EECS 16B Designing Information Devices and Systems II
Spring 2019 A. Sahai, J. Roychowdhury, K. Pister **Discussion 2B**

1 Transfer Functions

At this point, you are now more or less familiar with the basic tools of *frequency-domain analysis*. In the frequency domain, we represent voltages and currents using complex-valued *phasors*, and we represent capacitors and inductors with their complex-valued *impedances*. Frequency-domain analysis uses no differential equations: all equations are algebraic, just like DC analysis from 16A. In addition, frequency-domain analysis allows us to construct two very powerful tools for describing the *input-output* behavior of a circuit, which are the *transfer function* and the *Bode plot*.

Let's consider an example of why transfer functions are useful. Take a look at the following circuit:

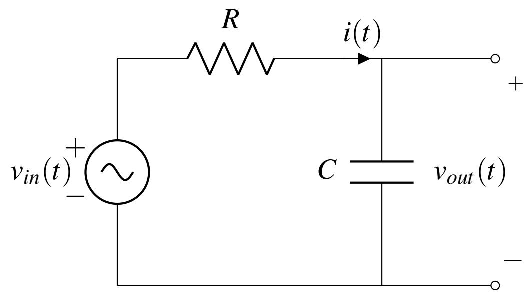


Figure 1: First order RC low-pass filter in the time domain.

It would be nice if we had an easy way to tell, for a given input $v_{in}(t)$, what the corresponding output $v_{out}(t)$ would be. If we wanted to use transient analysis, we would have to solve the differential equation

$$\frac{d}{dt}v_{out}(t) + \frac{1}{RC}v_{out}(t) - \frac{1}{RC}v_{in}(t)) = 0, \tag{1}$$

and match to some initial conditions for every different input $v_{in}(t)$. This is a fair bit of work, even in the simplest of circumstances.

On the other hand, in the frequency domain, we would represent the circuit like this:

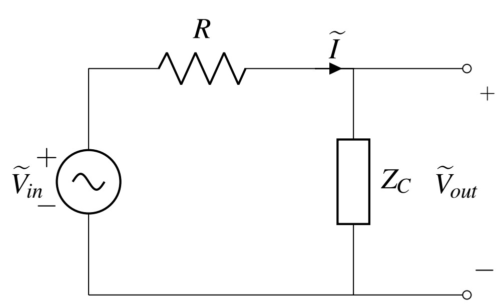


Figure 2: The same first order RC low-pass filter in the frequency domain.

$\widetilde{V}_{in}$ is a given voltage phasor with frequency ω. $\widetilde{V}_{out}$ is in the middle of a voltage divider with the resistor R and the impedance $Z_C = \frac{1}{j\omega C}$, and we can write

$$\widetilde{V}_{out} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \widetilde{V}_{in} = \frac{1}{1 + j\omega RC} \widetilde{V}_{in}. \tag{2}$$

If we define the transfer function $H(\omega)$ as

$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{1}{1 + j\omega RC}. \tag{3}$$

and we rewrite the equation as $\widetilde{V}_{out} = H(\omega)\widetilde{V}_{in}$, then we have achieved our goal. $H(\omega)$ is a simple function relating the input and output of the circuit. We can change our input $\widetilde{V}_{in}$ to a different sinusoid and quickly get our output response $\widetilde{V}_{out}$ with very little extra calculation.

We call such a function a *transfer function*. In general, a transfer function is a ratio between any two phasors in a circuit. Typically the two phasors are considered to be an *input*, which is known, and an *output*, which is to be determined. The transfer function tells us how the known input is *transferred* to the output.

The transfer function of a circuit can tell you in detail how the circuit will respond to inputs of different frequencies. If you have a specific ω in mind, say $\omega = 1000$ radians/s, for example, you can plug this value into the equation for $H(\omega)$ to understand the effect on the output. The transfer function is also useful to see how the circuit will respond to generally "low" or generally "high" frequencies, which you can do by taking the limit of $H(\omega)$ as ω goes to zero or infinity.

Let's try this on the example we just did. At low frequencies, we have

$$\lim_{\omega \to 0} H(\omega) = \frac{1}{1} = 1. \tag{4}$$

This means that, at low frequencies, we have $\widetilde{V}_{out} \approx \widetilde{V}_{in}$. On the other hand, at high frequencies, we have

$$\lim_{\omega \to \infty} H(\omega) = \lim_{\omega \to \infty} \frac{1}{j\omega RC} = 0. \tag{5}$$

So at high frequencies, we have $\widetilde{V}_{out} \approx 0$. In other words, this circuit lets low-frequency inputs pass through the circuit almost unaffected but stops high-frequency inputs from passing. For this reason, we refer to this type of circuit as a *low-pass filter*.

2 Plotting Transfer Functions and Bode Plots

It is often useful to be able to plot the transfer function of a circuit as a function of frequency. This allows us to visualize how our circuit reacts to any particular frequency. Because the frequency response is a complex number, we plot the magnitude and phase of the frequency response on separate plots. In this section, we will introduce a graphical tool to approximate the graph of a transfer function without using a computer. This tool is called a *Bode plot*, and it relies on the power of log-log plots by approximating $\log_{10}(|H(\omega)|)$ by a piecewise-linear function.

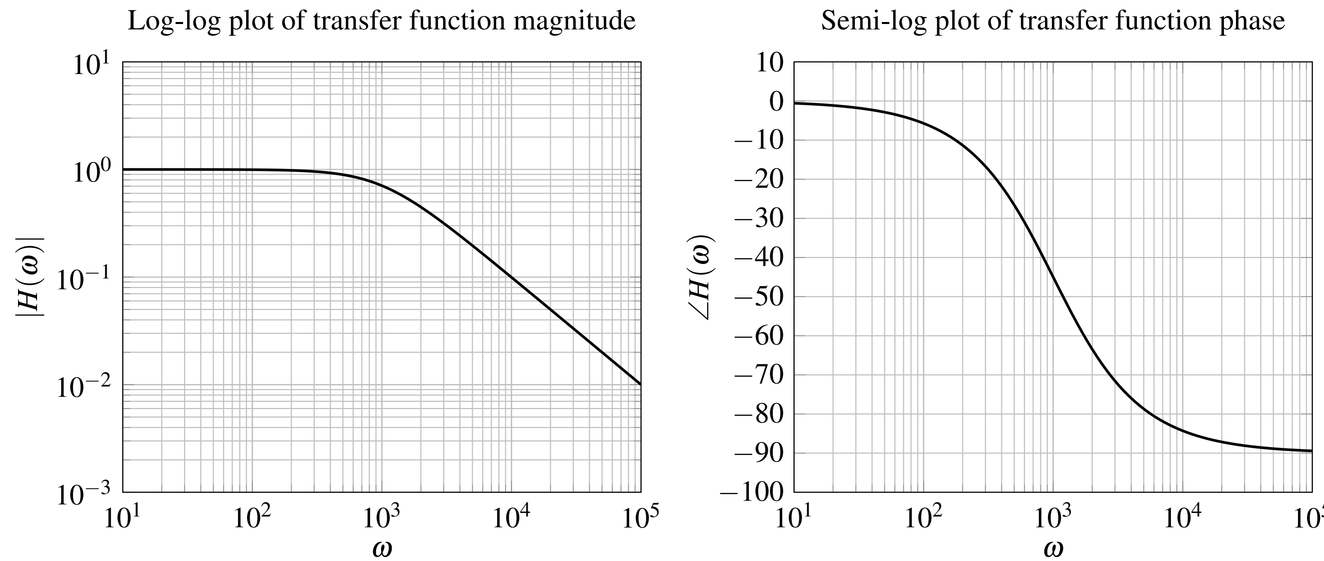
To show you how it works, we'll plot the transfer function for the RC low-pass filter we've been looking at. We need specific values to make the plot, so we'll take $R = 1\,\text{k}\Omega, C = 1\,\mu\text{F}$. We've analyzed this circuit before and found the transfer function to be:

$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{1}{1 + j\omega RC}. \tag{6}$$

Since $H(\omega)$ is complex, we need two separate plots to show the whole thing. We could either plot the real part and the imaginary part, or the magnitude and phase. It will be more useful to plot the magnitude and the phase, since that will make the effect of $H(\omega)$ on phasors more apparent. First, we'll find the magnitude and phase response from the transfer function:

$$|H(\omega)| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{|1|}{|j\omega RC + 1|} = \frac{1}{\sqrt{(\omega RC)^2 + 1}},$$
$$\angle H(\omega) = \angle \frac{1}{j\omega RC + 1} = \angle 1 - \angle(j\omega RC + 1) = -\tan^{-1}(\omega RC).$$

Just to give you a mental picture, using plotting software to plot the exact $|H(\omega)|$ and $\angle H(\omega)$ gives the following:



For now, we will focus on the plot of the magnitude of the transfer function, $|H(\omega)|$. This plot verifies our limit-based analysis of $H(\omega)$: The magnitude stays near 10^0 at low frequencies, then at some point it starts to decrease. The point at which $|H(\omega)|$ starts to decrease is called the *cutoff frequency* of the filter, and denoted ω_c.

How would we go about plotting this by hand? Well, first we would need to express $\log_{10}(|H(\omega)|)$ in terms of $\log_{10}(\omega)$. Taking the logarithm of both sides of our expression for $|H(\omega)|$ gives

$$\log_{10}(|H(\omega)|) = \log_{10}\left(\frac{1}{\sqrt{(\omega RC)^2 + 1}}\right) = -\frac{1}{2}\log_{10}((\omega RC)^2 + 1) \tag{7}$$

Here is where we will make the approximation. For frequencies such that $\omega RC \ll 1$, *i.e.*, for frequencies much lower than the cutoff frequency, we can ignore the $(\omega RC)^2$ term inside the logarithm.[1] This gives

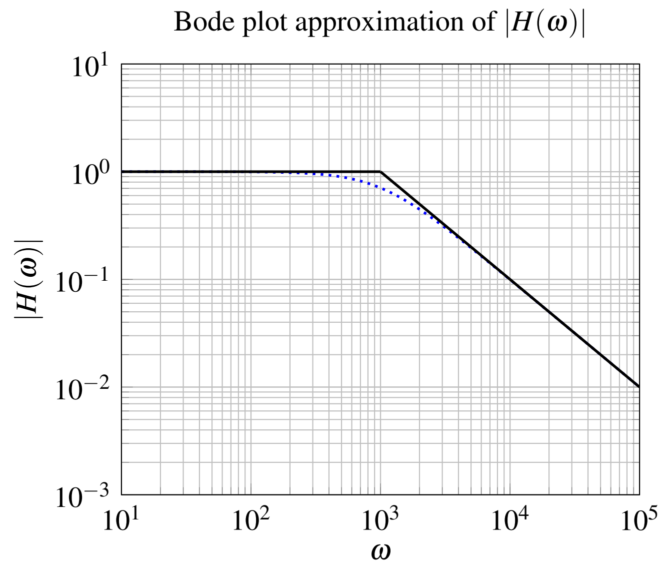
$$\log_{10}(|H(\omega)|) \approx -\frac{1}{2}\log_{10}(1) = 0, \text{ when } \omega RC \ll 1 \tag{8}$$

[1] Why? because $1 +$ something much smaller $\simeq 1$.

On the other hand, for frequencies such that $\omega RC \gg 1$, *i.e.*, for frequencies far beyond the cutoff, we can ignore the 1 inside the logarithm. This gives

$$\log_{10}(|H(\omega)|) \; - \frac{1}{2}\log_{10}\left((\omega RC)^2\right) = -\log_{10}(\omega) + \log_{10}\left(\frac{1}{RC}\right), \text{ when } \omega RC \gg 1. \tag{9}$$

If we combine these two approximations on a log-log plot, we get the following approximation of $|H(\omega)|$:



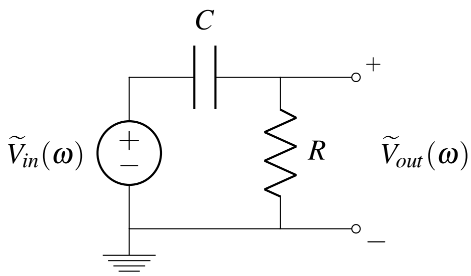
The true transfer function is shown as a dotted line.

In the coming weeks, you will learn how to construct Bode plots for all kinds of transfer functions. The purpose of this section is to help you gain an intuition for how the approximation is made: specifically, the approximation of $\log_{10}(|H(\omega)|)$ by a piecewise combination of linear functions.

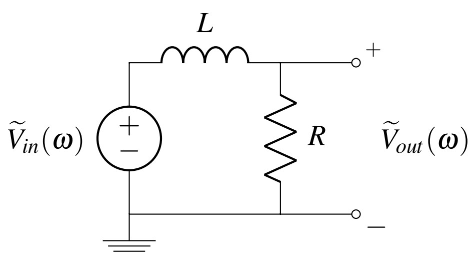
1. Transfer function practice

Now that you've seen how to derive a transfer function for a simple circuit, you'll be deriving some transfer functions on your own. To start off with, determine $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$ for the following circuits, and use the transfer function you find to describe how the circuit responds to low and high frequencies.

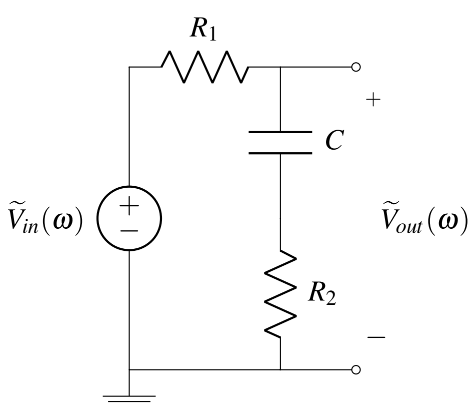
(a) Determine $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?



(b) Determine $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?

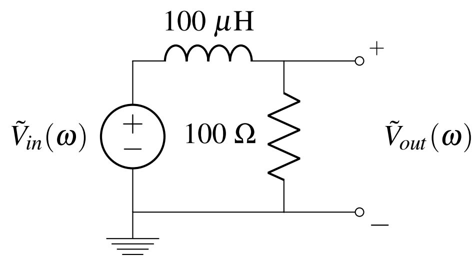


(c) Determine $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?

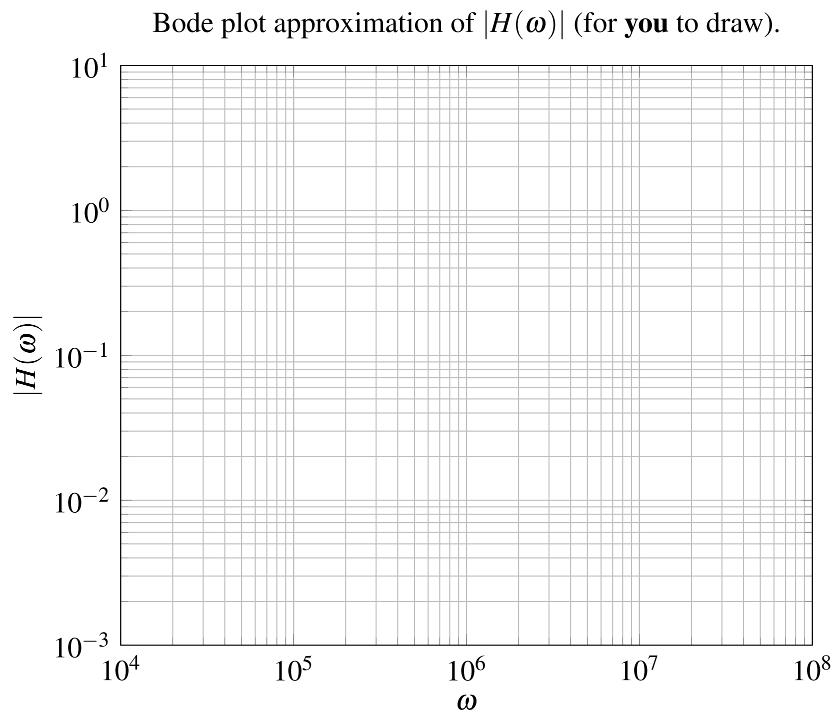


2. Bode plot practice

For this problem, we will try to follow the same reasoning from section 2 to construct a Bode plot of the following circuit:



Here is a blank log-log plot for you to draw your Bode plot in.



(a) Write an expression for $\log_{10}(|H(\omega)|)$. For now, you can keep it in terms of R and L.

(b) What is the cutoff frequency for this circuit? Mark it on the log-log plot with a vertical line.

(c) Find two approximations of $\log_{10}(|H(\omega)|)$: one valid for frequencies far below the cutoff frequency, and one valid for frequencies far above it. Draw both of them on the log-log plot, in the regions that they are valid.

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