1. Changing Coordinates and Systems of Differential Equations

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -5x_1(t) \tag{1}$$

$$\frac{d}{dt}x_2(t) = -2x_2(t) \tag{2}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 2$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

(b) Suppose we want to change variables to:

$$\breve{x}_1(t) = x_1(t) + x_2(t)$$
$$\breve{x}_2(t) = -x_1(t) + 2x_2(t)$$

Write out a system of differential equations relating the $\frac{d}{dt}\breve{x}_i(t)$ to the $\breve{x}_i(t)$. The system can have equations that involve both $i = 1, 2$ variables on the right hand side, but should only involve one of the $\frac{d}{dt}\breve{x}_i(t)$ on each of the left hand sides.

Change to these new coordinates using two methods.

Method 1: Direct Substitution Solve for $\breve{x}_i(t)$ by direct substitution from $x_i(t)$.

Method 2: Linear Algebra Change-of-Coordinates Solve for $\breve{x}_i(t)$ using a change of coordinates, like you saw in EE16A.

(c) How do the initial conditions for $x_i(t)$ translate into the initial conditions for $\breve{x}_i(t)$? What are the solutions for $\breve{x}_i(t)$?

(d) Consider a "new" system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}y_1(t) = -4y_1(t) + y_2(t) \tag{3}$$

$$\frac{d}{dt}y_2(t) = 2y_1(t) - 3y_2(t) \tag{4}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 3$.

Write out the differential equations and initial conditions in matrix/vector form.

(e) Find the eigenvalues λ_1, λ_2 and eigenspaces for the differential equation matrix above.

(f) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t)$, $z_{\lambda_2}(t)$. (These variables represent eigenbasis-aligned coordinates.)

(g) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

(h) Convert your solution back into the original coordinates to find $y_i(t)$.

(i) We can solve this equation using a slightly shorter approach by observing that the solutions for $y_i(t)$ will all be of the form

$$y_i(t) = \sum_k K_{i,k} e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix and the $K_{i,k}$ are constants derived from our initial conditions and the coordinate changes involved.

Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $y_i(t)$ as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} \\ \gamma e^{\lambda_1 t} + \kappa e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha,\ \beta,\ \gamma,\ \kappa$ are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} y_1(t) \\ \frac{d}{dt} y_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for $y_i(t)$ from this form of the derivative.

2. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations

$$\frac{d}{dt} x_1(t) = \lambda x_1(t) \tag{5}$$

$$\frac{d}{dt} x_2(t) = \overline{\lambda} x_2(t) \tag{6}$$

with initial conditions $x_1(0) = c_0$ and $x_2(0) = \overline{c}_0$, where λ and c_0 are complex numbers and $\overline{\lambda}$ and $\overline{c}_0$ are their complex conjugates, respectively.

Suppose now that we have the following different variables related to the original ones:

$$\check{x}_1(t) = a x_1(t) + \overline{a} x_2(t)$$
$$\check{x}_2(t) = b x_1(t) + \overline{b} x_2(t)$$

a and b are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written:

$$a = a_r + j a_i$$
$$\overline{a} = a_r - j a_i$$
$$b = b_r + j b_i$$
$$\overline{b} = b_r - j b_i$$

(a) First, assume that $\lambda = j$ in the equations for $x_1(t)$ and $x_2(t)$ above. Write out a system of differential equations using $\frac{d}{dt}\check{x}_i(t)$ and $\check{x}_i(t)$. (Just use coordinate changes this time. No need to work through direct substitutions unless you want to check your work that way.)

(b) How do the initial conditions for $x_i(t)$ translate into the initial conditions for $\check{x}_i(t)$?

(c) Find the eigenvalues $\lambda_1,\ \lambda_2$ and associated eigenspaces for the differential equation matrix for $\breve{x}_i(t)$ above.

(d) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t),\ z_{\lambda_2}(t)$. (These variables are in eigenbasis-aligned coordinates.)

(e) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

(f) Convert your solution back into the $\breve{x}_i(t)$ coordinates to find $\breve{x}_i(t)$.

(g) Repeat the above for general complex λ.

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