EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet — Discussion 10B

Calculating the Singular Value Decomposition

In lecture, we learned how to find the SVD of a wide matrix, A of dimension m x n ($n > m$). To decompose A into $U\Sigma V^T$ we took the following steps:

(a) Compute the symmetric matrix A^TA with dimension n x n.

(b) Find the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and eigenvectors $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ of A^TA. By the spectral theorem for real symmetric matrices, these eigenvectors are orthonormal.

(c) $\sigma_i = \sqrt{\lambda_i}$ where λ_i are the sorted in descending order eigenvalues of A^TA. We know these are all non-negative because $(A\vec{v}_i)^T(A\vec{v}_i) = \|A\vec{v}_i\|^2$ and $(A\vec{v}_i)^T(A\vec{v}_i) = \vec{v}_i^T(A^TA)\vec{v}_i = \lambda_i\vec{v}_i^T\vec{v}_i = \lambda_i$. The corresponding normalized eigenvectors $\vec{v}_i$ form the V matrix.

(d) Using σ_i and $\vec{v}_i$ we can find the corresponding vectors of the U matrix, $\vec{u}_i$ by computing $\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$. These are normalized since $\sigma_i = \|A\vec{v}_i\|$ by the argument above, and orthogonal since $(A\vec{v}_i)^T(A\vec{v}_j) = \vec{v}_i^T(A^TA)\vec{v}_j = \lambda_j\vec{v}_i^T\vec{v}_j = 0$ if $i \neq j$, since V is an orthonormal matrix.

Using this knowledge, if we were asked find the SVD of a tall matrix A of dimension m x n such that $m > n$, one option would be to take the transpose of this matrix to get a wide matrix, proceed as described above, and transpose the result.

In this discussion we will better understand how to compute the SVD for any matrix.

Questions

1. Understanding the SVD

We can compute the SVD for a wide matrix A with dimension m x n where $n > m$ using A^TA with the method described above. However, when doing so you may realize that A^TA is much larger than AA^T for such wide matrices. This makes it more efficient to find the eigenvalues for AA^T. In this question we will explore how to compute the SVD using AA^T instead of A^TA.

(a) What are the dimensions of AA^T and A^TA.

(b) Given that the $A = U\Sigma V^T$, find a symbolic expression for AA^T.

(c) Using the solution to the previous part explain how to find U and Σ from AA^T.

(d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, devise a way to find the corresponding vectors $\vec{v}_i$ in matrix V.

(e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V, verify that they are orthonormal.

(f) Now that we have found $\vec{v}_i$ you may notice that we only have $m < n$ vectors of dimension n. This is not enough for a basis. How would you complete the m vectors to form an orthonormal basis?

(g) Given that $A = U\Sigma V^T$ verify that the vectors you found to extend the $\vec{v}_i$ into a basis are in the nullspace of A.

(h) Using the previous parts of this question and what you learned from lecture write out a procedure on how to find the SVD for any matrix.

2. SVD Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) Find the SVD of A (compact form is fine).

(b) Find the rank of A.

(c) Find a basis for the nullspace of A.

(d) Find a basis for the range (or columnspace) of A.

(e) Repeat parts (a) - (d), but instead, create the SVD of A^T. What are the relationships between the answers for A and the answers for A^T?

3. Eigenvalue Decomposition and Singular Value Decomposition

We define the Eigenvalue Decomposition as follows:

If a matrix $A \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $\vec{p}_1, \ldots, \vec{p}_n$ with eigenvalues $\lambda_i, \ldots, \lambda_n$, then we can write:

$$A = P\Lambda P^{-1}$$

Where the columns of P consist of $\vec{p}_1, \ldots, \vec{p}_n$, and Λ is a diagonal matrix with diagonal entries $\lambda_i, \ldots, \lambda_n$.

For the sake of convenience, assume that these are sorted by their absolute values in descending order.

Consider a symmetric matrix $A = A^T \in \mathbb{R}^{n\times n}$. This is a symmetric matrix and thus has orthogonal eigenvectors and real eigenvalues (not necessarily non-negative).

Therefore its eigenvalue decomposition can be written as,

$$A = P\Lambda P^T$$

(a) First, assume $\lambda_i \geq 0, \forall i$. Find the SVD of A.

(b) Let one particular eigenvalue λ_j be negative, with the associated eigenvector being $\vec{p}_j$. Succinctly,

$$A\vec{p}_j = \lambda_j \vec{p}_j \text{ with } \lambda_j < 0$$

We are still assuming that,

$$A = P\Lambda P^T$$

i. What is the singular value σ_j associated to λ_j?

ii. What is the relationship between the left singular vector $\vec{u}_j$, the right singular vector v_j and the eigenvector $\vec{p}_j$?

Extra Practice

1. More SVD

Define the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a) Find the SVD of A (compact form is fine).

(b) Find the rank of A.

(c) Find a basis for the nullspace of A.

(d) Find a basis for the range (or columnspace) of A.

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