EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet **Fourier Series**

1 Periodic Waveforms

Periodic waveforms are signals $x(t)$ that repeat the same pattern in a set amount of time T, which is called the period of $x(t)$. Mathematically, we say $x(t)$ is T-periodic if

$$x(t+T) = x(t)$$

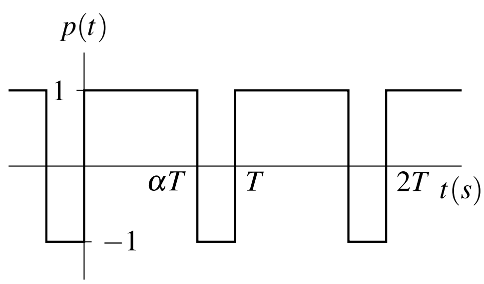
for all time t. The fundamental frequency f_0 of the signal is given by

$$f_0 = \frac{1}{T}.$$

One periodic signal we've used many times is the sinusoid. The natural period of a sinusoid is 2π. We define a cosine with period T by

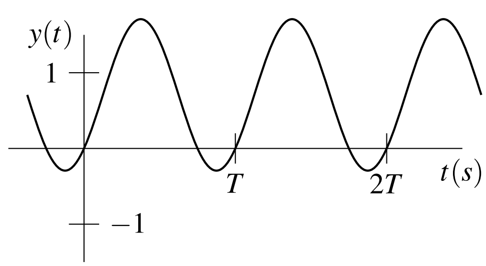
$$x(t) = \cos\left(\frac{2\pi}{T}t\right).$$

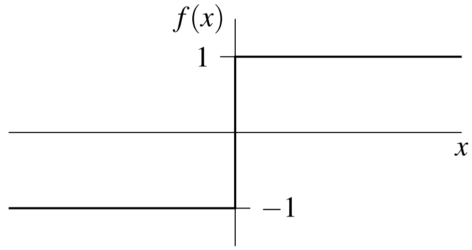
Periodic waveforms are extremely useful in many engineering contexts. An obvious example is alternating current (AC), which is a pure sinusoid. But often we use other periodic functions besides sinusoids. For example, pulse width modulation (PWM) waveforms have the form



where $\alpha \in [0,1]$ defines the duty-cycle of the signal.

Pulse width modulation is used to change the brightness of LED's, as well as in DC-DC power conversion. We can produce a PWM waveform $p(t)$ by passing a T-periodic sine $y(t)$ with an amplitude (DC) and phase offset through a nonlinear function $f(\cdot)$:





Here, $p(t) = f(y(t))$. As we can see above, applying this nonlinear function to a T-periodic signal results in a T-periodic waveform. This is in fact a more general property: a function applied to a T-periodic sinusoid will produce a (more complicated) T-periodic waveform.

Therefore, we might consider that an arbitrary T-periodic waveform is related to an underlying T-periodic sinusoid.

2 Fourier Series

The Fourier Series says we can represent *any* T-periodic waveform as the sum of many sine waves with frequencies at integer multiples of the fundamental frequency f_0:

$$f = 0, f_0, 2f_0, 3f_0, ...$$

where $f = 0$ is the DC component, $f = f_0$ is the fundamental frequency, and $f = 2f_0, 3f_0, ...$ are the harmonics.

Mathematically, we can write any T-periodic waveform $x(t)$ as

$$x(t) = \sum_{i=0}^{\infty} B_i \cos(2\pi i f_0 t + \theta_i) = \sum_{l=-\infty}^{\infty} A_l e^{j2\pi f_0 l t}, \tag{1}$$

where B_i is a real-valued amplitude, θ_i is a real-valued phase offset, and A_l is a complex-valued coefficient. This is called the Fourier Series representation of the signal. We calculate the coefficients A_l as

$$A_l = \frac{1}{T} \int_0^T e^{-j2\pi f_0 l t} x(t) dt \tag{2}$$

for each integer $l \in [-\infty, \infty]$.

We can see that the Fourier Series represents a waveform with an infinite number of sinusoids up to infinitely high frequencies. But what if we only want to represent a waveform with a finite number of sinusoids?

Let us truncate the Fourier Series representation of $x(t)$ to a summation of $N = 2M + 1$ sinusoids and use the fact that $f_0 = \frac{1}{T}$:

$$x(t) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{T} it}. \tag{3}$$

This format should remind you of the Discrete Fourier Transform, where we are representing a discrete waveform as a summation of discrete sinusoids - except, of course, that the Fourier Series is a representation of *continuous* waveforms, rather than *discrete* waveforms. We will explore this connection further below.

3 Questions

1. Fourier Series and the DFT

In this problem, you will discover how we can use the DFT of N samples of a continuous waveform $x(t)$ to calculate the truncated Fourier Series representation of $x(t)$.

Let's explore the connection between the truncated Fourier Series and the DFT further. As a reminder, the truncated Fourier Series has us represent continuous, T-periodic signal $x(t)$ as

$$x(t) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{T}it}. \tag{4}$$

(a) We take $N = 2M+1$ samples of the T-periodic function $x(t)$ across a single period T. That is, our samples are at

$$t = \frac{T}{N}k,\ k \in \{0,1,2,\cdots,N-1\}.$$

How can we represent the k^{th} sample of $x(t)$, x_k?

(b) Write the relationship between the x_k (samples) and X_i (Fourier Series coefficients) in matrix-vector form. That is, given

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = A \begin{bmatrix} X_{-M} \\ X_{-M+1} \\ \vdots \\ X_{M-1} \\ X_M \end{bmatrix}, \tag{5}$$

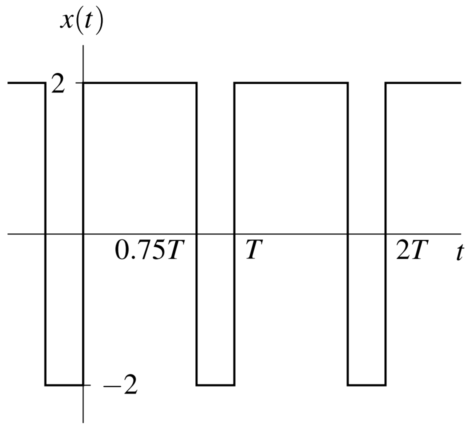
what is the matrix A? What is this matrix's relationship to the DFT matrix, F_N?

(c) If we reorder the X_i Fourier Series coefficients to be in "DFT order" (that is, in frequency order $f = 0, f_0, 2f_0, \cdots, Mf_0, -Mf_0, \cdots, -2f_0, -f_0$) show that

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = F_N^* \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_M \\ X_{-M} \\ \vdots \\ X_{-1} \end{bmatrix}. \tag{6}$$

2. Fourier Series

Suppose we have the T-periodic waveform $x(t)$:



That is, on the interval $[0,T)$, $x(t)$ is given by

$$x(t) = \begin{cases} 2, & 0 \leq t < \frac{3}{4}T \\ -2, & \frac{3}{4}T \leq t < T \end{cases}$$

and is T-periodic outside $t = [0,T)$.

(a) Calculate the Fourier Series coefficients A_l, given by

$$A_l = \frac{1}{T}\int_0^T e^{-j\frac{2\pi}{T}lt}x(t)dt$$

for integer $l \in [-\infty, \infty]$.

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