Discussion 12B

Overview

(a) First let's look at how to 'vectorize' a discrete time signal. Suppose we have the following arbitrary signal for $-1 \leq t \leq 3$:

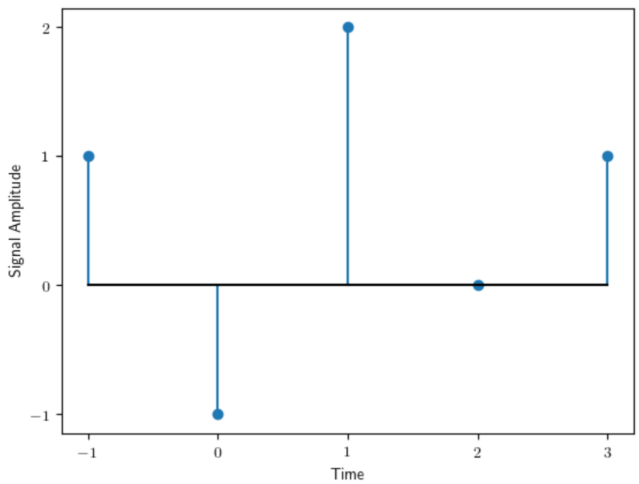


Figure 1: Discrete-time signal

We can stack the values at the various time steps into a vector as follows,

$$\vec{x} = \begin{bmatrix} x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Furthermore, if the start-time is not specified, we will assume the signals start at $t = 0$.

(b) **Real Inner products:** When dealing with vectors in $\mathbb{R}^n$ we defined the real inner product as

$$< \vec{a}, \vec{b} >= \vec{a}^T \vec{b}. \tag{1}$$

This inner product was symmetric, meaning that

$$< \vec{a}, \vec{b} >^T = (\vec{a}^T \vec{b})^T = \vec{b}^T \vec{a} =< \vec{b}, \vec{a} > . \tag{2}$$

And since for any scalar $x \in \mathbb{R}$ $x = x^T$, the inner product is also commutative.

We used this inner product to define projection such that $\vec{a}$ projected on $\vec{b}$ is

$$\text{proj}_{\vec{b}}\vec{a} = \frac{<\vec{b},\vec{a}>}{<\vec{b},\vec{b}>}\vec{b}. \tag{3}$$

So for vectors:

$$\vec{a} = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

where c is some constant,

$\text{proj}_{\vec{b}}\vec{a} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) **Hermitian:** Just take the complex conjugate of all the elements and transpose the matrix (or vector). Generally represented with a '*' or 'H'. For example,

$$A = \begin{bmatrix} 1 & j \\ 2 & 3j \end{bmatrix},$$

then,

$$A^* = A^H = \begin{bmatrix} 1 & 2 \\ -j & -3j \end{bmatrix}.$$

(d) **Complex Inner products:** When dealing with vectors in $\mathbb{C}^n$ we defined the complex inner product as

$$<\vec{a},\vec{b}> = \vec{a}^*\vec{b}. \tag{5}$$

Here $\vec{a^*}$ is the conjugate transpose of $\vec{a}$.

Notice that unlike the real inner product we defined in equation (1), this inner product is not symmetric. It is however conjugate symmetric. More concretely,

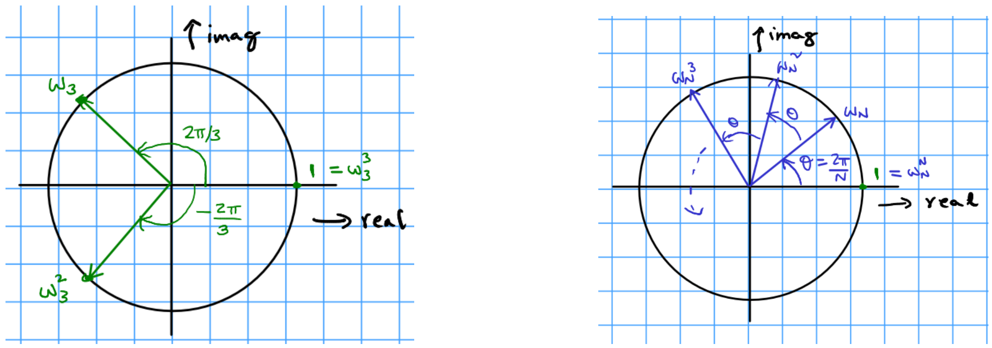
$$<\vec{a},\vec{b}>^* = (\vec{a}^*\vec{b})^* = \vec{b}^*\vec{a} = <\vec{b},\vec{a}>. \tag{6}$$

Due to this conjugate symmetry we see that the complex inner product is not commutative (*i.e.*, order matters).

We can use our complex inner product with our earlier definition of projection. Thus projecting vector $\vec{a} = \begin{bmatrix} u+vj \\ 0 \\ 0 \end{bmatrix}$ onto $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{<\vec{b},\vec{a}>}{<\vec{b},\vec{b}>}\vec{b} = (u+vj)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

(e) **Roots of Unity:** We all know that $\sqrt{1} = \pm 1$, but what about $\sqrt[3]{1}$? Well, the first thing that comes to mind is 1. But it should have 3 roots, right? With access to the complex plane, we can find the other two. Take your unit circle in the complex plane and split them into 3 sections (as in figure 2a).



(a) Cube roots of 1

(b) N^{th} roots of 1

Figure 2

Now, each of the arrows represent a root of unity. And we have $\sqrt[3]{1} = 1, e^{j\frac{2\pi}{3}}, e^{j\frac{4\pi}{3}}$. Extending this to the N roots, we can divide the unit circle into N sections (as in figure 2b.

Hence, we have $\sqrt[N]{1} = e^{j\frac{2\pi}{N}*k}$ for $k \in [0, N-1]$. Why does it end at $N-1$?

Furthermore, let's define $\omega_N = e^{j\frac{2\pi}{N}}$.

Contributors:

- Aditya Arun.
- Nikhil Shinde.