

1 Complex Numbers

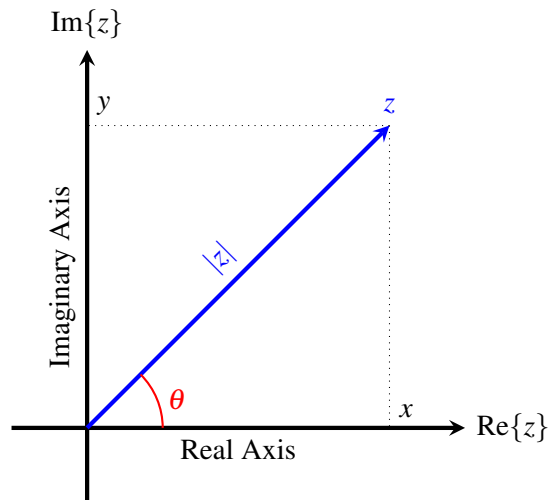


Figure 1: Complex number z represented as a vector in the complex plane.

A complex number z is an ordered pair (x, y) , where x and y are real numbers, written as $z = x + jy$ where $j = \sqrt{-1}$. A complex number can also be written in polar form as follows:

$$z = |z|e^{j\theta}$$

The magnitude z is denoted as $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

The phase or argument of a complex number is denoted as θ and is given by

$$\theta = \text{atan2}(y, x).$$

Here, $\text{atan2}(y, x)$ is a function that returns the angle from the positive x -axis to the vector from the origin to the point (x, y) ¹.

The complex conjugate of a complex number z is denoted by \bar{z} (or might also be written z^*) and is given by

$$\bar{z} = x - jy.$$

Euler's Identity is

$$e^{j\theta} = \cos(\theta) + j \sin(\theta).$$

¹See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at <https://en.wikipedia.org/wiki/Atan2>.

With this definition, the polar representation of a complex number will make more sense. Note that

$$|z|e^{j\theta} = |z|\cos(\theta) + j|z|\sin(\theta).$$

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case $|z|$ is the magnitude and $e^{j\theta}$ is the unit vector that defines the direction.

2 Useful Identities

Complex Number Properties

Rectangular vs. polar forms: $z = x + jy = |z|e^{j\theta}$

where $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, $\theta = \text{atan2}(y, x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex conjugate: $\bar{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z+w)} = \bar{z} + \bar{w}, \quad \overline{(z-w)} = \bar{z} - \bar{w}$$

$$\overline{(zw)} = \bar{z}\bar{w}, \quad \overline{(z/w)} = \bar{z}/\bar{w}$$

$$\bar{\bar{z}} = z \Leftrightarrow z \text{ is real}$$

$$\overline{(z^n)} = (\bar{z})^n$$

Complex Algebra

Let $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$, $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$.

(Note that we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{j(\theta_1 + \theta_2)}$

Division: $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1 - \theta_2)}$

Power: $z_1^n = |z_1|^n e^{jn\theta_1}$
 $z_1^{\frac{1}{2}} = \pm |z_1|^{\frac{1}{2}} e^{j\frac{\theta_1}{2}}$

Useful Relations

$$-1 = j^2 = e^{j\pi} = e^{-j\pi}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1}$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}}$$

$$\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1-j)}{\sqrt{2}}$$

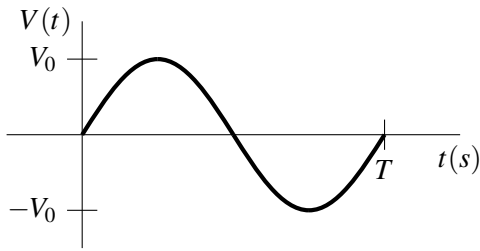
3 Phasors

We consider sinusoidal voltages and currents of a specific form:

$$\begin{array}{l|l} \text{Voltage} & v(t) = V_0 \cos(\omega t + \phi_v) \\ \text{Current} & i(t) = I_0 \cos(\omega t + \phi_i), \end{array}$$

where,

- V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.
- ω is the angular frequency of oscillation. ω is related to frequency by $\omega = 2\pi f$. Frequency f is the number of oscillation cycles that occur per second. If T is the *period* of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is $f = \frac{1}{T}$.



- (c) ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = \frac{V_0}{2}e^{j\omega t + j\phi_v} + \frac{V_0}{2}e^{-j\omega t - j\phi_v} = \frac{V_0}{2}e^{j\phi_v}e^{j\omega t} + \frac{V_0}{2}e^{-j\phi_v}e^{-j\omega t} \quad (1)$$

The coefficient of the $e^{j\omega t}$ term in Equation 1 is called the phasor form of this signal:

$$\tilde{V} = \frac{1}{2}V_0e^{j\phi_v}$$

The complex conjugate of the phasor is the coefficient of the $e^{-j\omega t}$ term in Equation 1 and is given by:

$$\tilde{V}^* = \frac{1}{2}V_0e^{-j\phi_v}$$

The phasor representation contains the magnitude and phase of the signal, but not the time-varying portion. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: We can only use phasors if we know that all of our signals are sinusoids.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where \tilde{V} is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \tilde{V}e^{j\omega t} + \tilde{V}^*e^{-j\omega t}$$

The standard forms for voltage and current phasors are given below:

$$\begin{array}{l|l} \text{Voltage} & \tilde{V} = \frac{1}{2}V_0e^{j\phi_v} \\ \text{Current} & \tilde{I} = \frac{1}{2}I_0e^{j\phi_i} \end{array}$$

We define the *impedance* of a circuit component to be $Z = \frac{\tilde{V}}{\tilde{I}}$, where \tilde{V} and \tilde{I} represent the voltage across and the current through the component, respectively.

3.1 Phasor Relationship for Resistors

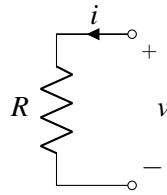


Figure 2: A simple resistor circuit

Consider a simple resistor circuit as in Figure 2, with current being

$$i(t) = I_0 \cos(\omega t + \phi)$$

By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R \cos(\omega t + \phi) \end{aligned}$$

In the phasor domain,

$$\tilde{V} = R\tilde{I}.$$

We usually refer to the impedance of the resistor, Z_R , in the phasor domain. Since $Z_R = R$, we can also write:

$$\tilde{V} = Z_R \tilde{I}.$$

3.2 Phasor Relationship for Capacitors

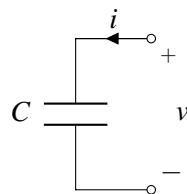


Figure 3: A simple capacitor circuit

Consider a capacitor circuit as in Figure 3, with voltage being

$$v(t) = V_0 \cos(\omega t + \phi)$$

By the capacitor equation,

$$\begin{aligned}
 i(t) &= C \frac{dv}{dt}(t) \\
 &= -CV_0\omega \sin(\omega t + \phi) \\
 &= -CV_0\omega \left(-\cos\left(\omega t + \phi + \frac{\pi}{2}\right) \right) \\
 &= CV_0\omega \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \\
 &= (\omega C)V_0 \cos\left(\omega t + \phi + \frac{\pi}{2}\right)
 \end{aligned}$$

In the phasor domain,

$$\tilde{I} = \omega C e^{j\frac{\pi}{2}} \tilde{V} = j\omega C \tilde{V}.$$

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted Z_C , which is given by:

$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

1. Inductor Impedance

Given the voltage-current relationship of an inductor $V = L \frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

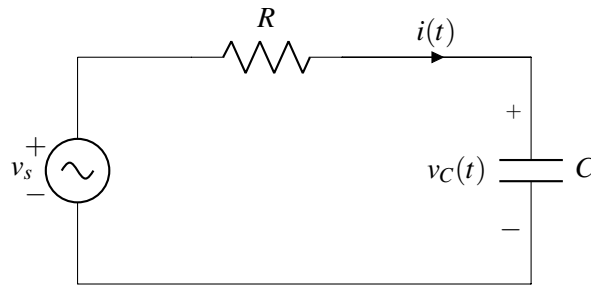
2. Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \tilde{X} e^{j\omega t} + \tilde{X}^* e^{-j\omega t}, \quad (2)$$

where \tilde{X} is a time-independent function called the **phasor** representation of $x(t)$ (recall that \bar{a} denotes the complex conjugate of a). Note that 1) \tilde{X} and \tilde{X}^* are complex conjugates of each other, 2) $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates of each other, and 3) that $\tilde{X} e^{j\omega t}$ and $\tilde{X}^* e^{-j\omega t}$ are also complex conjugates of each other.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12 \sin\left(\omega t - \frac{\pi}{4}\right), \quad (3)$$

with $\omega = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3} \text{k}\Omega$, and $C = 1 \mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

(a) **Step 1: Adopt cosine references (sine references, as shown in class, also work fine)**

All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert $v_s(t)$ into a cosine and write down its phasor representation \tilde{V}_s .

(b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor \tilde{V}_s . The current $i(t)$ is related to its phasor counterpart \tilde{I} by

$$i(t) = \tilde{I}e^{j\omega t} + \tilde{I}^*e^{-j\omega t}.$$

What are the impedances of the resistor, Z_R , and capacitor, Z_C ? We sometimes also refer to this as the "phasor representation" of R and C .

(c) **Step 3: Cast KCL and/or KVL equations in phasor domain**

Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

(d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for \tilde{I} and \tilde{V}_C . What is the polar form of \tilde{I} and \tilde{V}_C ? Polar form is given by $Ae^{j\theta}$, where A is a positive real number.

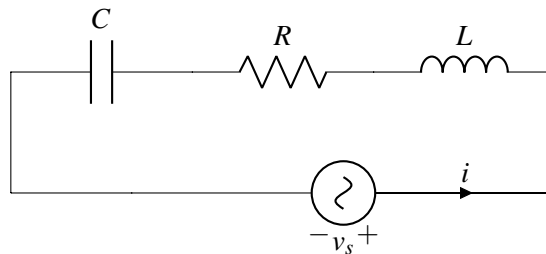
(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?

3. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = B \cos(\omega t - \phi)$$



(a) Write out the phasor representations of $v_s(t)$, R , C , and L .

(b) Use Kirchhoff's laws to write down a loop equation relating the phasors in the previous part.

(c) Solve the equation in the previous step for the current \tilde{I} . What is the magnitude and phase of the polar form of \tilde{I} ?

4. (Practice) Complex Algebra

(a) Express the following values in polar forms: -1 , j , $-j$, \sqrt{j} , and $\sqrt{-j}$.

(b) Represent $\sin \theta$ and $\cos \theta$ using complex numbers. (*Hint*: Use Euler's identity.)

(c) For complex number $z = x + jy$ how that $|z| = \sqrt{z\bar{z}}$, where \bar{z} is the complex conjugate of z .

For the next two parts, consider two complex numbers $V = 3 - j4$ and $I = -(2 + j3)$.

- (d) Express V and I in polar form.
- (e) Find VI , $V\bar{I}$, $\frac{V}{I}$, and \sqrt{I} .
- (f) For complex number $z = x + jy$, what are the roots of $z^2 = 1$? What about $z^3 = 1$? How many roots does $z^n = 1$ have? What is the general form for the solutions of $z^n = 1$?

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