Discrete Fourier Transform - Signal Analysis

In this discussion, we want to analyze a signal of the following form:

$$x(t) = A_0 + \sum_{l=1}^{M} A_l \cos\big(2\pi(l f_0)t + \theta_l\big)$$

where f_0 is a known frequency and $A_0 \in \mathbb{R}$, $\theta_l \in \mathbb{R}$ and $A_l \geq 0$ for $l = 1, \ldots, M$. By analyzing this signal, what we mean is that we want to identify A_l for $l = 0, \ldots, N-1$ and θ_l for $l = 1, \ldots, N-1$. We will also define for $l = 1, \ldots, N-1$

$$x_l : t \mapsto A_l \cos\big(2\pi(l f_0)t + \theta_l\big)$$

and $x_0 : t \mapsto A_0$, the DC component of our signal.

The aim of this discussion is to show that given well-chosen samples of this signal, we can get what we want by simply computing the DFT of the sample vector $\vec{x}$. We will denote the DFT matrix for N samples by F_N, throughout this discussion.

To show this, we will go through three parts:

- Going through the proof of the DFT matrix's orthogonality, which is a particularly useful property here
- Showing that the coefficients of $\vec{X}_l = F_N \vec{x}_l$ have distinct non zero components.
- Finally computing A_l, θ_l for each $l = 0, \ldots, M$.

1. Orthogonality of the DFT matrix

(a) Let $N \geq 2$. We want to prove the orthogonality of the rows of the DFT matrix, F_N. Let $\omega_N = e^{j\frac{2\pi}{N}}$ Remember that the $(i,k)^{th}$ entry of F_N is $\omega_N{}^{-i*k}$.

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \ldots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \ldots & \omega_N^{-(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \ldots & \omega_N^{-(N-1)(N-1)} \end{pmatrix} = \begin{pmatrix} \vec{u}_0^\top \\ \vdots \\ \vec{u}_{N-1}^\top \end{pmatrix} \quad (1)$$

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(b) Show that for any complex number z, $z^N - 1 = (z-1)(1 + z + \cdots + z^{N-1})$.

(c) Deduce from this that if z is such that $z^N = 1$ but $z \neq 1$, then $1 + z + \cdots + z^{N-1} = 0$.

(d) **Compute $\vec{u}_i^\star \vec{u}_k$, for any pair $(i,\ k)$.**

2. Discrete Fourier Transform of Harmonics

In this part, we suppose that our signal is simply the l-th harmonic:

$$x(t) = x_l(t) = A_l \cos\big(2\pi(l f_0)t + \theta_l\big)$$

i.e. it is a sinusoidal signal of frequency $l f_0$, but all we know about l is that it is in $1, \ldots, M$.

The task is to find A_l, θ_l and l at the same time. It turns out that the Discrete Fourier Transform allows us to solve this problem by taking $N = 2 * M + 1$ samples of this signal, at time points $\{0, \Delta, 2\Delta, ..., (N-1)\Delta\}$, where $\Delta = \frac{T}{N}$ and T is the smallest time period over which, all of the above signals are periodic. We suppose in the following that $l \geq 1$.

(a) **What is the smallest value of T** so that every cosine signal having a frequency of $\{f_0, 2f_0, ..., M f_0\}$ would be periodic?

(b) Before we move forward, let's remind ourselves of Euler's formula. **Write down cosine(x) in terms of e^{jx} and e^{-jx}.**

(c) Let's consider the vector of samples of $x_l(t)$ at time $k\Delta$ denoted as $\vec{x}_l$. **Compute the expression for the components of $\vec{x}_l$** using the previous expression for cosine and using $\omega_N = e^{j\frac{2\pi}{N}}$. Finally write $\vec{x}_l$ using the vectors $\vec{u}_l$, where $\vec{u}_l^\top = \begin{pmatrix} 1 & \omega_N^{-l} & \omega_N^{-2l} & \ldots & \omega_N^{-(N-1)l} \end{pmatrix}$.

(d) **Compute the Discrete Fourier Transform of $\vec{x}_l$**, i.e. $\vec{X}_l = F_N \vec{x}_l$. Start by writing F_N out using the $\vec{u}_k$ vectors for $k = 0, \ldots, N-1$. **Show that this allows to identify l, A_l and θ_l.**

3. Analyzing a harmonic signal

Let us go back to our original problem, analyzing $x(t) = A_0 + \sum_{l=1}^{M} A_l \cos\big(2\pi(l f_0)t + \theta_l\big)$.

(a) Show how to compute A_0.

(b) Show how to identify the quantities A_l and θ_l for each l.

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