EECS 16B Designing Information Devices and Systems II
Spring 2019 Discussion Worksheet

4B: Inputs

1. Differential equations with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs and the sampling of a continuous-time system of differential equations into a discrete-time view.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Suppose that our $u(t)$ of interest is *constructed* to be piecewise constant over durations of width Δ. In other words:

$$u(t) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta) \tag{2}$$

Given that we start at $x(i\Delta) = x_d[i]$, where do we end up at $x_d[i+1] = x((i+1)\Delta)$?

(b) Suppose that $x_d[0] = x_0$. **Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves x_0 and the $u[j]$ for $j = 0, 1, \ldots, i$.**

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{3}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.

(c) **For a given time t in real time, what is the i that corresponds to it?** Meaning that if we think of the computer viewing the system as perceiving a constant state for durations $[i\Delta, (i+1)\Delta)$, which is the discrete time index for the $x_d[i]$ that corresponds to t in real time.

(d) Now, we are going to turn this around. Suppose that the $u[i]$ is actually a sample of a desired control input $u_c(t)$ in continuous time. Namely that $u[i] = u_c(i\Delta)$. Recall what the a and b mean in (3) and **approximate $x(t)$ if we apply this piecewise constant control $u(t)$ to the system (1).** You can assume that Δ is small enough that $x(t)$ can't change too much over an interval of length Δ.

(e) **Draw a picture of what is going on and then further approximate the previous expression by considering $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ where needed and treating $\Delta \approx \frac{t}{n}$.** This is a meaningful approximation when we think about n large enough.

(f) **Take the limit of $\Delta \to 0$ by taking the limit $n \to \infty$. What is the expression you get for $x(t)$?**

(HINT: Remember your definition of definite integrals as limits of Riemann sums in calculus.)

(g) Stepping back. Suppose we have a system of differential equations with an input that we express in vector form:

$$\frac{d}{dt}\vec{x}_c(t) = A\vec{x}_c(t) + \vec{b}u(t) \tag{4}$$

where $\vec{x}_c(t)$ is n-dimensional.

Suppose further that the matrix A is invertible and has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Collect the eigenvectors together into a matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. If we apply a piecewise constant control input $u(t)$ as in (2), and sample the system $\vec{x}_d[i] = \vec{x}_c(i\Delta)$, **what are the corresponding A_d and $\vec{b}_d$ in:**

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b}_d u[i]. \tag{5}$$

(h) Leveraging what you learned in the previous part, **explain why if the input $u(t) = e^{st}$ is applied in (4), then any of the state variables within $\vec{x}_c(t)$ can be expressed as a sum $\sum_{k=1}^{n} (\alpha_k(s) e^{\lambda_k t} + \frac{\beta_k}{s-\lambda_k} e^{st})$ where the β_k do not depend on s or the initial condition even though the $\alpha_k(s)$ might depend on both s and the inital condition $\vec{x}_c(0)$.**

Contributors:

- Anant Sahai.
- Nikhil Shinde.
- Sanjit Batra.
- Aditya Arun.

