EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet **Discussion 13A**

Discrete Fourier Transform - Lecture Overview

In this discussion we will consider the

(a) Let us start by considering the DFT of the following discrete DC time domain signal.

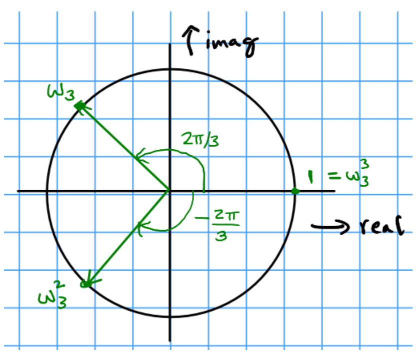
(b) **Hermitian**: Just take the complex conjugate of all the elements and transpose the matrix. Generally represented with a '*' or 'H'. For example,

$$A = \begin{bmatrix} 1 & j \\ 2 & 3j \end{bmatrix},$$

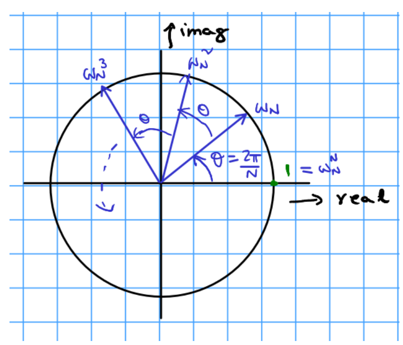
then,

$$A^* = A^H = \begin{bmatrix} 1 & 2 \\ -j & -3j \end{bmatrix}.$$

(c) **Roots of Unity**: We all know that $\sqrt{1} = \pm 1$, but what about $\sqrt[3]{1}$? Well, the first thing that comes to mind is 1. But it should have 3 roots, right? With access to the complex plane, we can find the other two. Take your unit circle in the complex plane and split them into 3 sections (as in figure 1a).



(a) Cube roots of 1



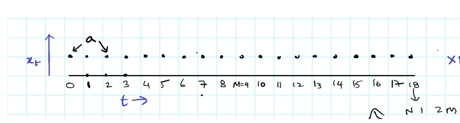
(b) N^{th} roots of 1

Figure 1

Now, each of the arrows represent a root of unity. And we have $\sqrt[3]{1} = 1, e^{j\frac{2\pi}{3}}, e^{j\frac{4\pi}{3}}$. Extending this to the N roots, we can divide the unit circle into N sections (as in figure 1b.

Hence, we have $\sqrt[N]{1} = e^{j\frac{2\pi}{N}*k}$ for $k \in [0, N-1]$. Why does it end at $N-1$?

Furthermore, let's define $\omega_N = e^{j\frac{2\pi}{N}}$.



(d) Next, let's look at how to 'vectorize' a discrete time signal. Suppose we have the following arbitrary signal for $-1 \leq t \leq 3$:

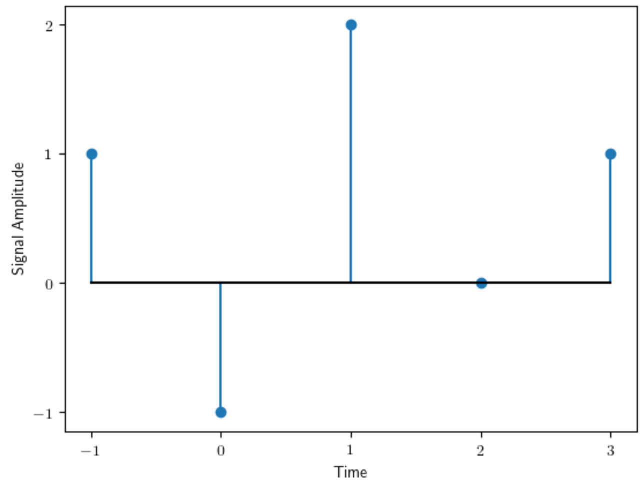


Figure 2: Discrete-time signal

We can stack the values at the various time steps into a vector as follows,

$$\vec{x} = \begin{bmatrix} x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Furthermore, if the start-time is not specified, we will assume the signals start at $t = 0$.

(e) Now, let's define the 'special' DFT matrix:

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 & \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \omega_N^{-3} & \cdots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega^{-4} & \omega_N^{-6} & \cdots & \omega_N^{-2(N-1)} \\ & & \vdots & & & \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \omega_N^{-3(N-1)} & \cdots & \omega_N^{-(N-1)(N-1)} \end{bmatrix}. \quad (1)$$

Now, given a signal, $\vec{x} \in \mathbb{C}^N$, *i.e.*, the signal covers N time-steps, we can find it's DFT as follows:

$$\vec{X} = F_N \vec{x}.$$

Note, we will use a small-case letter to represent a '**time-domain**' signal and a capitalized letter to represent the '**frequency-domain**' signal. We can also write the above matrix equation as a summation:

$$X_i = \sum_{j=0}^{N-1} \omega^{-ij} x_j.$$

(f) Before proceeding, we should understand some important properties of the DFT matrix

i. It is symmetric, *i.e.*, $F_N^T = F_N$.

ii. It has full rank, and hence $F_N^{-1} = \frac{1}{N}F_N^*$. For the proof, please refer to the lecture notes. F_N^{-1} is aptly named the inverse-DFT (IDFT). Hence we can write,

$$\vec{x} = \frac{1}{N}F_N^*\vec{X}.$$

Or, as a summation,

$$x_i = \frac{1}{N}\sum_{j=0}^{N-1}\omega^{ij}X_j.$$

iii. Furthermore, the columns are orthogonal. Each column has norm $\sqrt{N}$.

(g) Let's assume we have a signal $\vec{x} \in \mathbb{R}^N$ and it's DFT $\vec{X}$, then it must follow the 'complex-conjugacy' property,

i. $X_{N-i} = \overline{X_i}, \quad \forall i \in [1, N-1]$,

ii. And, $X_0 \in \mathbb{R}$.

For the proof, please see the lecture notes.

Questions

1. Converse of the Complex Conjugacy Property

We have seen that if a vector $\vec{x}$ is real, then the entries of its DFT $\vec{X}$ obeys a complex conjugacy property, *i.e.*, $X_i = \overline{X}_{N-i}$ for $i = 1, \cdots, N-1$, and X_0 is real.

The following is the converse of this complex-conjugacy property, *i.e.*,

Let $\vec{X} \in \mathbb{C}^N$. If $X_{N-i} = \overline{X_i}$, for all $i \in [1, N-1]$ and $X_0 \in \mathbb{R}$, then $\vec{x} = IDFT(\vec{X})$ is a real vector.

Prove this result.

2. DFT of pure sinusoids

In this question, let's motivate a different way of looking of looking at the inverse-DFT matrix we are now familiar with. Since, it is a orthogonal matrix, it's columns form a basis. Let's define $\vec{u}_k$ as the k^{th} column of $\frac{1}{N}F_N^*$. More concretely,

$$\vec{u}_k = \frac{1}{N}\begin{bmatrix} 1 \\ \omega_N^k \\ \omega_N^{2k} \\ \vdots \\ \omega_N^{(N-1)k} \end{bmatrix}$$

(a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

Compute $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.

(b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

Repeat what you did above. **What are $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.**

(c) Let's do this more generally. **For the signal $x(t) = \cos\left(\frac{2\pi k}{N}t\right)$, compute $\vec{X}$ of its vector form in discrete time, $\vec{x}$, of length $n = N$:**

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(N-1)\right) \end{bmatrix}^T.$$

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