \vec{u}_r$ are orthonormal vectors in $\mathbb{R}^m$; $\vec{v}_1, \ldots, \vec{v}_r$ are orthonormal vectors in $\mathbb{R}^n$.

- the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are always real and positive.

We can rewrite the decomposition in the form

$$A = U_1 S V_1^T,$$

where

- U_1 is an $[m \times r]$ matrix whose columns consist of $\vec{u}_1, \ldots, \vec{u}_r$ (orthonormal vectors in $\mathbb{R}^m$). Consequently,
$$U_1^T U_1 = I_{r \times r}$$
- V_1 is an $[n \times r]$ matrix whose columns consist of $\vec{v}_1, \ldots, \vec{v}_r$ (orthonormal vectors in $\mathbb{R}^n$). Consequently,
$$V_1^T V_1 = I_{r \times r}$$
- U_1 characterizes the column space of A and V_1 characterizes the row space of A.
- S is an $[r \times r]$ matrix whose diagonal entries are the singular values of A arranged in descending order. The singular values are the square roots of the nonzero eigenvalues of $A^T A$ (or, identically, AA^T).

The full matrix form of SVD is

$$A = U \Sigma V^T$$

where $U^T U = I_{m \times m}, V^T V = I_{n \times n}, \Sigma \in \mathbb{R}^{m \times n},$ which contains S and elsewhere zero.

Questions

1. **Eigenvectors are Orthogonal**

   Prove the following: For any symmetric matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

   Hint: Use the definition of an eigenvalue to show that $\lambda_1(\vec{v}_1^T \vec{v}_2) = \lambda_2(\vec{v}_1^T \vec{v}_2)$.

2. **Frobenius Norm**

   In this problem we will investigate the properties of the Frobenius norm.

   Much like the norm of a vector $\vec{x} \in \mathbb{R}^N$ is $\|x\| = \sqrt{\sum_{i=1}^N x_i^2}$, the Frobenius norm of a matrix $A \in \mathbb{R}^{N \times N}$ is defined as
$$\|A\|_F = \sqrt{\sum_{i=1}^N \sum_{j=1}^N |A_{ij}|^2}.$$

   Note that matrices have other types of norms as well.

(a) With the above definitions, show that,

$$\|A\|_F = \sqrt{Tr\{A^T A\}}.$$

Note: The trace of a matrix is the sum of its diagonal entries. For example, let $A \in \mathbb{R}^{N\times N}$, then,

$$Tr\{A\} = \sum_{i=1}^{N} A_{ii}$$

(b) Show that if U and V are orthonormal matrices, then

$$\|UA\|_F = \|AV\|_F = \|A\|_F.$$

(c) Show that $\|A\|_F = \sqrt{\sum_{i=1}^{N} \sigma_i^2}$, where $\sigma_1, \cdots, \sigma_N$ are the singular values of A.

3. SVD Short Questions

Assume we have the compact form of the SVD of

$$A = U_1 S V_1^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T.$$

(a) Compute $AV_1V_1^T$.

(b) What is the subspace that spans the column space of A?

4. (Optional) Symmetric Matrix Properties

Given the symmetric matrix

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

find the eigenvalues and eigenvectors of T.

Show that the eigenvalues are real and the eigenvectors are orthonormal.

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